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Asymptotic behavior of semi-canonical third-order delay differential equations with a superlinear neutral term

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Abstract This paper deals with the asymptotic behavior of solutions to a class of semi-canonical third-order delay differential equations with a superlinear neutral term. First, we transform the semi-canonical equation into canonical type and then by applying a Riccati type transformation and integral conditions, we obtain some new sufficient conditions ensuring that every solution of the studied equation either oscillates or converges to zero asymptotically. The results are illustrated via two examples.

1 Introduction

In this paper, we study the oscillation and asymptotic behavior of solutions to the semi-canonical third-order delay differential equations with a superlinear neutral term of the form

$$(r_2(t)(r_1(t)(z'(t))^{\alpha})')' + q(t)y^{\lambda}(\sigma(t)) = 0, \quad t \geq t_0 > 0, \tag{1.1}$$

where $z(t) = y(t) + p(t)y^{\beta}(\tau(t))$, and the constants α, β and λ are the ratios of odd positive integers with $\beta \geq 1$. In the sequel, without further mention, we assume that:

(C_1) $r_2, r_1, p, q : [t_0, \infty) \to \mathbb{R}$ are continuous functions with $r_2(t) > 0$, $r_1(t) > 0$, $p(t) \geq 1$, $p(t) \not\equiv 1$ for large t, $q(t) \geq 0$, and $q(t)$ is not identically zero for large t;

(C_2) $\tau, \sigma : [t_0, \infty) \to \mathbb{R}$ are continuous functions such that $\tau(t) \leq t$, $\sigma(t) \leq \tau(t)$, τ is strictly increasing, and $\lim_{t\to\infty} \tau(t) = \lim_{t\to\infty} \sigma(t) = \infty$;

(C_3) equation (1.1) is in semi-canonical form, i.e.,

$$\int_{t_0}^{\infty} \frac{1}{r_2(t)} dt < \infty \quad \text{and} \quad \int_{t_0}^{\infty} \frac{1}{r_1^{1/\alpha}(t)} dt = \infty. \tag{1.2}$$

We note that if assumption (1.2) in (C_3) is replaced by

$$\int_{t_0}^{\infty} \frac{1}{r_2(t)} dt = \int_{t_0}^{\infty} \frac{1}{r_1^{1/\alpha}(t)} dt = \infty,$$

we say that equation (1.1) is in canonical form.

By a *solution* of (1.1), we mean a function $y \in C([t_y, \infty), \mathbb{R})$ for some $t_y \geq t_0$ such that $z \in C^1([t_y, \infty), \mathbb{R})$, $r_1(z')^{\alpha} \in C^1([t_y, \infty), \mathbb{R})$, $r_2(r_1(z')^{\alpha})' \in C^1([t_y, \infty), \mathbb{R})$ and y satisfies (1.1) on $[t_y, \infty)$. We only consider those solutions y of (1.1) that exist on some half-line $[t_y, \infty)$ and satisfy the condition

$$\sup\{|y(t)| : T_1 \leq t < \infty\} > 0 \text{ for any } T_1 \geq t_y;$$

further, we tacitly assume that (1.1) possesses such solutions. Such a solution $y(t)$ of (1.1) is said to be *oscillatory* if it has arbitrarily large zeros on $[t_y, \infty)$, and it is called *nonoscillatory* otherwise.

The investigation of qualitative properties of solutions of neutral type differential equations is not only of theoretical interest but also has important practical applications. This is due to the fact that such equations arise in number of applied problems in natural sciences, engineering and control. For instance, see [10, 23] for some particular applications of differential equations with a nonlinear neutral term.

Oscillatory and asymptotic behavior of solutions to various classes of third-order neutral type differential equations have attracted great interest of researches; see, e.g., [1, 2, 3, 4, 5, 6, 7, 8, 9, 12, 14, 15, 16, 17, 18, 19, 20, 21, 22] and the references cited therein. A commonly used assumption is that the neutral term is linear, that is, $\beta = 1$, see, e.g., [1, 2, 3, 4, 9, 12, 14, 15, 18, 20, 22] and the references therein for more details. However, very few results are available for equations with nonlinear neutral term (i.e., $\beta \neq 1$). For the results concerned with a sublinear neutral term (i.e., $0 < \beta < 1$), see [6, 7, 19]; and for a superlinear neutral term (i.e., $\beta > 1$), see [8, 16, 21].

Very recently in [16], the authors studied the equation

$$\left(a(t)\left[(x(t)+p(t)x^{\beta}(\tau(t)))''\right]^{\alpha}\right)' + q(t)x^{\delta}(\sigma(t)) = 0 \tag{1.3}$$

under the assumption

$$\int_{t_0}^{t} \frac{1}{a^{1/\alpha}(s)} ds < \infty \quad \text{as } t \to \infty, \quad \text{and } \beta \geq 1, \tag{1.4}$$

without changing the form of the equation.

Therefore, our aim in this paper is first to transform (1.1) into canonical type equation and then use a Riccati type transformation as well as integral criteria to analyze the behavior of its solutions. To the best of our knowledge, there are no results for (1.1) using the above mentioned technique, and so this paper contributes further in the oscillation theory of third-order neutral differential equations.

