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Spectral properties for classes of operators related to Perinormal operators

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Abstract In this paper, we give several examples of n-perinormal operators for each $n \geq 3$ such as (1) n-perinormal whose restriction to its invariant subspace is not n-perinormal, (2) n-perinormal which is not $(n-1)$-perinormal and (3) an invertible n-perinormal operator whose inverse is not n-perinormal. There are several papers studying n-perinormal operators which are using the assertions that a restriction of n-perinormal operator to its invariant subspace, the inverse of n-perinormal operator is also n-perinormal even if $n \geq 3$. We remark that if $n = 2$ then 2-perinormal is equal to quasihyponormal, and since a restriction of quasihyponormal to any invariant subspace is always quasihyponormal, so it is 2-perinormal. And every invertible 2-perinormal is invertible hyponormal, so the inverse of it is also hyponormal and 2-perinormal. We also show that Weyl's theorem holds for every n-perinormal and some results related to the Riesz idempotent of n-perinormal. Moreover, We study fundamental structural characteristics of class of (n,k)-quasiperinormal operators. Also, we show that, if T is (n,k)-quasiperinormal, then $T-\lambda$ has finite ascent for all $\lambda \in \mathbb{C}$. Further, we give a necessary and sufficient condition for $T \otimes S$ to be in a class of (n,k)-quasiperinormal .

1 Introduction

Let $\mathcal{H}$ be a complex (separable) infinite dimensional Hilbert space and $\mathcal{B}(\mathcal{H})$ be the set of all bounded linear operators on $\mathcal{H}$. An operator $T \in \mathcal{B}(\mathcal{H})$ is called to be hyponormal iff $T^*T \geq TT^*$, p-hyponormal for a $p > 0$ iff $(T^*T)^p \geq (TT^*)^p$. An operator T is called to be n-perinormal for an $n \geq 2$ iff $T^{*n}T^n \geq (T^*T)^n$. This class was introduced by Fujii, Izumino and Nakamoto [13].

Definition 1.1. An operator $T \in \mathcal{B}(\mathcal{H})$ is said to be n-perihyponormal if

$$T^{*n}T^n \geq (TT^*)^n$$

for $n \geq 1$.

Observe that 1-perihyponormal is equal to hyponormal. It is easy to see that $(n-1)$-perihyponormal is always n-perinormal. In general, the converse is not true, however, if an n-perinormal operator has dense range then it is $(n-1)$-perihyponormal.

An operator T is said to be $*$-paranormal if

$$\|T^*x\|^2 \leq \|T^2x\|\,\|x\|$$

for all $x \in \mathcal{H}$. This class of operators was introduced by S. M. Patel [32]. S. C. Arora and J. K. Thukral [2] proved that $*$-paranormal operators are normaloid, i.e., the operator norm $\|T\|$ of T equals to the spectral radius $r(T) = \sup\{|z| : z \in \sigma(T)\}$ of T where $\sigma(T)$ denotes the spectrum of T. Also we say that T belongs to the class $\mathfrak{P}(n)$ for an integer $n \geq 2$ if

$$\|Tx\|^n \leq \|T^nx\|\|x\|^{n-1}$$

for all $x \in \mathcal{H}$. We remark that an operator in $\mathfrak{P}(2)$ is called class (N) by V. Istracescu, T. Saito and T. Yoshino in [19] and paranormal by T. Furuta in [15], and an operator in $\mathfrak{P}(n)$ is called n-paranormal [3] and also called $(n-1)$-paranormal, e.g., [9], [26]. In order to avoid confusion we use notation $\mathfrak{P}(n)$. S. M. Patel [32] proved that $*$-paranormal operators belong to the class $\mathfrak{P}(3)$. Fujii, Izumino and Nakamoto proved that every n-perinormal operator belongs to the class $\mathfrak{P}(n)$. After, we shall show that every n-perihyponormal belongs to the class $\mathfrak{P}(n+1)$.

The Riesz idempotent E_λ of an operator T with respect to an isolated point λ of $\sigma(T)$ is defined as follows.

$$E_\lambda = \frac{1}{2\pi i}\int_{\partial D_\lambda} (z-T)^{-1}\,dz \tag{1.1}$$

It satisfies $\sigma(T|_{E_\lambda\mathcal{H}}) = \{\lambda\}$ and $\sigma(T|_{(1-E_\lambda)\mathcal{H}}) = \sigma(T) \setminus \{\lambda\}$, where the integral is taken by the positive direction and D_λ is a closed disk with center λ and small enough radius r such as $D_\lambda \cap \sigma(T) = \{\lambda\}$. In [40], Uchiyama proved that for every paranormal operator T and each isolated point λ of $\sigma(T)$ the Riesz idempotent E_λ satisfies that

$$E_0 = \ker T$$
$$E_\lambda = \ker(T-\lambda) = \ker(T-\lambda)^* \text{ and } E_\lambda \text{ is self-adjoint if } \lambda \neq 0.$$

We shall show that for every $*$-paranormal operator T and each isolated point $\lambda \in \sigma(T)$ the Riesz idempotent E_λ of T with respect to λ is self-adjoint with the property that $E_\lambda\mathcal{H} = \ker(T-\lambda) = \ker(T-\lambda)^*$.

If $T \in \mathcal{B}(\mathcal{H})$, we denote $\operatorname{ker} T$ and $\operatorname{ran} T$ for the kernel of T and the range of T respectively. We also denote the spectrum of T, the point spectrum of T, the Weyl spectrum of T and the set of all eigenvalues of T with finite multiplicity which are isolated in the spectrum by $\sigma(T)$, $\sigma_p(T)$, $w(T)$ and $\pi_{00}(T)$ respectively. An operator $T \in \mathcal{B}(\mathcal{H})$ is called to be Fredholm if $\operatorname{ran} T$ is closed and both of $\ker T$ and $\ker T^*$ are finite dimensional subspaces. For arbitrary Fredholm operator T, the index of T is definded by

$$\operatorname{ind}(T) := \dim\ker T - \dim\ker T^*.$$

An operator $T \in \mathcal{B}(\mathcal{H})$ is called to be Weyl iff T is a Fredholm operator with $\operatorname{ind}(T) = 0$. And the Weyl spectrum of T is defined by

$$w(T) := \{\lambda \in \mathbb{C} | T-\lambda \text{ is not Weyl}\}.$$

We say that the Weyl's theorem holds for an operator $T \in \mathcal{B}(\mathcal{H})$ if

$$\sigma(T) \setminus w(T) = \pi_{00}(T).$$

In this paper, we show that the Weyl's theorem holds for n-perinormal operators.

2 Preliminaries and Definitions

We will introduce basic concepts and notations in this section that will serve as the foundation for the research.

An operator $T \in \mathcal{B}(\mathcal{H})$ is called $*$-paranormal iff

$$\|T^*x\|^2 \leq \|T^2x\|\|x\| \quad (\forall x \in \mathcal{H}),$$

and T is called n-paranormal iff

$$\|Tx\|^n \leq \|T^nx\|\|x\|^{n-1} \quad (\forall x \in \mathcal{H}),$$

for each $n \geq 2$. We denote the set of all n-paranormal operators on $\mathcal{H}$ by $\mathfrak{P}(n)$.

Theorem 2.1. *[39] If T is $*$-paranormal then the following assertions hold.*

(i) $T \in \mathfrak{P}(3)$.

(ii) T is isoloid, i.e., every isolated point of $\sigma(T)$ is an eigen value of T.

(iii) Weyl's theorem holds for T, i.e., $\sigma(T) \setminus w(T) = \pi_{00}(T)$,

(iv) If λ is isolated point of $\sigma(T)$ then the Riesz idempotent

$E_\lambda = \frac{1}{2\pi i}\int_{|z-\lambda|=r} (z-T)^{-1}\, dz$ *with respect to λ is self-adjoint which satisfies*

$$E_\lambda\mathcal{H} = \ker(T-\lambda) = \ker(T-\lambda)^*,$$

where $r > 0$ is small enough such as $\{z : |z-\lambda| \leq r\} \cap \sigma(T) = \{\lambda\}$ and the integral is taken by positive direction.

(v) T is normaloid, i.e., $\|T\| = r(T)$.

(vi) If T is invertible then

$$\|T^{-1}\| \leq r(T^{-1})^3 r(T)^2.$$

Theorem 2.2. *[39] If $T \in \mathfrak{P}(n)$ for an $n \geq 2$ then the following assertions hold.*

(a) T is isoloid, i.e., every isolated point of $\sigma(T)$ is an eigen value of T.

(b) Weyl's theorem holds for T.

(c) If λ is isolated point of $\sigma(T)$ then the Riesz idempotent

$E_\lambda = \int_{|z-\lambda|=r} (z-T)^{-1}\, dz$ *with respect to λ satisfies*

$$E_\lambda\mathcal{H} = \ker(T-\lambda),$$

where $r > 0$ is small enough such as $\{z : |z-\lambda| \leq r\} \cap \sigma(T) = \{\lambda\}$ and the integral is taken by positive direction.

(d) Any restriction $T|_{\mathcal{M}}$ of T to an arbitrary T-invariant subspace $\mathcal{M}$ also belongs to $\mathfrak{P}(n)$.

(e) T *is normaloid, i.e.,* $\|T\| = r(T)$.

(f) *If* T *is invertible then*

$$\|T^{-1}\| \le r(T^{-1})^{\frac{n(n-1)}{2}} r(T)^{\frac{(n+1)(n-2)}{2}}.$$

Definition 2.3. [44] An operator $T \in \mathcal{B}(\mathcal{H})$ is said to be (n,k)-quasiparanormal if

$$\left\|T(T^k x)\right\| \le \left\|T^{n+k+1}x\right\|^{\frac{1}{n+1}} \left\|T^k x\right\|^{\frac{n}{n+1}} \text{ for } x \in \mathcal{H}.$$

Remark 2.4. It follows from Definition 2.3 that T is n-paranormal should be $(n,0)$-quasiparanormal if n-paranormal is defined by

$$\|Tx\| \le \left\|T^{n+1}x\right\|^{\frac{1}{n+1}} \|x\|^{\frac{n}{n+1}} \text{ for } x \in \mathcal{H}.$$

However, [6] defined n-paranormal as

$$\|Tx\| \le \|T^n x\|^{\frac{1}{n}} \|x\|^{\frac{n-1}{n}} \text{ for } x \in \mathcal{H},$$

this means $(n-1)$-paranormal in Yuan's definition.