While considering nonoscillatory solutions, we deal only with eventually positive solutions of (1.1) due to the fact, under our assumptions on α, β and λ, if y is a solution, so is $-y$.

Lemma 1.1. *Assume that y is an eventually positive solution of (1.1). Then there exists a $t_1 \geq t_0$ such that, for $t \geq t_1$, the corresponding function z satisfies one of the following three possibilities:*

$(\mathcal{N}_0)$: $z(t) > 0,\ z'(t) < 0,\ (r_1(t)(z'(t))^{\alpha})' > 0,\ (r_2(t)(r_1(t)(z'(t))^{\alpha})')' \leq 0,$

$(\mathcal{N}_1)$: $z(t) > 0,\ z'(t) > 0,\ (r_1(t)(z'(t))^{\alpha})' < 0,\ (r_2(t)(r_1(t)(z'(t))^{\alpha})')' \leq 0,$

$(\mathcal{N}_2)$: $z(t) > 0,\ z'(t) > 0,\ (r_1(t)(z'(t))^{\alpha})' > 0,\ (r_2(t)(r_1(t)(z'(t))^{\alpha})')' \leq 0,$

Proof. Let $y(t)$ be an eventually positive solution of (1.1), say $y(t) > 0$, $y(\tau(t)) > 0$ and $y(\sigma(t)) > 0$ for $t \geq t_1$ for some $t_1 \geq t_0$. It follows from (1.1) that

$$(r_2(t)(r_1(t)(z'(t))^{\alpha})')' = -q(t)y^{\lambda}(\sigma(t)) \leq 0$$

for $t \geq t_1$. Then, $r_2(t)(r_1(t)(z'(t))^{\alpha})'$ is decreasing for $t \geq t_1$. Therefore, $(r_1(t)(z'(t))^{\alpha})'$ and $z'(t)$ are eventually of one sign. Then, there exists a sufficiently large $t_2 \geq t_1$ so that $(r_1(t)(z'(t))^{\alpha})'$ and $z'(t)$ are of fixed sign for $t \geq t_2$. Therefore, we consider the following cases:

(i) $(r_1(t)(z'(t))^{\alpha})' < 0$ and $z'(t) < 0$;

(ii) $(r_1(t)(z'(t))^{\alpha})' < 0$ and $z'(t) > 0$;

(iii) $(r_1(t)(z'(t))^{\alpha})' > 0$ and $z'(t) < 0$;

(iV) $(r_1(t)(z'(t))^{\alpha})' > 0$ and $z'(t) > 0$,

for $t \geq t_2$. For the case (i), we see that

$$z(t) = z(t_2) + \int_{t_2}^{t} \frac{(r_1(s)(z'(s))^{\alpha})^{1/\alpha}}{r_1^{1/\alpha}(s)} ds \leq z(t_2) + (r_1(t_2)(z'(t_2))^{\alpha})^{1/\alpha} \int_{t_2}^{t} \frac{1}{r_1^{1/\alpha}(s)} ds.$$

Hence, by (1.2), we see that $\lim_{t\to\infty} z(t) = -\infty$, which contradicts the fact that z is a positive, and so the case (i) is impossible. This completes the proof. □

So, if we want to obtain oscillation criteria for (1.1), we have to eliminate the above mentioned three classes $(\mathcal{N}_0)$-$(\mathcal{N}_2)$, which may lead to three conditions. To reduce the above mentioned classes to two, we assume a simple condition that yields to a canonical form and this essentially simplifies the investigation of (1.1).

2 Main Results

To prove our main theorems, we require the following lemmas. To present the results in a compact form, we adopt the following notation:

$$\Pi(t) := \int_t^\infty \frac{1}{r_2(s)}ds,\ d(t) := r_2(t)\Pi^2(t),\ c(t) := \frac{r_1(t)}{\Pi(t)},\ F(t) := \Pi(t)q(t);$$

$$\mu(t,t_1) := \int_{t_1}^t \frac{1}{d(s)}ds \text{ for } t \geq t_1 \geq t_0,\ \eta(t,t_2) := \int_{t_2}^t \left(\frac{\mu(s,t_1)}{c(s)}\right)^{1/\alpha} ds \text{ for } t \geq t_2 \geq t_1;$$

$$Q(t) := \begin{cases} 1, & \text{if } \frac{\lambda}{\beta} - \alpha = 0, \\ d_1, & \text{if } \frac{\lambda}{\beta} - \alpha > 0, \\ d_2\eta^{\frac{\lambda}{\beta}-\alpha}(t,t_2), & \text{if } \frac{\lambda}{\beta} - \alpha < 0, \end{cases}$$

for all constants $d_1 > 0$ and $d_2 > 0$;

$$G_1(t) := \frac{1}{p(\tau^{-1}(t))}\left(1 - \frac{m^{\frac{1}{\beta}-1}}{p^{1/\beta}(\tau^{-1}(\tau^{-1}(t)))}\right) \geq 0, \tag{2.1}$$