Definition 2.5. An operator $T \in \mathcal{B}(\mathcal{H})$ is said to be

(i) a class (M,n) or n-perinormal if $T^{*n}T^n \ge (T^*T)^n$ for positive integer n such that $n \ge 2$ [13].

(ii) a n-$*$-perinormal (briefly, $T \in (M^*,n)$) if $|T^n|^2 \ge |T^*|^{2n}$ for $n \ge 2$ [6].

Definition 2.6. Let $T \in \mathcal{B}(\mathcal{H})$. We say that an operator T is a (n,k)-quasiperinormal (briefly, $T \in \mathbb{QP}(n,k)$) if

$$T^{*k}(|T^n|^2 - |T|^{2n})T^k \ge 0$$

for positive integer $n \ge 2$ and integer $k \ge 0$. And we say that T is a (n,k)-$*$-quasiperinormal (briefly, $T \in \mathbb{QP}^*(n,k)$) if

$$T^{*k}(|T^n|^2 - |T^*|^{2n})T^k \ge 0$$

for positive integer $n \ge 2$ and integer $k \ge 0$.

Definition 2.7. Let $T \in \mathcal{B}(\mathcal{H})$. We say that an operator T belongs to class (U,n) if

$$(T^{*n}T^n)^{\frac{2}{n}} \ge (T^*T)^2$$

for positive integer $n \ge 2$.

3 class $T \in \mathbb{QP}(n,k)$ and class $T \in \mathbb{QP}^*(n,k)$ operators

The following lemma is very important in the sequel

Lemma 3.1. (Hölder-McCarthy Inequality) Let $T \ge 0$. Then the following assertions hold.

(i) $\langle T^r x, x\rangle \geq \langle Tx, x\rangle^r \|x\|^{2(1-r)}$for $r > 1$ and $x \in \mathcal{H}$.

(ii) $\langle T^r x, x\rangle \leq \langle Tx, x\rangle^r \|x\|^{2(1-r)}$for $r \in [0, 1]$ and $x \in \mathcal{H}$.

Proposition 3.2. *Let $T \in \mathcal{B}(\mathcal{H})$. If T is an n-perinormal operator with $n \geq 2$, then we have*

$$\|T^n x\| \, \|x\|^{n-1} \geq \|Tx\|^n$$

for all $x \in \mathcal{H}$, and hence T is n-paranormal operator.

Proof. Assume that T is a n-perinormal operator. Then $T^{n*}T^n \geq (T^*T)^n$ and so for all $x \in \mathcal{H}$, we have

$$\begin{aligned} \|T^n x\|^2 &= \langle T^{n*}T^n x, x\rangle \geq \left\|(T^*T)^{n/2}x\right\|^2 = \langle (T^*T)^n x, x\rangle \\ &\iff \|T^n x\|^2 \geq \langle T^*Tx, x\rangle^n \|x\|^{2(1-n)} \quad \text{(by Hölder Mc-Carthy inequality)} \\ &\iff \|T^n x\| \, \|x\|^{n-1} \geq \|Tx\|^n . \end{aligned}$$

■

Proposition 3.3. *Let $T \in \mathcal{B}(\mathcal{H})$. Then $T \in \mathbb{QP}(n, k)$ with $n \geq 2$ and $k \geq 0$ if and only if $\left\|T^{n+k}x\right\| \geq \left\|(T^*T)^{n/2}T^k x\right\|$ holds for every $x \in \mathcal{H}$.*

Proof. We have

$$\begin{aligned} T \in \mathbb{QP}(n, k) &\iff T^{*k}(|T^n|^2 - |T|^{2n})T^k \geq 0 \\ &\iff \left\langle (T^{*k}(|T^n|^2 - |T|^{2n})T^k)x, x\right\rangle \geq 0, \text{ for all } x \in \mathcal{H} \end{aligned}$$

$$\begin{aligned} &\iff \left\langle T^{n+k}x, T^{n+k}x\right\rangle - \left\langle (T^*T)^{n/2}T^k x, (T^*T)^{n/2}T^k x\right\rangle \geq 0, \text{ for all } x \in \mathcal{H} \\ &\iff \left\|T^{n+k}x\right\|^2 \geq \left\|(T^*T)^{n/2}T^k x\right\|^2, \text{ for all } x \in \mathcal{H}. \end{aligned}$$

■

Remark 3.4. It follows from Proposition 3.3 that

(i) $T \in \mathbb{Q}P^*(1, k)$ is k-quasihyponormal.

(ii) T belongs to class (M, n) with $n \geq 2$ if and only if $\|T^n x\| \geq \left\|(T^*T)^{n/2}x\right\|$ holds for every $x \in \mathcal{H}$.

Proposition 3.5. *Let $T \in \mathcal{B}(\mathcal{H})$. Then $T \in \mathbb{QP}(n, k)$ if and only if*

$$|T^{n+k}|^2 + 2\lambda T^{*k}|T|^{2n}T^k + \lambda^2|T^{n+k}|^2 \geq 0$$

for all $\lambda \in \mathbb{R}$.

Proof. Let $n \in N$, $\lambda \in \mathbb{R}$, $x \in \mathcal{H}$ and $k \in \mathbb{Z}$ such that $k \geq 0$. Then we get

$$\begin{aligned} T \in \mathbb{QP}(n,k) &\iff \left\|T^{n+k}x\right\|^2 \geq \left\|(T^*T)^{n/2}T^kx\right\|^2 \\ &\iff 4\left\|(T^*T)^{n/2}T^kx\right\|^4 \leq 4\left\|T^{n+k}x\right\|^2\left\|T^{n+k}x\right\|^2 \\ &\iff \left\|T^{n+k}x\right\|^2 + 2\lambda\left\|(T^*T)^{n/2}T^kx\right\|^2 + \lambda^2\left\|T^{n+k}x\right\|^2 \geq 0 \\ &\iff \left\langle T^{n+k}x, T^{n+k}x\right\rangle + 2\lambda\left\langle (T^*T)^{n/2}T^kx, (T^*T)^{n/2}T^kx\right\rangle + \lambda^2\left\langle T^{n+k}x, T^{n+k}x\right\rangle \geq 0 \\ &\iff \left\langle \left(|T^{n+k}|^2 + 2\lambda T^{*k}|T|^{2n}T^k + \lambda^2|T^{n+k}|^2\right)x, x\right\rangle \geq 0 \end{aligned}$$

and so

$$|T^{n+k}|^2 + 2\lambda T^{*k}|T|^nT^k + \lambda^2|T^{n+k}|^2 \geq 0.$$

■

Proposition 3.6. *Let $T \in \mathcal{B}(\mathcal{H})$. Then $T \in \mathbb{QP}^*(n,k)$ with $n \geq 2$ and $k \geq 0$ if and only if $\left\|T^{n+k}x\right\| \geq \left\|(TT^*)^{n/2}T^kx\right\|$ holds for every $x \in \mathcal{H}$.*

Proof. We have

$$\begin{aligned} T \in \mathbb{QP}^*(n,k) &\iff T^{*k}(|T^n|^2 - |T^*|^{2n})T^k \geq 0 \\ &\iff \left\langle (T^{*k}(|T^n|^2 - |T^*|^{2n})T^k)x, x\right\rangle \geq 0, \text{ for all } x \in \mathcal{H} \\ &\iff \left\langle T^{n+k}x, T^{n+k}x\right\rangle - \left\langle (TT^*)^{n/2}T^kx, (TT^*)^{n/2}T^kx\right\rangle \geq 0, \text{ for all } x \in \mathcal{H} \\ &\iff \left\|T^{n+k}x\right\|^2 \geq \left\|(TT^*)^{n/2}T^kx\right\|^2, \text{ for all } x \in \mathcal{H}. \end{aligned}$$

■

An operator $T \in \mathcal{B}(\mathcal{H})$ is called k-quasihyponormal operator if $T^{*k}(|T|^2 - |T^*|^2)T^k \geq 0$ for $k \geq 0$.

From Proposition 3.6 it follows that:

Corollary 3.7. *Let $T \in \mathcal{B}(\mathcal{H})$ and $n = 1$, then it follows that T is a k-quasihyponormal operator.*

By the same arguments of the proof of Proposition 3.5, we can prove the following result.

Corollary 3.8. *Let $T \in \mathcal{B}(\mathcal{H})$. Then $T \in \mathbb{QP}^*(n,k)$ if and only if*

$$|T^{n+k}|^2 + 2\lambda T^{*k}|T^*|^{2n}T^k + \lambda^2|T^{n+k}|^2 \geq 0$$

for all $\lambda \in \mathbb{R}$.

Proposition 3.9. *Let $T \in \mathbb{QP}(2,k)$, then T is a k-quasiparanormal operator.*

Proof. Let $T \in \mathbb{QP}(2,k)$, then we get

$$T^{*k}|T^2|^2T^k \geq T^{*k}|T|^4T^k \iff \langle T^{*k}(T^{*2}T^2 - (T^*T)^2)T^k x, x\rangle \geq 0, \text{ for all } x \in \mathcal{H}$$
$$\iff \langle T^{k+2}x, T^{k+2}x\rangle - \langle T^*T^{k+1}x, T^*T^{k+1}x\rangle \geq 0, \text{ for all } x \in \mathcal{H}$$
$$\iff \left\|T^{k+2}x\right\|^2 \geq \left\|T^*T^{k+1}x\right\|^2, \text{ for all } x \in \mathcal{H}. \tag{3.1}$$

On the other hand,

$$\left\|T^{k+1}x\right\|^2 = |\langle T^{k+1}x, T^{k+1}x\rangle| = |\langle T^*TT^k x, T^k x\rangle| \leq \left\|T^*T^{k+1}x\right\| \left\|T^k x\right\|. \tag{3.2}$$

Now from relations (3.1) and (3.2) follows that

$$\left\|T^{k+1}x\right\|^2 \leq \left\|T^{k+2}x\right\| \left\|T^k x\right\|$$

for every $x \in \mathcal{H}$. That is, T is a k-quasiparanormal operator. ■

Remark 3.10. In [28], quasi-$A(n,k)$ class operators ($T \in \mathcal{B}(\mathcal{H})$: $T^{*k}(|T^n| - |T|^n)T^k \geq 0$ for integers $n \geq 2$ and $k \geq 0$) has been studied by Lee and Yun. It follows from the definition of class (M,n) and Löwner-Heinz inequality that if $T \in (M,n)$, then T is a quasi-$A(n,0)$ class operator.

Proposition 3.11. *Let $T \in \mathcal{B}(\mathcal{H})$ be a class (M,n) operator and T^n be a compact operator for some $n \in \mathbb{N}$. Then T is also compact and normal.*

Proof. Assume that T is a class (M,n) operator for $n \geq 2$. Hence

$$\left\|(T^*T)^{n/2}x\right\| \leq \|T^n x\| \quad \text{for every } x \in \mathcal{H}. \tag{3.3}$$

Let $\{x_m\} \in \mathcal{H}$ be weakly convergent sequence with limit 0 in $\mathcal{H}$. From the compactness of T^n and the relation (3.3) we get the following relation:

$$\left\|(T^*T)^{n/2}x_m\right\| \to 0, \ m \to \infty.$$

From the last relation it follows that T^*T is compact operator and so T is compact. Since T is compact $\sigma(T)$ is finite set or countable infinite with 0 as the unique limit point of it. Let $\sigma(T) \setminus \{0\} = \{\lambda_l\}$ with

$$|\lambda_1| \geq |\lambda_2| \geq \cdots \geq |\lambda_l| \geq |\lambda_{l+1}| \geq \cdots \geq 0, \text{ and } \lambda_l \to 0 \ (l \to \infty).$$

By the compactness of T or isoloidness of T, $\lambda_l \in \sigma_p(T)$ and $dim\, \ker(T - \lambda_l) < \infty$ for all l. Since $\ker(T - \lambda_l) \subset \ker(T - \lambda_l)^*$, $\mathcal{M} := \bigoplus_{l=1}^{\infty} \ker(T - \lambda_l)$ reduces T, and T is of the form

$$T = \left(\bigoplus_{l=1}^{\infty} \lambda_l\right) \oplus T' \text{ on } \mathcal{H} = \mathcal{M} \oplus \mathbb{P}^{\perp}.$$

By the construction, T' is n-perinormal and $\sigma(T') = \{0\}$ hence $T' = 0$. This shows that

$$T = \left(\bigoplus_{l=1}^{\infty} \lambda_l\right) \oplus 0$$

and it is normal. ■

Proposition 3.12. *Let $T \in \mathcal{B}(\mathcal{H})$. If $T \in \mathbb{QP}^*(n,k)$, then $T \in \mathbb{QP}(k+1,n)$.*

Proof. Let us suppose that $T \in \mathbb{QP}^*(n,k)$. Then for $n \geq 2$ and $k \geq 0$, it follows that

$$T^{*k}|T^n|^2T^k \geq T^{*k}|T^*|^{2n}T^k.$$

This is equivalent with:

$$\left\langle T^{*k}(|T^n|^2 - |T^*|^{2n})T^k x, x\right\rangle \geq 0,$$

for every $x \in \mathcal{H}$. Further:

$$\begin{aligned}\left\langle T^{*k}(T^{*(n+1)}T^{n+1} - (T^*T)^{n+1})T^k x, x\right\rangle &= \left\langle T^{*(k+1)}(T^{*n}T^n - (TT^*)^n)T^{k+1}x, x\right\rangle \\ &= \left\langle [T^{*k}(|T^n|^2 - |T^*|^{2n})T^k]Tx, Tx\right\rangle \geq 0.\end{aligned}$$

From this it follows that $T^{*(k+1)}(|T^n|^2 - |T|^{2n})T^{k+1} \geq 0$ and so $T \in \mathbb{QP}(k+1,n)$. ■

Proposition 3.13. *Let $\mathcal{H} = \bigoplus_{i\in\mathbb{N}} \mathcal{H}_i$, $\mathcal{H}_i \cong \mathcal{H}_j$ and $T = \oplus_{i\in\mathbb{N}}T_i$, where $\mathbb{QP}(n,k) \ni T_i : \mathcal{H}_i \to \mathcal{H}_i$, $T \in \mathcal{B}(\mathcal{H})$, then $T \in \mathbb{QP}(n,k)$.*

Proof. Assume that $T_i \in \mathbb{QP}(n,k)$ for each $i \in \mathbb{N}$. Then

$$T_i^{*k}(T_i^{*n}T_i^n)T_i^k \geq T_i^{*k}(T_i^*T_i)^nT_i^k,\ \ i \in \mathbb{N}.$$

Hence

$$\begin{aligned}T^{*k}(T^{*n}T^n)T^k &= (\oplus_{i\in\mathbb{N}}T_i)^{*k}\left((\oplus_{i\in\mathbb{N}}T_i)^{*n}(\oplus_{i\in\mathbb{N}}T_i)^n\right)(\oplus_{i\in\mathbb{N}}T_i)^k \\ &= (\oplus_{i\in\mathbb{N}}T_i^{*k})[(\oplus_{i\in\mathbb{N}}T_i^{*n})(\oplus_{i\in\mathbb{N}}T_i^n)](\oplus_{i\in\mathbb{N}}T_i^k) \\ &= \oplus_{i\in\mathbb{N}}T_i^{*k}(T_i^{*n}T_i^n)T_i^k \geq \oplus_{i\in\mathbb{N}}T_i^{*k}(T_i^*T_i)^nT_i^k \\ &= \oplus_{i\in\mathbb{N}}T_i^{*k} \oplus_{i\in\mathbb{N}} (T_i^*T_i)^n \oplus_{i\in\mathbb{N}} T_i^k \\ &= (\oplus_{i\in\mathbb{N}}T_i)^{*k}(\oplus_{i\in\mathbb{N}}T_i^*T_i)^n(\oplus_{i\in\mathbb{N}}T_i)^k \\ &= T^{*k}(T^*T)^nT^k\end{aligned}$$

and so $T \in \mathbb{QP}(n,k)$. ■

Theorem 3.14. *If T is (n,k)-quasiperinormal, then T is $(n-1,k)$-quasiparanormal.*

Proof. Since

$$\begin{aligned}\left\|T^{n+k}x\right\|^2 &= \left\langle T^{*k}T^{*n}T^nT^kx, x\right\rangle = \left\langle T^{*k}|T^n|^2T^kx, x\right\rangle \\ &\geq \left\langle T^{*k}|T|^{2n}T^kx, x\right\rangle \\ &= \left\langle |T|^{2n}T^kx, T^kx\right\rangle \\ &\geq \left\langle |T|^2T^kx, T^kx\right\rangle^n \left\|T^kx\right\|^{2(1-n)} = \left\|T^{k+1}x\right\|^{2n}\left\|T^kx\right\|^{2(1-n)},\end{aligned}$$

we have

$$\left\|T(T^kx)\right\| \leq \left\|T^{n+k}x\right\|^{\frac{1}{n}}\left\|T^kx\right\|^{\frac{n-1}{n}}.$$

■

4 Examples

If $T \in \mathcal{B}(\mathcal{H})$ is hyponormal (or p-hyponormal for $0 < p \leq 1$ or 2-perinormal or belongs to $\mathfrak{P}(n)$) then the restriction $T|_{\mathcal{M}}$ to any T-invariant subspace $\mathcal{M}$ is also hyponormal (p-hyponormal, 2-perinormal or belongs to $\mathfrak{P}(n)$ respectively). This result is important to prove the Weyl's theorem for these operators. However, the following example tells us n-perinormal does not have that property in general for $n \geq 3$.

Let A, B be 2×2 positive invertible matrices which satisfy $A \leq B$ and $A^n \not\leq B^n$ for all $n \geq 2$. Let $\mathcal{H} = \bigoplus_{k=-\infty}^{\infty} \mathbb{C}^2$. We definde an invertible operator T on $\mathcal{H}$ by

$$T(x_k) = (y_k), \quad y_k = \begin{cases} A^{1/2} x_{k-1} & (k \leq 0) \\ B^{1/2} x_{k-1} & (k \geq 1) \end{cases} \qquad (x_k) \in \mathcal{H}.$$

For examples, put $A = \begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix}$, $B = \begin{pmatrix} 3 & 1 \\ 1 & 1 \end{pmatrix}$. Then, A and B are invertible, and

$$B - A = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \geq 0. \quad \therefore\ B \geq A.$$

$$B^2 - A^2 = \begin{pmatrix} 5 & 1 \\ 1 & 0 \end{pmatrix} \not\geq 0. \quad \therefore\ B^2 \not\geq A^2.$$

$$B^n \not\geq A^n\ (n \geq 2) \quad \text{(by the Heinz inequality)}.$$

Proposition 4.1. *Let T be as above. Then the following assertions hold:*

(i) T is an invertible hyponormal operator, hence it is 2-perinormal.

(ii) T is not n-perinormal for all $n \geq 3$.

(iii) T satisfies $(T^{*m}T^m)^{\frac{1}{m}} \geq TT^*$ for all $m \geq 2$.

Proof. (i) Since $T^*T - TT^* = \left(\bigoplus_{k \leq -1} 0\right) \oplus \overset{(0)}{(B - A)} \oplus \left(\bigoplus_{k \geq 1} 0\right) \geq 0$. Hence, T is hyponormal.

(ii) We shall show that T is not m-perihyponormal for all $m \geq 2$, i.e., $T^{*m}T^m \not\geq (TT^*)^m$. Since

$$\begin{aligned} &T^{*m}T^m - (TT^*)^m \\ &= \left(\bigoplus_{k \leq -m} 0\right) \oplus \left(\bigoplus_{k=-m+1}^{-1} \left(A^{\frac{-k}{2}} B^{m+k} A^{\frac{-k}{2}} - A^m\right)\right) \oplus \overset{(0)}{(B^m - A^m)} \oplus \left(\bigoplus_{k \geq 1} 0\right) \not\geq 0, \end{aligned}$$

T is not m-perihyponormal for all $m \geq 2$. The invertibility of T implies that T is not n-perinormal for all $n \geq 3$.