for all constants $m > 0$;

$$G_2(t) := \frac{1}{p(\tau^{-1}(t))}\left[1 - \frac{\eta(\tau^{-1}(\tau^{-1}(t)),t_2)}{\eta(\tau^{-1}(t),t_2)}\frac{1}{p^{1/\beta}(\tau^{-1}(\tau^{-1}(t)))}\right] \geq 0, \tag{2.2}$$

for sufficiently large $t_2 \geq t_1 \geq t_0$ and for sufficiently large t;

$$\Omega(t) := g(t)F(t)G_2^{\lambda/\beta}(\sigma(t))Q(\delta(t))$$

for some nondecreasing function $g \in C^1([t_0,\infty),(0,\infty))$, where $\delta(t) := \tau^{-1}(\sigma(t))$.

Lemma 2.1 ([11]). *If $X > 0$ and $0 < \gamma \leq 1$, then*

$$X^\gamma \leq \gamma X + (1-\gamma),$$

and equality holds when $\gamma = 1$.

Lemma 2.2. *Assume that*

$$\int_{t_0}^\infty \left(\frac{\Pi(t)}{r_1(t)}\right)^{1/\alpha} dt = \infty. \tag{2.3}$$

Then the semi-canonical operator

$$\mathcal{L}z = (r_2(t)(r_1(t)(z'(t))^\alpha)')'$$

has the following unique canonical representation

$$\mathcal{L}z = \frac{1}{\Pi(t)}(d(t)(c(t)(z'(t))^\alpha)')'.$$

Proof. The proof is similar to Theorems 2.1 in [17] and hence the details are omitted. □

Now it follows from Lemma 2.2 that (1.1) can be written in the canonical form as below

$$(d(t)(c(t)(z'(t))^\alpha)')' + F(t)y^\lambda(\sigma(t)) = 0, \tag{2.4}$$

and the following result is immediate.

Theorem 2.3. *Assume* (2.3) *holds. Then semi-canonical equation* (1.1) *possesses a solution* $y(t)$ *if and only if canonical equation* (2.4) *has the solution* $y(t)$.

Corollary 2.4. *Assume that* (2.3) *holds. Then semi-canonical equation* (1.1) *has an eventually positive solution if and only if canonical equation* (2.4) *has an eventually positive solution.*

From Corollary 2.4, it is clear that the investigation of the oscillation of (1.1) is reduced to that of (2.4), and therefore, if y is an eventually positive solution of (1.1), then from Kiguradze's lemma [13] the corresponding function z satisfies one of the following two cases:

$(\mathcal{N}_0^*)$: $z(t) > 0,\ c(t)(z'(t))^\alpha < 0,\ d(t)(c(t)(z'(t))^\alpha)' > 0,\ (d(t)(c(t)(z'(t))^\alpha)')' \leq 0,$

$(\mathcal{N}_2^*)$: $z(t) > 0,\ c(t)(z'(t))^\alpha > 0,\ d(t)(c(t)(z'(t))^\alpha)' > 0,\ (d(t)(c(t)(z'(t))^\alpha)')' \leq 0,$

for sufficiently large t.

Lemma 2.5. *Let condition* (2.3) *holds and assume that* y *is an eventually positive solution of* (1.1) *and* $z \in (\mathcal{N}_0^*)$. *If, for all constants* $m > 0$,

$$\int_{t_0}^{\infty} F(t)G_1^{\lambda/\beta}(\sigma(t))dt = \infty, \tag{2.5}$$

or

$$\int_{t_0}^{\infty} F(t)G_1^{\lambda/\beta}(\sigma(t))dt < \infty \tag{2.6}$$

and

$$\int_{t_0}^{\infty} \frac{1}{c^{1/\alpha}(v)} \left(\int_v^{\infty} \frac{1}{d(u)} \int_u^{\infty} F(s)G_1^{\lambda/\beta}(\sigma(s))dsdu \right)^{1/\alpha} dv = \infty, \tag{2.7}$$

then $\lim_{t\to\infty} y(t) = \lim_{t\to\infty} z(t) = 0$.

Proof. Let $y(t)$ be an eventually positive solution of (1.1), say $y(t) > 0$, $y(\tau(t)) > 0$ and $y(\sigma(t)) > 0$ for $t \geq t_1$ for some $t_1 \geq t_0$. Then, by Corollary 2.4, $y(t)$ is also an eventually positive solution of (2.4). From the definition of z, we see that $p(t)y^\beta(\tau(t)) \leq z(t)$ and so

$$y^\beta(\tau(t)) \leq \frac{z(t)}{p(t)}. \tag{2.8}$$

From (2.8) and the fact that τ is strictly increasing, it is easy to see that

$$y(\tau^{-1}(t)) \leq \frac{z^{1/\beta}(\tau^{-1}(\tau^{-1}(t)))}{p^{1/\beta}(\tau^{-1}(\tau^{-1}(t)))}.$$