(iii) Let $p = m + k,\ q = m,\ r = \frac{-k}{2}$ for an $m \geq 2$ and $-m + 1 \leq k \leq -1$. Then $(1 + 2r)q = (1 - k)m \geq p + 2r = m + k - k = m$. Since $A \leq B$, we have

$$\left(A^{\frac{-k}{2}} B^{m+k} A^{\frac{-k}{2}}\right)^{1/m} \geq \left(A^{\frac{-k}{2}} A^{m+k} A^{\frac{-k}{2}}\right)^{1/m} = A, \quad (-m + 1 \leq k \leq -1)$$

by Furuta inequality. Hence,

$$\begin{aligned}&(T^{*m}T^m)^{\frac{1}{m}} - TT^* \\ &= \left(\bigoplus_{k \leq -m} 0\right) \oplus \left(\bigoplus_{k=-m+1}^{-1} \left\{\left(A^{\frac{-k}{2}} B^{m+k} A^{\frac{-k}{2}}\right)^{1/m} - A\right\}\right) \oplus \overset{(0)}{(B - A)} \oplus \left(\bigoplus_{k \geq 1} 0\right) \geq 0.\end{aligned}$$

Therefore $(T^{*m}T^m)^{\frac{1}{m}} \geq TT^*$ for all $m \geq 2$. ■

Example 4.2. Let T be as above and define an operator S on $\mathcal{K} = \mathcal{H} \oplus \mathcal{H}$ by

$$S = \begin{pmatrix} T & X_m \\ 0 & 0 \end{pmatrix},$$

where $X_m = \left((T^{*m}T^m)^{\frac{1}{m}} - TT^*\right)^{1/2}$ for an $m \geq 2$. Then

$$\begin{aligned} S^{*m}S^m &= \begin{pmatrix} T^{*m} & 0 \\ X_m T^{*(m-1)} & 0 \end{pmatrix} \begin{pmatrix} T^m & T^{m-1}X_m \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} T^{*m}T^m & * \\ * & * \end{pmatrix}, \\ SS^* &= \begin{pmatrix} T & X_m \\ 0 & 0 \end{pmatrix} \begin{pmatrix} T^* & 0 \\ X_m & 0 \end{pmatrix} = \begin{pmatrix} TT^* + X_m^2 & 0 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} (T^{*m}T^m)^{1/m} & 0 \\ 0 & 0 \end{pmatrix}, \\ (SS^*)^m &= \begin{pmatrix} (T^{*m}T^m)^{1/m} & 0 \\ 0 & 0 \end{pmatrix}^m = \begin{pmatrix} T^{*m}T^m & 0 \\ 0 & 0 \end{pmatrix}. \end{aligned}$$

Thus,

$$\begin{aligned} S^{*(m+1)}S^{m+1} - (S^*S)^{m+1} &= S^* \left\{S^{*m}S^m - (SS^*)^m\right\} S \\ &= \begin{pmatrix} T^* & 0 \\ X_m & 0 \end{pmatrix} \begin{pmatrix} 0 & * \\ * & * \end{pmatrix} \begin{pmatrix} T & X_m \\ 0 & 0 \end{pmatrix} \\ &= \begin{pmatrix} T^* & 0 \\ X_m & 0 \end{pmatrix} \begin{pmatrix} 0 & 0 \\ * & * \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} = 0, \end{aligned}$$

This implies that S is $(m + 1)$-perinormal. Put $\mathcal{M} = \operatorname{ran} S$. Then $\mathcal{M} = \mathcal{H} \oplus \{0\}$ is closed and the restriction $S|_{\mathcal{M}}$ to its invariant subspace $\mathcal{M}$ is equal to T which is not n-perinormal for all $n \geq 3$ by Proposition 4.1.

Remark 4.3. (i)If $m = 2$ then the above S is 3-perinormal which is not 2-perinormal.
(ii) S is $(m + 1)$-perinormal which is not m-perinormal for $m \geq 3$.

Proof. (i) Suppose S is 2-perinormal, i.e., S satisfies $S^{*2}S^2 \geq (S^*S)^2$. Put $P = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$. Then

$$\begin{pmatrix} T^{*2}T^2 & 0 \\ 0 & 0 \end{pmatrix} = PS^{*2}S^2P \geq P\left(S^*S\right)^2 P = \begin{pmatrix} (T^*T)^2 + T^*X_2^2T & 0 \\ 0 & 0 \end{pmatrix}.$$

Hence $T^{*2}T^2 \geq (T^*T)^2 + T^*X_2^2T$. Since T is invertible,

$$\begin{aligned} T^*T &\geq TT^* + X_2^2 = TT^* + (T^{*2}T^2)^{1/2} - TT^* = (T^{*2}T^2)^{1/2} \\ &\geq \left\{(T^*T)^2\right\}^{1/2} \quad (\because T \text{ is hyponormal, so it is 2-perinormal}) \\ &= T^*T. \end{aligned}$$

Thus $T^*T = (T^{*2}T^2)^{1/2}$, $(T^*T)^2 = T^{*2}T^2$ and $T^*T = TT^*$. It follows that T is normal, however, T is not normal. Hence, S is not 2-perinormal.
(ii) Suppose S is m-perinormal, i.e., S satisfies $S^{*m}S^m \geq (S^*S)^m$. Then $0 \leq S^{*m}S^m - (S^*S)^m = S^*\left(S^{*m-1}S^{m-1} - (SS^*)^{m-1}\right)S$ and $0 \leq P\left(S^{*m-1}S^{m-1} - (SS^*)^{m-1}\right)P$. Hence

$$\begin{aligned} PS^{*m-1}S^{m-1}P &= \begin{pmatrix} T^{*m-1}T^{m-1} & 0 \\ 0 & 0 \end{pmatrix} \geq P(SS^*)^{m-1}P \\ &= \begin{pmatrix} TT^* + X_m^2 & 0 \\ 0 & 0 \end{pmatrix}^{m-1} = \begin{pmatrix} TT^* + (T^{*m}T^m)^{1/m} - TT^* & 0 \\ 0 & 0 \end{pmatrix}^{m-1} \\ &= \begin{pmatrix} (T^{*m}T^m)^{\frac{m-1}{m}} & 0 \\ 0 & 0 \end{pmatrix}. \end{aligned}$$

Hence $T^{*m-1}T^{m-1} \geq (T^{*m}T^m)^{\frac{m-1}{m}}$. It follows that

$$\begin{aligned} 0 \leq & T^{*m-1}T^{m-1} - (T^{*m}T^m)^{\frac{m-1}{m}} \\ = & \left(\bigoplus_{k\leq -m} 0\right) \oplus \left(A^{m-1} - \left\{A^{\frac{m-1}{2}}BA^{\frac{m-1}{2}}\right\}^{\frac{m-1}{m}}\right) \\ & \oplus \left(\bigoplus_{k=-m+2}^{-1} \left(A^{\frac{-k}{2}}B^{m-1+k}A^{\frac{-k}{2}} - \left\{A^{\frac{-k}{2}}B^{m+k}A^{\frac{-k}{2}}\right\}^{\frac{m-1}{m}}\right)\right) \oplus \left(\bigoplus_{k\geq 0} 0\right), \end{aligned}$$

and hence

$$A^{m-1} \geq \left\{A^{\frac{m-1}{2}}BA^{\frac{m-1}{2}}\right\}^{\frac{m-1}{m}} \geq \left\{A^{\frac{m-1}{2}}AA^{\frac{m-1}{2}}\right\}^{\frac{m-1}{m}} = A^{m-1}.$$

This implies that $A^{m-1} = \left\{A^{\frac{m-1}{2}}BA^{\frac{m-1}{2}}\right\}^{\frac{m-1}{m}}$ and $A = B$ which is a contradiction. Therefore, S is not m-perinormal. ■

If $T \in \mathcal{B}(\mathcal{H})$ is invertible hyponormal or p-hyponormal for $0 < p$ then the inverse T^{-1} of T is is also hyponormal or p-hyponormal respectively. However, in general, the inverse of invertible n-perinormal is not necessarily n-perinormal for $n \geq 3$. We give an example of invertible 3-perinormal operator whose inverse is not 3-perinormal.

Example 4.4. Let A, B be 2×2 positive invertible matrices which satisfy $A \leq B \leq 1$ and $A^n \not\leq B^n$ for all $n \geq 2$. Let $\mathcal{H} = \bigoplus_{k=-\infty}^{\infty} \mathbb{C}^2$. We define an invertible operator T on $\mathcal{H}$ by

$$T(x_k) = (y_k), \quad y_k = \begin{cases} A^{1/2}x_{k-1} & (k \leq 0) \\ B^{1/2}x_{k-1} & (k = 1) \\ x_{k-1} & (k \geq 2) \end{cases} \qquad (x_k) \in \mathcal{H}.$$

For examples, put $A = \frac{1}{4}\begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix}$, $B = \frac{1}{4}\begin{pmatrix} 3 & 1 \\ 1 & 1 \end{pmatrix}$. Then, A and B are invertible, and

$$B - A = \frac{1}{4}\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \geq 0. \quad \therefore\ A \leq B \leq 1.$$

$$B^2 - A^2 = \frac{1}{16}\begin{pmatrix} 5 & 1 \\ 1 & 0 \end{pmatrix} \not\geq 0. \quad \therefore\ A^2 \not\leq B^2.$$

$$A^n \not\leq B^n\ (n \geq 2) \quad \text{(by the Heinz inequality)}.$$

Proposition 4.5. *Let $T \in \mathcal{B}(\mathcal{H})$. Then the following assertions hold:*

(i) T is an invertible 2-perihyponormal operator, hence T is 3-perinormal.

(ii) T^{-1} is not 3-perinormal.