Using this in the definition of z, we obtain

$$y^\beta(t) \geq \frac{1}{p(\tau^{-1}(t))} \left[z(\tau^{-1}(t)) - \frac{z^{1/\beta}(\tau^{-1}(\tau^{-1}(t)))}{p^{1/\beta}(\tau^{-1}(\tau^{-1}(t)))} \right]. \tag{2.9}$$

From the fact that $\tau(t) \leq t$ and τ is strictly increasing, we observe that τ^{-1} is increasing and $t \leq \tau^{-1}(t)$. Thus,

$$\tau^{-1}(t) \leq \tau^{-1}(\tau^{-1}(t)). \tag{2.10}$$

Since z is positive and decreasing, it follows from (2.10) that

$$z\left(\tau^{-1}(t)\right) \geq z\left(\tau^{-1}(\tau^{-1}(t))\right).$$

Using this in (2.9), we obtain

$$y^\beta(t) \geq \frac{z(\tau^{-1}(t))}{p(\tau^{-1}(t))} \left[1 - \frac{z^{\frac{1}{\beta}-1}(\tau^{-1}(t))}{p^{1/\beta}(\tau^{-1}(\tau^{-1}(t)))} \right]. \tag{2.11}$$

Since z is positive and decreasing, there exists a constant ℓ such that

$$\lim_{t\to\infty} z(t) = \ell < \infty,$$

where $\ell \geq 0$. If $\ell := m > 0$, then there exists $t_2 \geq t_1$ such that

$$z(t) \geq \ell \quad \text{for } t \geq t_2. \tag{2.12}$$

From (2.11) and (2.12), we observe that

$$y^{\beta}(t) \geq G_1(t)z(\tau^{-1}(t)).$$

Using this in (2.4) gives

$$(d(t)(c(t)(z'(t))^{\alpha})')' \leq -F(t)G_1^{\lambda/\beta}(\sigma(t))z^{\lambda/\beta}(\delta(t)) \leq -\ell^{\lambda/\beta}F(t)G_1^{\lambda/\beta}(\sigma(t)) \tag{2.13}$$

for $t \geq t_3$ for some $t_3 \geq t_2$. Integrating (2.13) from t_3 to ∞ yields

$$\int_{t_3}^{\infty} F(t)G_1^{\lambda/\beta}(\sigma(t))dt \leq \frac{d(t_3)(c(t_3)(z'(t_3))^{\alpha})'}{\ell^{\lambda/\beta}} < \infty,$$

which contradicts (2.5), and so we have $\ell = 0$. Therefore, $\lim_{t\to\infty} z(t) = 0$. Since $0 < y(t) \leq z(t)$ on $[t_1, \infty)$, we easily deduce that $\lim_{t\to\infty} y(t) = 0$.

Now we consider the case (2.5) is not satisfied, i.e., (2.6) holds. Then, integrating (2.13) from $t \geq t_3$ to ∞ consecutively two times, we are lead to

$$\frac{1}{c^{1/\alpha}(t)}\left(\int_t^{\infty}\frac{1}{d(u)}\int_u^{\infty}F(s)G_1^{\lambda/\beta}(\sigma(s))dsdu\right)^{1/\alpha} \leq -\frac{z'(t)}{\ell^{\lambda/\alpha\beta}}.$$

One more integration of the last inequality from t_3 to ∞ yields

$$\int_{t_3}^{\infty}\left(\frac{1}{c(v)}\int_v^{\infty}\frac{1}{d(u)}\int_u^{\infty}F(s)G_1^{\lambda/\beta}(\sigma(s))dsdu\right)^{1/\alpha}dv \leq \frac{z(t_3)}{\ell^{\lambda/\alpha\beta}},$$

which contradicts (2.7), and thus $\ell = 0$. This completes the proof of the lemma. □

Lemma 2.6. *Let $y(t)$ be an eventually positive solution of (2.4) with the corresponding function $z \in (\mathcal{N}_2^*)$. Then the following:*

(i) $z'(t) \geq \left(\frac{\mu(t,t_1)}{c(t)}\right)^{1/\alpha}(d(t)(c(t)(z'(t))^{\alpha})')^{1/\alpha}$ *for $t \geq t_1$,*

(ii) $\frac{c^{1/\alpha}(t)z'(t)}{\mu^{1/\alpha}(t,t_1)}$ *is decreasing for $t \geq t_2$,*

(iii) $z(t) \geq \frac{c^{1/\alpha}(t)z'(t)}{\mu^{1/\alpha}(t,t_1)}\eta(t,t_2)$ *for $t \geq t_2$,*

(iv) $\frac{z(t)}{\eta(t,t_2)}$ *is decreasing for $t \geq t_3$*

hold for all sufficiently large $t_1 \geq t_0$ and for $t_3 > t_2 > t_1$.