Proof. (i) We shall show that $T^{*2}T^2 \geq (TT^*)^2$. Since $0 < A \leq B \leq 1$, we have $A^2 \leq A \leq B$ and

$$\begin{aligned} & T^{*2}T^2 - (TT^*)^2 \\ = & \left(\bigoplus_{k \leq -2} 0\right) \oplus \left(A^{1/2}BA^{1/2} - A^2\right) \oplus (B \overset{(0)}{-} A^2) \oplus (1 - B^2) \oplus \left(\bigoplus_{k \geq 2} 0\right) \geq 0, \end{aligned}$$

because $A^{1/2}BA^{1/2} - A^2 = A^{1/2}(B - A)A^{1/2} \geq 0$, $B - A^2 \geq 0$ and $1 - B \geq 0$. The invertibility of T implies that T is 3-perinormal.
(ii) We shall show that T^{-1} is not 2-perihyponormal. Since

$$T^{-1}(x_k) = (y_k), \quad y_k = \begin{cases} A^{-1/2}x_{k+1} & (k \leq -1) \\ B^{-1/2}x_1 & (k = 0) \\ x_{k+1} & (k \geq 1) \end{cases} \qquad (x_k) \in \mathcal{H},$$

we obtain

$$T^{-2*}T^{-2}-(T^{-1}T^{-1*})^2$$
$$=\left(\bigoplus_{k\le -1}0\right)\oplus\left(\overset{(0)}{A^{-2}-B^{-2}}\right)\oplus\left(B^{-1/2}A^{-1}B^{-1/2}-1\right)\oplus\left(B^{-1}-1\right)\oplus\left(\bigoplus_{k\ge 3}0\right).$$

Hence, T^{-1} is 2-perihyponormal iff $A^{-2}\ge B^{-2}$ which is equivalent to $B^2\ge A^2$. However, the last inequality does not hold. Hence T^{-1} is not 2-perihyponormal and therefore T^{-1} is not 3-perinormal. ■

Remark 4.6. In Proposition 4.5, if we choose A,B such as $0<A\le B\le 1$, $A^{n-2}\le B^{n-2}$ and $A^{n-1}\not\le B^{n-1}$ then the operator T is n-perinormal but the inverse T^{-1} is not n-perinormal for each $n\ge 4$, because

$$T^{*n-1}T^{n-1}-(TT^*)^{n-1}$$
$$=\left(\bigoplus_{k\le -n+1}0\right)\oplus\left(\bigoplus_{k=-n+2}^{-1}\left(A^{\frac{-k}{2}}BA^{\frac{-k}{2}}-A^{n-1}\right)\right)\oplus\left(\overset{(0)}{B-A^{n-1}}\right)\oplus\left(1-B^{n-1}\right)\oplus\left(\bigoplus_{k\ge 2}0\right),$$

and

$$T^{-(n-1)*}T^{-(n-1)}-(T^{-1}T^{-1*})^{n-1}$$
$$=\left(\bigoplus_{k\le -1}0\right)\oplus\left(\overset{(0)}{A^{-(n-1)}-B^{-(n-1)}}\right)\oplus\left(\bigoplus_{k=1}^{n-1}\left(B^{\frac{-1}{2}}A^{-(n-k-1)}B^{\frac{-1}{2}}-1\right)\right)\oplus\left(\bigoplus_{k\ge n}0\right).$$

5 Complementary Results

The following lemma is very important in the sequel

Lemma 5.1. *[11, Hansen's Inequality] If $A,B\in\mathcal{B}(\mathcal{H})$ satisfying $A\ge 0$ and $\|B\|\le 1$, then*

$$(B^*AB)^\alpha\ge B^*A^\alpha B\qquad \textit{for all } \alpha\in(0,1].$$

Lemma 5.2. *(1) Every 2-perihyponormal operator is $*$-paranormal, hence it is 3-paranormal. (2) Every m-perihyponormal operator is $(m+1)$-paranormal for each $m\ge 3$.*

Proof. (1) By the assumption, for every $x\in\mathcal{H}$,

$$\|(TT^*)x\|^2\le\langle (TT^*)^2x,x\rangle\le\langle T^{*2}T^2x,x\rangle=\|T^2x\|,\quad \therefore \|TT^*x\|\le\|T^2x\|.$$

It follows that

$$\|T^*x\|^2=\langle TT^*x,x\rangle\le\|TT^*x\|\|x\|\le\|T^2x\|\|x\|$$

for every $x\in\mathcal{H}$.
(2) Let $x\in\mathcal{H}$ be arbitrary.

$$\begin{aligned}\|T^mx\|^2&=\langle T^{*m}T^mx,x\rangle\ge\langle (TT^*)^mx,x\rangle\\&\ge\langle TT^*x,x\rangle^m\|x\|^{2(1-m)}\quad\text{(by (1.1))}\\&=\|T^*x\|^{2m}\|x\|^{2(1-m)}.\end{aligned}$$

Hence, $\|T^*x\|^m \le \|T^m x\|\|x\|^{m-1}$ and

$$\|Tx\|^2 = \langle T^*Tx, x\rangle \le \|T^*Tx\|\|x\| \le \sqrt[m]{\|T^{m+1}x\|\|Tx\|^{m-1}}\|x\|.$$

It follows that

$$\begin{aligned}\|Tx\|^{2m} &\le \|T^{m+1}x\|\|Tx\|^{m-1}\|x\|^m,\\ \|Tx\|^{m+1} &\le \|T^{m+1}x\|\|x\|^m.\end{aligned}$$

This implies that $T \in \mathfrak{P}(m+1)$. ∎

Lemma 5.3. *[13] Every n-perinormal is n-paranormal.*

Lemma 5.4. *[31, 45] If T is n-perinormal, $\lambda \in \sigma_p(T)\setminus\{0\}$ and $x \in \ker(T-\lambda)$, then*

$$(T-\lambda)^*x = 0.$$

As we see in the previous section, the restricsion of n-perinormal to its invariant subspace is not necessarily n-perinormal for $n \ge 3$. However, we have a weak result as follows.

Lemma 5.5. *If T is n-perinormal for $n \ge 2$ and $\mathcal{M}$ is a T-invariant closed subspace. Then the restriction $T_1 := T|_{\mathcal{M}}$ of T to $\mathcal{M}$ belongs to class (U,n).*

Proof. Let P be the orthogonal projection onto $\mathcal{M}$. Then T and P are of the forms

$$T = \begin{pmatrix} T_1 & A \\ 0 & B\end{pmatrix},\ P = \begin{pmatrix} 1 & 0 \\ 0 & 0\end{pmatrix} \text{ on } P\mathcal{H}\oplus(1-P)\mathcal{H}.$$

Since $TP = PTP$ and $T^{*n}T^n \ge (T^*T)^n$ it follows that

$$P(T^{*n}T^n)P = (TP)^{*n}(TP)^n \ge P(T^*T)^nP$$

$$\begin{aligned}\begin{pmatrix} (T_1^{*n}T_1^n)^{\frac{2}{n}} & 0 \\ 0 & 0\end{pmatrix} &= \begin{pmatrix} T_1^{*n}T_1^n & 0 \\ 0 & 0\end{pmatrix}^{\frac{2}{n}} = \{P(T^{*n}T^n)P\}^{\frac{2}{n}} = \{(TP)^{*n}(TP)^n\}^{\frac{2}{n}}\\ &\ge \{P(T^*T)^nP\}^{\frac{2}{n}}\\ &\ge P(T^*T)^2P \quad \text{(by Hansen's inequality)}\\ &= (TP)^*(TT^*)(TP) \ge (TP)^*(TPT^*)(TP)\\ &= \{(TP)^*(TP)\}^2 = \begin{pmatrix} (T_1^*T_1)^2 & 0 \\ 0 & 0\end{pmatrix}.\end{aligned}$$

This completes the proof. ∎

The following lemma is an extension of Lemma 5.3.

Lemma 5.6. *If T belongs to class (U,n) and $\lambda \in \sigma_p(T)\setminus\{0\}$. Then* $\ker(T-\lambda) \subset \ker(T-\lambda)^*$.

Proof. Let P be the orthogonal projection onto $\ker(T-\lambda)$. Then $TP=\lambda P$, $T^nP=\lambda^nP$, $PT^*=\overline{\lambda}P$, $PT^{*n}=\overline{\lambda}^nP$. And T and P are of the forms $T=\begin{pmatrix}\lambda & A\\ 0 & B\end{pmatrix}$, $P=\begin{pmatrix}1 & 0\\ 0 & 0\end{pmatrix}$ on $P\mathcal{H}\oplus(1-P)\mathcal{H}$.

$$\begin{aligned}\begin{pmatrix}|\lambda|^4 & 0\\ 0 & 0\end{pmatrix}=\begin{pmatrix}|\lambda|^{2n} & 0\\ 0 & 0\end{pmatrix}^{\frac{2}{n}} &= \{P(T^{*n}T^n)P\}^{\frac{2}{n}}\\ &\geq P\left(T^{*n}T^n\right)^{\frac{2}{n}}P \quad \text{(by Hansen's inequality)}\\ &\geq P(T^*T)^2P=|\lambda|^2P(TT^*)P=\begin{pmatrix}|\lambda|^2\left(|\lambda|^2+AA^*\right) & 0\\ 0 & 0\end{pmatrix}.\end{aligned}$$

This implies that $|\lambda|^4\geq|\lambda|^4+|\lambda|^2AA^*$, so $AA^*=0$ and $A=0$ by $\lambda\neq 0$. Thus $\ker(T-\lambda)$ reduces T and the proof is complete. ■

Proposition 5.7. *If $T\in\mathcal{B}(\mathcal{H})$ belongs to class (U,n), then T is n-paranormal. In particular, T is isoloid and normaloid.*

Proof. By Heinz inequality, we have $(T^{*n}T^n)^{\frac{1}{n}}\geq T^*T$. Therefore,

$$\begin{aligned}\|Tx\|^2=\langle T^*Tx,x\rangle &\leq \langle (T^{*n}T^n)^{\frac{1}{n}}x,x\rangle\\ &\leq \langle T^{*n}T^nx,x\rangle^{\frac{1}{n}}\|x\|^{2(1-\frac{1}{n})} \quad \text{(by Lemma 3.1)}\\ &= \|T^nx\|^{\frac{2}{n}}\|x\|^{2(1-\frac{1}{n})}.\\ \therefore\ \|Tx\|^n &\leq \|T^nx\|\|x\|^{n-1}.\end{aligned}$$

■

6 Spectral properties of $\mathbb{QP}(n,k)$

Lemma 6.1. *Let $T\in\mathbb{QP}(n,k)$ and T^k do not have a dense range. Then*

$$T=\begin{pmatrix}T_1 & T_2\\ 0 & T_3\end{pmatrix} \quad on \quad \mathcal{H}=\overline{\text{ran}(T^k)}\oplus\ker(T^{*k}),$$

where $T_1=T|_{\overline{\text{ran}(T^k)}}$ is the restriction of T to $\overline{\text{ran}(T^k)}$, and T_1 is a class (U,n) and T_3 is nilpotent of nilpotency n. Moreover, $\sigma(T)=\sigma(T_1)\cup\{0\}$.