Proof. Let $y(t)$ be an eventually positive solution of (2.4), say $t \geq t_1$ for some $t_1 \geq t_0$. Since $z \in (\mathcal{N}_2^*)$, we see that

$$c(t)(z'(t))^{\alpha} \geq \int_{t_1}^{t}\frac{d(s)(c(s)(z'(s))^{\alpha})'}{d(s)}ds \geq \mu(t,t_1)d(t)(c(t)(z'(t))^{\alpha})'. \tag{2.14}$$

From (2.14), we have

$$z'(t) \geq \left(\frac{\mu(t,t_1)}{c(t)}\right)^{1/\alpha}(d(t)(c(t)(z'(t))^{\alpha})')^{1/\alpha} \quad \text{for } t \geq t_1,$$

i.e., (i) holds. From (2.14), we see that, for $t \geq t_2 > t_1$,

$$\left(\frac{c(t)(z'(t))^{\alpha}}{\mu(t,t_1)}\right)' = \frac{\mu(t,t_1)d(t)(c(t)(z'(t))^{\alpha})' - c(t)(z'(t))^{\alpha}}{d(t)\mu^2(t,t_1)} \leq 0,$$

i.e., $c(t)(z'(t))^\alpha/\mu(t,t_1)$ is decreasing, and so $c^{1/\alpha}(t)z'(t)/\mu^{1/\alpha}(t,t_1)$ is decreasing, i.e., (ii) holds. Using the fact that $c^{1/\alpha}(t)z'(t)/\mu^{1/\alpha}(t,t_1)$ is decreasing, we get, for $t \geq t_2$,

$$z(t) \geq \int_{t_2}^{t} z'(s)ds \geq \int_{t_2}^{t} \frac{\mu^{1/\alpha}(s,t_1)}{c^{1/\alpha}(s)} \frac{c^{1/\alpha}(s)z'(s)}{\mu^{1/\alpha}(s,t_1)} ds \geq \frac{c^{1/\alpha}(t)z'(t)}{\mu^{1/\alpha}(t,t_1)}\eta(t,t_2), \tag{2.15}$$

i.e., (iii) holds. From (2.15), we see that, for $t \geq t_3 > t_2$,

$$\left(\frac{z(t)}{\eta(t,t_2)}\right)' = \frac{c^{1/\alpha}(t)z'(t)\eta(t,t_2) - \mu^{1/\alpha}(t,t_1)z(t)}{c^{1/\alpha}(t)\eta^2(t,t_2)} \leq 0,$$

i.e., (iv) holds. This complete the proof. □

Lemma 2.7. *Let condition* (2.3) *holds and assume that y is an eventually positive solution of* (1.1) *and $z \in (\mathcal{N}_2^*)$. Then for all sufficiently large $t_1 \in [t_0,\infty)$, for some $t_2 \in [t_1,\infty)$, and $t_3 \in [t_2,\infty)$, the following inequality, for $t \geq t_3$,*

$$(d(t)(c(t)(z'(t))^\alpha)')' + F(t)G_2^{\lambda/\beta}(\sigma(t))Q(\delta(t))z^\alpha(\delta(t)) \leq 0 \tag{2.16}$$

holds.

Proof. Let $y(t)$ be an eventually positive solution of (1.1), say $y(t) > 0$, $y(\tau(t)) > 0$ and $y(\sigma(t)) > 0$ for $t \geq t_1$ for some $t_1 \geq t_0$. Then, by Corollary 2.4, $y(t)$ is also an eventually positive solution of (2.4) for $t \geq t_1$. As in the proof of Lemma 2.5, we again see that (2.8) and (2.10) hold. Applying Lemma 2.1 to (2.8), we have

$$y(\tau(t)) \leq \frac{\frac{1}{\beta}z(t) + (1-\frac{1}{\beta})}{p^{1/\beta}(t)}.$$

Using this in the definition of z, we get

$$y^\beta(t) \geq \frac{1}{p(\tau^{-1}(t))}\left[z(\tau^{-1}(t)) - \frac{\frac{1}{\beta}z(\tau^{-1}(\tau^{-1}(t))) + \left(1-\frac{1}{\beta}\right)}{p^{1/\beta}(\tau^{-1}(\tau^{-1}(t)))}\right]. \tag{2.17}$$

Since $z \in (\mathcal{N}_2^*)$, we have from Lemma 2.6 (iv) and (2.10) that

$$z(\tau^{-1}(\tau^{-1}(t))) \leq \frac{\eta(\tau^{-1}(\tau^{-1}(t)),t_2)}{\eta(\tau^{-1}(t),t_2)}z(\tau^{-1}(t)) \tag{2.18}$$

for $t \geq t_3$ for some $t_3 \geq t_2 \geq t_1$. Also from (2.3), we see that $z(t) \to \infty$ as $t \to \infty$, and therefore there exists $t_2 \geq t_1$ such that $z(\tau^{-1}(\tau^{-1}(t)) > 1$ for $t \geq t_2$. Using this in (2.17), we get

$$\begin{aligned} y^\beta(t) &\geq \frac{1}{p(\tau^{-1}(t))}\left[z(\tau^{-1}(t)) - \frac{\frac{1}{\beta}z(\tau^{-1}(\tau^{-1}(t))) + \left(1-\frac{1}{\beta}\right)z(\tau^{-1}(\tau^{-1}(t)))}{p^{1/\beta}(\tau^{-1}(\tau^{-1}(t)))}\right] \\ &= \frac{1}{p(\tau^{-1}(t))}\left[z(\tau^{-1}(t)) - \frac{z(\tau^{-1}(\tau^{-1}(t)))}{p^{1/\beta}(\tau^{-1}(\tau^{-1}(t)))}\right]. \end{aligned} \tag{2.19}$$