Proof. Consider $\mathcal{H}=\overline{\text{ran}(T^k)}\oplus\ker(T^{*k})$. Since $\overline{\text{ran}(T^k)}$ is an invariant subspace of T, T has the matrix representation

$$T=\begin{pmatrix}T_1 & T_2\\ 0 & T_3\end{pmatrix}$$

with respect to $\mathcal{H} = \overline{\text{ran}(T^k)} \oplus \ker(T^{*k})$. Let P be the orthogonal projection onto $\overline{\text{ran}(T^k)}$. Then $T_1 \oplus 0 = TP = PTP$ and $T_1^* T_1 \oplus 0 = PT^*TP$. Since $T \in \mathbb{QP}(n,k)$, we have

$$P\left(|T^n|^2 - |T|^{2n}\right)P \geq 0.$$

Using the facts $TP = PTP$, $PT^* = PT^*P$, we have

$$P(T^{*n}T^n)P = (TP)^{*n}(TP)^n \geq P(T^*T)^n P$$

$$\begin{aligned}
\begin{pmatrix} (T_1^{*n}T_1^n)^{\frac{2}{n}} & 0 \\ 0 & 0 \end{pmatrix} &= \begin{pmatrix} T_1^{*n}T_1^n & 0 \\ 0 & 0 \end{pmatrix}^{\frac{2}{n}} = \{P(T^{*n}T^n)P\}^{\frac{2}{n}} = \{(TP)^{*n}(TP)^n\}^{\frac{2}{n}} \\
&\geq \{P(T^*T)^n P\}^{\frac{2}{n}} \\
&\geq P(T^*T)^2 P \quad \text{(by Hansen's inequality)} \\
&= (TP)^*(TT^*)(TP) \geq (TP)^*(TPT^*)(TP) \\
&= \{(TP)^*(TP)\}^2 = \begin{pmatrix} (T_1^*T_1)^2 & 0 \\ 0 & 0 \end{pmatrix}.
\end{aligned}$$

On the other hand, if $u = \begin{pmatrix} u_1 \\ u_2 \end{pmatrix} \in \mathcal{H}$,

$$\left\langle T_3^k u_2, u_2 \right\rangle = \left\langle T^k(I-P)u, (I-P)u \right\rangle = \left\langle (I-P)u, T^{*k}(I-P)u \right\rangle = 0.$$

which implies that $T_3^k = 0$. It is well known that $\sigma(T_1) \cup \sigma(T_3) = \sigma(T) \cup \mathcal{C}$, where $\mathcal{C}$ is the union of certain of the holes in $\sigma(T)$ which happen to be subset of $\sigma(T_1) \cap \sigma(T_3)$ and $\sigma(T_1) \cap \sigma(T_3)$ has no interior points. Therefore, we have

$$\sigma(T) = \sigma(T_1) \cup \sigma(T_3) = \sigma(T_1) \cup \{0\}.$$

■

Lemma 6.2. *Let $T \in \mathbb{QP}(n,k)$ and $\mathcal{W}$ be its invariant subspace. Then the restriction $A_1 := T|_{\overline{T^k\mathcal{W}}}$ of T to $\overline{T^k\mathcal{W}}$ satisfies*

$$(A_1^{*n}A_1^n)^{\frac{2}{n}} \geq (A_1^*A_1)^2.$$

That is, A_1 belongs to class (U,n).

Proof. Let P be the orthogonal projection of $\mathcal{H}$ onto $\mathcal{W}$ and Q be the orthogonal projection of $\mathcal{H}$ onto $\overline{T^k\mathcal{W}}$. Since $\overline{T^k\mathcal{W}} \subset \mathcal{W}$, $Q \leq P$ holds. Decompose

$$T = \begin{pmatrix} A & B \\ 0 & C \end{pmatrix} \text{ on } \mathcal{H} = \mathcal{W} \oplus \mathcal{W}^\perp,$$

and

$$A = \begin{pmatrix} A_1 & A_2 \\ 0 & A_3 \end{pmatrix} \text{ on } \mathcal{W} = \overline{T^k\mathcal{W}} \oplus \left(\mathcal{W} \ominus \overline{T^k\mathcal{W}}\right).$$

Then we have $A \oplus 0 = TP = PTP$ and $(A \oplus 0)Q = TQ = QTQ = Q(A \oplus 0)Q = A_1 \oplus 0 \oplus 0$. Since $T \in \mathbb{QP}(n,k)$, we have

$$PT^{*k}(|T^n|^2 - |T|^{2n})T^kP \geq 0.$$

This implies that

$$Q(|T^n|^2 - |T|^{2n})Q \geq 0.$$

Hence,

$$\begin{aligned}
Q|T^n|^2Q = QT^{*n}T^nQ = Q(A^{*n}A^n \oplus 0)Q &= (Q(A \oplus 0)Q)^{*n}(Q(A \oplus 0)Q)^n \\
A_1^{*n}A_1^n \oplus 0 &\geq Q|T|^{2n}Q \\
\therefore\ (A_1^{*n}A_1^n)^{\frac{2}{n}} \oplus \{0\} &\geq \left(Q|T|^{2n}Q\right)^{\frac{2}{n}} \\
&\geq Q(|T|^{2n})^{\frac{2}{n}}Q = Q(T^*T)^2Q \quad \text{(by Hansen's inequality)} \\
&= (QT^*)TT^*(TQ) \\
&\geq (QT^*)(TQ)(QT^*)(TQ) = (A_1^*A_1)^2 \oplus \{0\}.
\end{aligned}$$

■

Theorem 6.3. *If $T \in \mathbb{QP}(n,k)$ and $(T-\lambda)x = 0$ for some $\lambda \neq 0$, then $(T-\lambda)^*x = 0$.*

Proof. Let P be the orthogonal projection onto $\ker(T-\lambda)$. Then $TP = \lambda P$, $T^mP = \lambda^mP$, $PT^* = \overline{\lambda}P$, $PT^{*m} = \overline{\lambda}^mP$ for all $m \geq 1$. And T and P are of the forms $T = \begin{pmatrix} \lambda & A \\ 0 & B \end{pmatrix}$, $P = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$ on $P\mathcal{H} \oplus (1-P)\mathcal{H}$. Since $T \in \mathbb{QP}(n,k)$, T satisfies $T^{*(n+k)}T^{n+k} = T^{*k}(T^{*n}T^n)T^k \geq T^{*k}(T^*T)^nT^k$ and hence

$$|\lambda|^{2(n+k)}P = PT^{*(n+k)}T^{n+k}P \geq PT^{*k}(T^*T)^nT^kP = |\lambda|^{2k}P(T^*T)^nP,$$

and

$$\begin{aligned}
|\lambda|^4P = \left(|\lambda|^{2n}P\right)^{\frac{2}{n}} &\geq (P(T^*T)^nP)^{\frac{2}{n}} \geq P(T^*T)^2P \quad \text{(by Hansen's inequality)} \\
&= |\lambda|^2P(TT^*)P = |\lambda|^2(|\lambda|^2 + AA^*)
\end{aligned}$$

This implies that $|\lambda|^4 \geq |\lambda|^4 + |\lambda|^2AA^*$, so $AA^* = 0$ and $A = 0$ by $\lambda \neq 0$. Thus $\ker(T-\lambda)$ reduces T and the proof is complete. ■

A complex number λ is said to be in the point spectrum $\sigma_p(T)$ of T if there is a nonzero $x \in \mathcal{H}$ such that $(T-\lambda)x = 0$. If in addition, $(T^* - \overline{\lambda})x = 0$, then λ is said to be in the joint point spectrum $\sigma_{jp}(T)$ of T.

Corollary 6.4. *If $T \in \mathbb{QP}(n,k)$, then $\sigma_{jp}(T) \setminus \{0\} = \sigma_p(T) \setminus \{0\}$.*

Corollary 6.5. *If $T \in \mathbb{QP}(n,k)$ and $\alpha, \beta \in \sigma_p(T)$ with $\alpha \neq \beta$. Then $\ker(T-\alpha) \perp \ker(T-\beta)$.*

Proof. Without loss of the generality, we may assume $\beta \neq 0$. Let $x \in \ker(T-\alpha)$ and $y \in \ker(T-\beta)$. Then $Tx = \alpha x$, $Ty = \beta y$ and $T^*y = \overline{\beta}y$. Therefore

$$\alpha \langle x,y\rangle = \langle \alpha x, y\rangle = \langle Tx, y\rangle = \langle x, T^*y\rangle = \left\langle x, \overline{\beta}y\right\rangle = \beta \langle x,y\rangle .$$

Hence $\alpha\langle x,y\rangle = \beta\langle x,y\rangle$ and so $(\alpha-\beta)\langle x,y\rangle = 0$. But $\alpha \neq \beta$, hence $\langle x,y\rangle = 0$. Consequently $\ker(T-\alpha) \perp \ker(T-\beta)$. ■

Theorem 6.6. *If T is a class (M,n) operator, then T is normaloid*

Proof. If T is a class (M,n) operator, then T is n-paranormal operator and so the result follows by [26, Proposition 1]. ■

Theorem 6.7. *Let $T \in \mathcal{B}(\mathcal{H})$. If $T \in \mathbb{QP}(n,k)$ with dense range, then T is class (M,n) operator.*

Proof. Since T has dense range, $\overline{\text{ran}(T^k)} = \mathcal{H}$. Then there exists a sequence $\{x_m\} \subset \mathcal{H}$ such that $\lim\limits_{n\longrightarrow\infty} T^k x_m = y$. Since $T \in \mathbb{QP}(n,k)$, we have

$$\left\langle T^{*k}|T^n|^2T^kx_m, x_m\right\rangle \geq \left\langle T^{*k}|T|^{2n}T^kx_m, x_m\right\rangle$$
$$\left\langle |T^n|^2T^kx_m, T^kx_m\right\rangle \geq \left\langle |T|^{2n}T^kx_m, T^kx_m\right\rangle \text{ for all } m \in \mathbb{N}$$

By the continuity of the inner product, we have

$$\left\langle (|T^n|^2 - |T|^{2n})y, y\right\rangle \geq 0.$$

Therefore T is a class (M,n) operator. ■

Corollary 6.8. *Let $T \in \mathcal{B}(\mathcal{H})$. If $T \in \mathbb{QP}(n,k)$ and not class (M,n), then T has not dense range.*

Lemma 6.9. *Let $T \in \mathcal{B}(\mathcal{H})$. If T is a class (M,n) and $\sigma(T) = \{\lambda\}$, then $T = \lambda$*

Proof. Since T is a class (M,n), T is n-paranormal. Hence the result follows from [39]. ■

Theorem 6.10. *Let $T \in \mathcal{B}(\mathcal{H})$. If $T \in \mathbb{QP}(n,k)$ and $\sigma(T) = \{\lambda\}$, then $T = \lambda$ if $\lambda \neq 0$ and $T^{k+1} = 0$ if $\lambda = 0$.*

Proof. If the range of T^k is dense, then T is of class $(M.n)$. Hence $T = \lambda$ by Lemma 6.9. If the range of T^k is not dense, then

$$T = \begin{pmatrix} T_1 & T_2 \\ 0 & T_3 \end{pmatrix} \text{ on } \mathcal{H} = \overline{\text{ran}(T^k)} \oplus \ker(T^{*k})$$

where T_1 satisfies the relation $(T_1^{*n}T_1^n)^{\frac{2}{n}} \geq (T_1^*T_1)^2$, $T_3^k = 0$ and $\sigma(T) = \sigma(T_1) \cup \{0\}$ by Lemma 6.1. In this case, $\lambda = 0$. Hence $T_1 = 0$ by Proposition 5.7, Lemma 6.1 and Lemma 6.9. Thus

$$T^{k+1} = \begin{pmatrix} 0 & T_2 \\ 0 & T_3 \end{pmatrix}^{k+1} = \begin{pmatrix} 0 & T_2T_3^k \\ 0 & T_3^{k+1} \end{pmatrix} = 0.$$

■

Corollary 6.11. *If $T \in \mathbb{QP}(n,k)$ and $(T-\alpha)x = 0$, $(T-\beta)x = 0$ with $\alpha^{n+1} \neq \beta^{n+1}$, then $\langle x, y\rangle = 0$.*

Proof. We may assume $\beta \neq 0$. Then

$$\alpha^{n+1}\langle x, y\rangle = \langle T^{n+1}x, y\rangle = \left\langle x, T^{*(n+1)}y\right\rangle = \beta^{n+1}\langle x, y\rangle$$

and so $\langle x, y\rangle = 0$. ■

The space of all functions that are analytical in the open neighborhoods of $\sigma(T)$ shall be denoted as $Hol(\sigma(T))$. Following [10], we state that $T \in \mathcal{B}(\mathcal{H})$ possesses the single-valued extension property (SVEP) at point $\lambda \in \mathbb{C}$ if the only analytic function $f : O_\lambda \longrightarrow \mathcal{H}$ that satisfies the equation $(T-\mu)f(\mu) = 0$ is the constant function $f \equiv 0$ Every point of the resolvent $\rho(T) := \mathbb{C} \setminus \sigma(T)$ has SVEP for $T \in \mathcal{B}(\mathcal{H})$, as is well known. Furthermore, it is clear that $T \in \mathcal{B}(\mathcal{H})$ has SVEP at every point in the border $\partial\sigma(T)$ of the spectrum from the identity theorem for analytic functions. Any isolated point of $\sigma(T)$ at T has SVEP, in particular. Laursen established in [29, Proposition 1.8] that if T is of finite ascent, then T possesses SVEP.

If each isolated point of $\sigma(T)$ is an eigenvalue of T, then an operator $T \in \mathcal{B}(\mathcal{H})$ is said to be isoloid. If $iso\sigma(T) \subseteq \pi(T)$, where $iso\sigma(T)$ is the set of isolated points of the spectrum $\sigma(T)$ of T, and $\pi(T)$ is the set of all poles of T, then an operator $T \in \mathcal{B}(\mathcal{H})$ is said to be polaroid.

A necessary and sufficient condition for $\lambda \in \pi(T)$ is that $\text{asc}(T-\lambda) = \text{dsc}(T-\lambda) < \infty$, where the ascent of T, $\text{asc}(T)$, is the least non-negative integer n such that $\ker(T^n) = \ker(T^{n+1})$ and the descent of T , $\text{dsc}(T)$, is the least non-negative integer n such that $\text{ran}(T^n) = \text{ran}(T^{n+1})$. In general, if T is polaroid then it is isoloid. However, the converse is not true. Consider the following example. Let $T \in \ell^2(\mathbb{N})$ be defined by

$$T(x_1, x_2, \cdots) = (\frac{x_2}{2}, \frac{x_3}{3}, \cdots).$$

Then T is a compact quasinilpotent operator with $\dim\ker(T) = 1$, and so T is isoloid. However, since T does not have finite ascent, T is not polaroid.

Recall that $T \in \mathcal{B}(\mathcal{H})$ is said to have finite ascent if $\ker(T^n) = \ker(T^{n+1})$ for some positive integer n.

Theorem 6.12. *Let $T \in \mathcal{B}(\mathcal{H})$. If T is a class (M,n), then T has SVEP.*

Proof. Since n-perinormal operator T is finite ascent by [31] , hence T has SVEP. ■

Corollary 6.13. *If T is a (n,k)-quasiperinormal, then T has SVEP.*

Proof. Let f be an analytic function on an open set D such that $(T-\alpha)f(\alpha) = 0$ for $\alpha \in D$. Let $\alpha = re^{i\theta} \neq 0$ and $\alpha_m = r^{1+\frac{1}{m}}e^{i\theta}$. Then

$$\|f(\alpha)\|^2 = \lim\langle f(\alpha), f(\alpha_m)\rangle = 0$$

by Corollary 6.11. ■

Corollary 6.14. *Suppose that T is non-zero (n,k)-quasiperinormal and it has no nontrivial T-invariant closed subspace. Then T is of class (M,n) operator.*

Proof. Since T has no non-trivial invariant closed subspace, it has no non-trivial hyperinvariant subspace. But $\ker(T^k)$ and $\overline{\operatorname{ran}(T^k)}$ are hyperinvariant subspaces, and $T \neq 0$, hence, $\ker(T^k) \neq \mathcal{H}$ and $\overline{\operatorname{ran}(T^k)} \neq \{0\}$. Therefore $\ker(T^k) = \{0\}$ and $\overline{\operatorname{ran}(T^k)} = \mathcal{H}$. In particular, T has dense range. It follows from Corollary 6.7 that T is of class (M, n) operator. ■

Theorem 6.15. *If $T \in \mathbb{QP}(n,k)$, then $\ker(T-\lambda) = \ker(T-\lambda)^2$ if $\lambda \neq 0$ and $\ker(T^{k+1}) = \ker(T^{k+2})$ if $\lambda = 0$. Consequently, $T - \lambda$ has finite ascent for all $\lambda \in \mathbb{C}$.*

Proof. Assume $0 \neq \lambda \in \sigma_p(T)$ because the case $\lambda \notin \sigma_p(T)$ is obvious. Let $0 \neq x \in \ker(T-\lambda)^2$, $x = x_1 \oplus x_2 \in \mathcal{H} = \overline{\operatorname{ran}(T^k)} \oplus \ker(T^k)$ and

$$T = \begin{pmatrix} T_1 & T_2 \\ 0 & T_3 \end{pmatrix} \text{ on } \mathcal{H} = \overline{\operatorname{ran}(T^k)} \oplus \ker(T^k).$$

Then

$$\begin{aligned} 0 = (T-\lambda)^2 x &= \begin{pmatrix} T_1 - \lambda & T_2 \\ 0 & T_3 - \lambda \end{pmatrix}^2 \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \\ &= \begin{pmatrix} (T_1-\lambda)^2 x_1 + ((T_1-\lambda)T_2 + T_2(T_3-\lambda))x_2 \\ (T_3-\lambda)^2 x_2 \end{pmatrix}. \end{aligned}$$

Consequently, $x_2 = 0$ because $T_3 - \lambda$ is invertible by Lemma 6.1. Thus $(T_1-\lambda)^2 x_1 = 0$ and $(T_1 - \lambda)x_1 \in \ker(T_1-\lambda) \subset \ker(T_1-\lambda)^*$ by Theorem 6.3. Therefore

$$\|(T_1-\lambda)x\|^2 = \langle (T_1-\lambda)^*(T_1-\lambda)x, x\rangle = \langle 0, x\rangle = 0,$$

so $(T_1 - \lambda)x = 0$ and

$$(T-\lambda)x = (T-\lambda)(x_1 \oplus 0) = (T_1-\lambda)x_1 = 0.$$

If $\lambda = 0$, $x \in \ker(T^{n+k})$, then

$$\begin{aligned} 0 &= \left\|T^{n+k}\right\|^2 = \left\langle T^{*k}T^{*n}T^nT^k x, x\right\rangle = \left\langle T^{*k}|T^n|^2T^k x, x\right\rangle \\ &\geq \left\langle T^{*k}|T|^{2n}T^k x, x\right\rangle = \left\||T|^nT^k x\right\|^2. \end{aligned}$$

Hence $|T|^nT^kx = 0$ and $|T|T^kx = 0$. Hence $T.T^kx = U|T|T^kx = 0$. This implies that $\ker(T^{n+k}) = \ker(T^{k+1})$ and $\ker(T^{k+1}) = \ker(T^{k+2}) = \cdots$.