From (2.18) and (2.19), we observe that

$$y^\beta(t) \geq G_2(t)z(\tau^{-1}(t)) \quad \text{for } t \geq t_3. \tag{2.20}$$

Since $\lim_{t\to\infty}\sigma(t) = \infty$, we can choose $t_4 \geq t_3$ such that $\sigma(t) \geq t_3$ for all $t \geq t_4$. Thus, from (2.20) we have

$$y^\beta(\sigma(t)) \geq G_2(\sigma(t))z(\delta(t))) \quad \text{for } t \geq t_4. \tag{2.21}$$

Substituting (2.21) in (2.4), we get

$$(d(t)(c(t)(z'(t))^\alpha)')' + F(t)G_2^{\lambda/\beta}(\sigma(t))z^{\lambda/\beta-\alpha}(\delta(t))z^\alpha(\delta(t)) \tag{2.22}$$

for $t \geq t_4$. Since $z(t)$ is increasing and $\frac{z(t)}{\eta(t,t_2)}$ is decreasing, there exist constants $M_1 > 0$ and $M_2 > 0$ such that

$$z(t) \geq M_1 \quad \text{and} \quad z(t) \leq M_2 \eta(t,t_2) \tag{2.23}$$

for $t \geq t_3$. Combining (2.22) and (2.23), there exist two constants $d_1 > 0$ and $d_2 > 0$ such that for $t \geq t_4$, inequality (2.22) can be written as

$$(d(t)(c(t)(z'(t))^\alpha)')' + F(t)G_2^{\lambda/\beta}(\sigma(t))Q(\delta(t))z^\alpha(\delta(t)) \leq 0, \tag{2.24}$$

i.e., inequality (2.16) holds. This completes the proof of the lemma. □

Theorem 2.8. *Let* (2.3) *holds and σ be nondecreasing, and assume that either* (2.5) *or* (2.6) *and* (2.7) *hold for all constants $m > 0$. If there exists a nondecreasing function $g \in C^1([t_0,\infty),(0,\infty))$ such that for all constants $d_1 > 0$, $d_2 > 0$, for all sufficiently large $t_1 \geq t_0$, for some $t_2 \in [t_1,\infty)$, and $T \in [t_2,\infty)$,*

$$\limsup_{t\to\infty} \int_T^t \left[\Omega(s)\frac{\eta^\alpha(\delta(s),t_2)}{\mu(s,t_1)} - \frac{g'(s)}{\mu(s,t_1)}\right] ds = \infty, \tag{2.25}$$

then every solution y of (1.1) *either oscillates or satisfies* $\lim_{t\to\infty} y(t) = 0$.

Proof. Assume that y is an eventually positive of (1.1), say $y(t) > 0$, $y(\tau(t)) > 0$ and $y(\sigma(t)) > 0$ for $t \geq t_1$ for some $t_1 \geq t_0$. Then, by Corollary 2.4, y is also an eventually positive solution of (2.4) and the corresponding function $z \in (\mathcal{N}_0^*)$ or $z \in (\mathcal{N}_2^*)$ for all $t \geq t_1$

If $z \in (\mathcal{N}_0^*)$, by Lemma 2.5, we see that $y(t) \to 0$ as $t \to \infty$. Next, we consider the case $z \in (\mathcal{N}_2^*)$. Then Lemma 2.6 (i), (iii) and (iv), and (2.24) hold for all $t \geq t_4$. For $t \geq t_4$, let us define

$$w(t) = g(t)\frac{d(t)(c(t)(z'(t))^\alpha)'}{c(t)(z'(t))^\alpha}. \tag{2.26}$$

Clearly, $w(t) > 0$, and by (2.24) and (2.26), we get

$$w'(t) \leq g'(t)\frac{d(t)(c(t)(z'(t))^\alpha)'}{c(t)(z'(t))^\alpha} - \Omega(t)\frac{z^\alpha(\delta(t))}{c(t)(z'(t)^\alpha} - \frac{g(t)d(t)((c(t)(z'(t))^\alpha)')^2}{(c(t)(z'(t))^\alpha)^2}. \tag{2.27}$$

It follows from Lemma 2.6 (i) and (2.27) that

$$w'(t) \leq \frac{g'(t)}{\mu(t,t_1)} - \Omega(t)\frac{z^\alpha(\delta(t))}{z^\alpha(t)}\frac{z^\alpha(t)}{c(t)(z'(t))^\alpha}. \tag{2.28}$$

Since $\delta(t) \leq t$, it follows from Lemma 2.6 (iv) that

$$\frac{z(\delta(t))}{z(t)} \geq \frac{\eta(\delta(t),t_2)}{\eta(t,t_2)}. \tag{2.29}$$

Combining the inequalities Lemma 2.6 (iii), (2.28) and (2.29), we get

$$w'(t) \leq \frac{g'(t)}{\mu(t,t_1)} - \Omega(t)\frac{\eta^\alpha(\delta(t),t_2)}{\mu(t,t_1)}.$$

Integrating the latter inequality from $t_4 \geq t_3$ to t, we arrive at

$$\int_{t_4}^t \left[\Omega(s)\frac{\eta^\alpha(\delta(s),t_2)}{\mu(s,t_1)} - \frac{g'(s)}{\mu(s,t_1)}\right] ds \leq w(t_4),$$

which contradicts (2.25). This completes the proof of the theorem. □

Theorem 2.9. *Let* (2.3) *holds and σ be nondecreasing, and assume that either* (2.5) *or* (2.6) *and* (2.7) *hold for all constants $m > 0$. If there exists a nondecreasing function $g \in C^1([t_0,\infty),(0,\infty))$ such that for all constants $d_1 > 0$, $d_2 > 0$, for all sufficiently large $t_1 \geq t_0$, for some $t_2 \in [t_1,\infty)$, and $T \in [t_2,\infty)$,*

$$\limsup_{t\to\infty} \int_T^t \left[\Omega(s)\frac{\eta^\alpha(\delta(s),t_2)}{\mu(s,t_1)} - \frac{(g'(s))^2 d(s)}{4g(s)}\right] ds = \infty, \tag{2.30}$$

then every solution y of (1.1) *either oscillates or satisfies* $\lim_{t\to\infty} y(t) = 0$.

Proof. Assume that y is an eventually positive of (1.1), say $y(t) > 0$, $y(\tau(t)) > 0$ and $y(\sigma(t)) > 0$ for $t \geq t_1$ for some $t_1 \geq t_0$. Then, by Corollary 2.4, y is also an eventually positive solution of (2.4) and the corresponding function $z \in (\mathcal{N}_0^*)$ or $z \in (\mathcal{N}_2^*)$ for all $t \geq t_1$. The proof of $z \in (\mathcal{N}_0^*)$ is the same as in Theorem 2.8. Next consider $z \in (\mathcal{N}_2^*)$. Defining again w by (2.26) and proceeding as in the proof of Theorem 2.8, we again arrive at (2.27). In view of (2.26), inequality (2.27) becomes

$$w'(t) \leq \frac{g'(t)}{g(t)} w(t) - \Omega(t) \frac{z^\alpha(\delta(t))}{c(t)(z'(t))^\alpha} - \frac{w^2(t)}{g(t)d(t)} \tag{2.31}$$

for $t \geq t_4$. Since $z \in (\mathcal{N}_2^*)$, Lemma 2.6 (iii) and (2.29) hold. Using these in (2.31), we have for $t \geq t_4$,

$$w'(t) \leq \frac{g'(t)}{g(t)} w(t) - \Omega(t) \frac{\eta^\alpha(\delta(t), t_2)}{\mu(t, t_1)} - \frac{w^2(t)}{g(t)d(t)}. \tag{2.32}$$

Applying the completing the square in (2.32), we get

$$w'(t) \leq -\Omega(t) \frac{\eta^\alpha(\delta(t), t_2)}{\mu(t, t_1)} + \frac{(g'(t))^2 d(t)}{4g(t)}.$$

Integrating the latter inequality from t_4 to t, yields

$$\int_{t_4}^{t} \left[\Omega(s) \frac{\eta^\alpha(\delta(s), t_2)}{\mu(s, t_1)} - \frac{(g'(s))^2 d(s)}{4g(s)} \right] ds \leq w(t_4),$$

which contradicts (2.30). The proof is complete. □

Remark 2.10. We would like to note that the results of this paper can be applied to the case where $p(t) \to \infty$ as $t \to \infty$ for $\beta > 1$, and to the cases where $p(t)$ is a bounded function and/or $p(t) \to \infty$ as $t \to \infty$ for $\beta = 1$.

3 Examples

In this section, we provide two examples to illustrate the importance of our results.

Example 3.1. Consider the third-order semi-canonical differential equation with a superlinear neutral term

$$\left(t^2 \left(\frac{1}{t} \left[\left(y(t) + 2^{22} t y^3 \left(\frac{t}{2} \right) \right)' \right]^{1/3} \right)' \right)' + 8ty^3 \left(\frac{t}{4} \right) = 0, \quad t \geq 1. \tag{3.1}$$

Here $r_2(t) = t^2$, $r_1(t) = 1/t$, $p(t) = 2^{22}t$, $q(t) = 8t$, $\tau(t) = t/2$, $\sigma(t) = t/4$, $\alpha = 1/3$, $\beta = 3$ and $\lambda = 3$. A simple calculation shows that conditions $(C_1) - (C_3)$ hold, and

$$\Pi(t) = 1/t, \quad d(t) = c(t) = 1, \quad F(t) = 8, \quad \mu(t, 1) = t - 1, \quad \delta(t) = t/2,$$

$$\eta(t, 1) = (t-1)^4/4, \quad Q(t) = d_1 > 0, \quad \tau^{-1}(t) = 2t, \quad \tau^{-1}(\tau^{-1}(t)) = 4t.$$

The transformed equation is

$$\left(\left[\left(y(t) + 2^{22} t y^3 \left(\frac{t}{2} \right) \right)' \right]^{1/3} \right)'' + 8y^3 \left(\frac{t}{4} \right) = 0, \quad t \geq 1,$$

which is clearly canonical. Now,

$$G_1(t) = \frac{1}{2^{23} t} \left(1 - \frac{1}{m^{2/3} 2^8 t^{1/3}} \right), \quad G_2(t) = \frac{1}{2^{23} t} \left(1 - \left(\frac{4t - 1}{2t - 1} \right)^4 \frac{1}{2^8 t^{1/3}} \right).$$

Since $(4t-1)^4/(2t-1)^4$ is decreasing, $G_2(t)$ can be written as

$$G_2(t) \geq \frac{1}{2^{23}t}\left(1-\frac{3^4}{2^8 t^{1/3}}\right) \quad \text{for large } t.$$

Condition (2.3) is clearly satisfied. Condition (2.5) becomes

$$\int_1^\infty \frac{1}{2^{18}t}\left(1-\frac{1}{m^{2/3}2^{22/3}t^{1/3}}\right)dt = \infty.$$

Letting $g(t)=1$, condition (2.30) becomes

$$\limsup_{t\to\infty}\int_T^t \left[\Omega(s)\frac{\eta^\alpha(\delta(s),t_2)}{\mu(s,t_1)} - \frac{(g'(s))^2 d(s)}{4g(s)}\right]ds$$
$$\geq \limsup_{t\to\infty}\int_T^t \frac{d_1}{2^{20}s}\left(1-\frac{3^4}{2^{22/3}s^{1/3}}\right)\frac{(s-2)^{4/3}}{(s-1)}ds = \infty,$$

that is, condition (2.30) holds. Now, all assumptions of Theorems 2.9 are fulfilled. Therefore, by Theorem 2.9, every solution y of (3.1) is either oscillatory or satisfies $\lim_{t\to\infty} y(t)=0$.

Example 3.2. Consider the third-order semi-canonical differential equation with a superlinear neutral term

$$\left(t^2\left(\frac{1}{t}\left[\left(y(t)+32y\left(\frac{t}{3}\right)\right)'\right]^5\right)'\right)' + 16ty^5\left(\frac{t}{6}\right) = 0, \quad t \geq 1. \tag{3.2}$$

Here $r_2(t)=t^2$, $r_1(t)=1/t$, $p(t)=32$, $q(t)=16t$, $\tau(t)=t/3$, $\sigma(t)=t/6$, $\alpha=5$, $\beta=1$ and $\lambda=5$. A simple calculation shows that conditions $(C_1)-(C_3)$ hold,

$$\tau^{-1}(t)=3t, \quad \tau^{-1}(\tau^{-1}(t))=9t, \quad \delta(t)=t/2, \quad \Pi(t)=1/t, \quad d(t)=c(t)=1,$$

$$F(t)=16, \quad \mu(t,1)=t-1, \quad \eta(t,1)=\frac{5}{6}(t-1)^{6/5}, \quad Q(t)=1.$$

The transformed equation is

$$\left(\left[\left(y(t)+32y\left(\frac{t}{3}\right)\right)'\right]^5\right)'' + 16y^5\left(\frac{t}{6}\right) = 0, \quad t \geq 1,$$

which is clearly canonical. Now

$$G_1(t)=\frac{1}{32}\left(1-\frac{1}{32}\right), \quad G_2(t)=\frac{1}{32}\left(1-\left(\frac{9t-1}{3t-1}\right)^{6/5}\frac{1}{32}\right).$$

Since $(9t-1)^{6/5}/(3t-1)^{6/5}$ is decreasing, $G_2(t)$ can be written as

$$G_2(t) \geq \frac{1}{32}\left(1-\frac{2^{12/5}}{32}\right).$$

As in the Example 3.1, it is easy to see that all conditions of Theorem 2.8 are satisfied. Therefore, by Theorem 2.8, every solution y of (3.2) is either oscillatory or satisfies $\lim_{t\to\infty} y(t)=0$.

4 Conclusion

In this paper, we have established new type of oscillation criteria for third-order semi-canonical differential equations with a superlinear neutral term by transforming it to canonical type equations. Our results simplify the examination of semi-canonical type third-order differential equations. Further our results are new from that of in [8, 16, 21] since our equation is different. We provide two examples to illustrate the main results. It would be interest to investigate (1.1) with different assumptions on the neutral coefficient. Further our results ensure only that every solution of (1.1) is either oscillatory or converges to zero as $t \to \infty$. Therefore, it is interesting to obtain criteria which guarantee that every solution of (1.1) is just oscillatory. These interesting problems open at the moment.

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