If $\lambda = 0$ and $x \in \ker(T^{k+1})$, then it follows from Theorem 3.14 that

$$\left\|T^k x\right\| = \left\|T(T^{k-1}x)\right\| \leq \left\|T^{n+k-1}x\right\|^{\frac{1}{n}} \left\|T^{k-1}x\right\|^{\frac{n-1}{n}} = 0.$$

Hence $T^kx = 0$. Then $x \in \ker(T^k)$. ■

7 Weyl's theorem and the self-adjointness of any Riesz idempotent with respect to an arbitrary isolated point of $\sigma(T)$

Theorem 7.1. *Let T be n-perinormal and λ is an isolated point of $\sigma(T)$ then the Riesz idempotent E_λ satisfies the followings;*

(i) $E_0(\mathcal{H}) = \ker T$ $(\lambda = 0)$

(ii) $E_\lambda(\mathcal{H}) = \ker(T-\lambda) = \ker(T-\lambda)^*,\ E_\lambda = E_\lambda^*$ $(\lambda \neq 0)$.

for each $n \geq 2$.

Proof. (i) Both of $E_0\mathcal{H}$ and $(1-E_0)\mathcal{H}$ are T-invariant closed subspaces which satisfy that $\sigma(T|_{E_0\mathcal{H}}) = \{0\}$ and $\sigma(T|_{(1-E_0)\mathcal{H}}) = \sigma(T)\setminus\{0\}$. Since $T \in \mathfrak{P}(n)$, the restrictions $T|_{E_0\mathcal{H}}, T|_{(1-E_0)\mathcal{H}} \in \mathfrak{P}(n)$ and $\|T|_{E_0\mathcal{H}}\| = r(T|_{E_0\mathcal{H}}) = 0$ by Theorem 2.2 (*e*) and hence $T|_{E_0\mathcal{H}} = 0$. This implies that $E_0\mathcal{H} \subset \ker T$. Conversely, let $x = y+z \in \ker T$ be arbitrary where $y \in E_0\mathcal{H}$ and $z \in (1-E_0)\mathcal{H}$. Since $T|_{E_0\mathcal{H}} = 0$ and $T|_{(1-E_0)\mathcal{H}}$ is invertible,

$$0 = Tx = Ty + Tz = (T|_{E_0\mathcal{H}})y + (T|_{(1-E_0)\mathcal{H}})z = (T|_{(1-E_0)\mathcal{H}})z$$

implies $z = 0$ and hence $x = y \in E_0\mathcal{H}$. Therefore $E_0\mathcal{H} = \ker T$ holds.
(ii) Both of $E_\lambda\mathcal{H}$ and $(1-E_\lambda)\mathcal{H}$ are T-invariant closed subspaces which satisfy that $\sigma(T|_{E_\lambda\mathcal{H}}) = \{\lambda\}$ and $\sigma(T|_{(1-E_\lambda)\mathcal{H}}) = \sigma(T)\setminus\{\lambda\}$. Since, $T \in \mathfrak{P}(n)$ the restrictions $T|_{E_\lambda\mathcal{H}}, T|_{(1-E_\lambda)\mathcal{H}} \in \mathfrak{P}(n)$ and $\|T|_{E_\lambda\mathcal{H}}\| = r(T|_{E_\lambda\mathcal{H}}) = |\lambda|$ by Theorem 2.2(*e*) and also $|\lambda|^{-1} \leq \left\|(T|_{E_0\mathcal{H}})^{-1}\right\| \leq |\lambda|^{-\frac{n(n-1)}{2}+\frac{(n+1)(n-2)}{2}} = |\lambda|^{-1}$ by Theorem 2.2(*f*). Hence $U = \dfrac{1}{\lambda}T|_{E_\lambda\mathcal{H}}$ is invertible isometry with the spectrum $\sigma(U) = \{1\}$, so U is unitary and $U = 1$ on $E_\lambda\mathcal{H}$. This implies that $T|_{E_\lambda} = \lambda E_\lambda$ and $(T-\lambda)E_\lambda = 0$. It follows that $(T-\lambda)^*E_\lambda = 0$ by Lemma 5.4 or Lemma 5.6, and hence $E_\lambda\mathcal{H}$ is a reducing subspace of T. Since $(z-T)^*E_\lambda = (\overline{z}-\overline{\lambda})E_\lambda$ and $(z-T)^{-1*}E_\lambda = \overline{\left(\dfrac{1}{z-\lambda}\right)}E_\lambda$, it follows that

$$\begin{aligned}0 \leq E_\lambda^*E_\lambda &= -\frac{1}{2\pi i}\int_{|z-\lambda|=r}(z-T)^{*-1}E_\lambda\,d\overline{z}\\ &= -\frac{1}{2\pi i}\int_{|z-\lambda|=r}\overline{\left(\frac{1}{z-\lambda}\right)}E_\lambda\,d\overline{z} = \overline{\left(\frac{1}{2\pi i}\int_{|z-\lambda|=r}\frac{1}{z-\lambda}\,dz\right)}E_\lambda = E_\lambda.\end{aligned}$$

Hence $E_\lambda = E_\lambda^*$. Thus T is of the form $T = \lambda \oplus T'$ on $\mathcal{H} = E_\lambda\mathcal{H} \oplus (1-E_\lambda)\mathcal{H}$ with $\lambda \notin \sigma(T')$. Therefore the assertion $E_\lambda\mathcal{H} = \ker(T-\lambda) = \ker(T-\lambda)^*$ holds. ■

Theorem 7.2. *Weyl's theorem hold for any n-perinormal operators.*

Proof. We first show that $\sigma(T)\setminus w(T) \subset \pi_{00}(T)$. Let $\lambda \in \sigma(T)\setminus w(T)$ be arbitrary. Then $T-\lambda$ is Fredholm operator with the index $ind(T-\lambda) = 0$ and $(T-\lambda)$ is not invertible.

Case (i). $\lambda = 0$. Then $\ker T \neq \{0\}$ is finite dimension and $\operatorname{ran} T$ is closed. Thus the range of T^* is closed and T is of the form

$$T = \begin{pmatrix} 0 & A \\ 0 & T' \end{pmatrix} \text{ on } \ker T \oplus \operatorname{ran} T^*.$$

Since A is a finite rank operator, it follows that T' is Fredholm with the index $\text{ind}(T') = \text{ind}(T) = \{0\}$. Let $x \in \ker T'$ be arbitrary. Then $T^2(0 \oplus x) = T(Ax \oplus T'x) = T(Ax \oplus 0) = 0 \oplus 0 = 0$, so $T^n(0 \oplus x) = 0$. Since T is n-perinormal, $\ker T^n = \ker T$ and hence $x \in \ker T \cap \text{ran}\, T^* = \{0\}$. Therefore T' is Weyl with $\ker T' = \{0\}$, so it is invertible. This implies that 0 is isolated in $\sigma(T) = \{0\} \cup \sigma(T')$ and $0 \in \pi_{00}(T)$.

Case (ii). $\lambda \neq 0$. Then $\ker(T - \lambda)$ is finite dimensional subspace which reduces T and ran $(T - \lambda)$ is closed, and hence T is of the form $T = \lambda \oplus T'$ on $\mathcal{H} = \ker(T - \lambda) \oplus \text{ran}\,(T - \lambda)^*$. Since $T' - \lambda$ is Fredholm with the index $\text{ind}(T' - \lambda) = 0$ and $\ker(T' - \lambda) = \{0\}$, it follows that $T' - \lambda$ is invertible and hence λ is isolated in $\sigma(T) = \{\lambda\} \cup \sigma(T')$. Therefore $\lambda \in \pi_{00}(T)$. Thus $\sigma(T) \setminus w(T) \subset \pi_{00}(T)$ holds.

Next, we show that $\pi_{00}(T) \subset \sigma(T) \setminus w(T)$.

Let $\lambda \in \pi_{00}(T)$ be arbitraray. Then λ is isolated in $\sigma(T)$ and $\ker(T - \lambda) \neq \{0\}$ is finite dimension.

Case (i). $\lambda = 0$. Since T is n-perinormal, $T|_{E_0(\mathcal{H})}$ is class (U, n) by Lemma 5.5 and $\sigma(T|_{E_0(\mathcal{H})}) = \{0\}$. Hence $T|_{E_0(\mathcal{H})} = 0$ by Proposition 5.7. Then the Riesz idempotent E_0 with respect to 0 for T satisfies that $T|_{E_0\mathcal{H}} = 0$ and $T' := T|_{(1-E_0)\mathcal{H}}$ is invertible (so, it is Weyl) and $T' \in \mathfrak{P}(n)$. And $T = 0 + T'$ on $\mathcal{H} = E_0\mathcal{H} + (1 - E_0)\mathcal{H}$ is also Weyl. Therefore $0 \in \sigma(T) \setminus w(T)$.

Case (ii). $\lambda \neq 0$. Then $\ker(T - \lambda)$ is finite dimensional subspace which reduces T and $T = \lambda \oplus T'$ on $\mathcal{H} = \ker(T - \lambda) \oplus \text{ran}\,(T - \lambda)^*$, where T' is n-perinormal (hence $T' \in \mathfrak{P}(n)$). If $\lambda \in \sigma(T')$ then λ is isolated in $\sigma(T')$ and $\lambda \in \sigma_p(T')$. This is a contradiction because $\ker(T' - \lambda) \subset \text{ran}\,(T - \lambda)^* \cap \ker(T - \lambda) = \{0\}$. Thus $T' - \lambda$ is invertible and $T - \lambda = 0 \oplus (T' - \lambda)$ implies that $T - \lambda$ is Fredholm with the index $\text{ind}(T - \lambda) = \text{ind}(T' - \lambda) = 0$, so $T - \lambda$ is Weyl. Therefore $\lambda \in \sigma(T) \setminus w(T)$ holds. ■

8 Riesz Idempotent for $\mathbb{QP}(n, k)$ operators

Let μ be an isolated instance of T. Following that, the Riesz idempotent E of T with respect to μ is defined as

$$E := \frac{1}{2\pi i} \int_{\partial D} (\mu - T)^{-1}\, d\mu,$$

where D is a closed disc with a center at μ and no other points of the points of the spectrum of T. It is understood that $E^2 = E, ET = TE, \sigma(T|_{\text{ran}(E)}) = \{\mu\}$ and $\ker(T - \mu) \subseteq \text{ran}(E)$. In [37],, Stampfli demonstrated that E is self-adjoint and $\text{ran}(E) = \ker(T - \mu)$ if T meets the growth condition G_1. Recently, Stampfli's result for quasi-class A operators, paranormal operators, and k-quasi-$*$-paranormal operators was obtained by Jeon and Kim [20], Uchiyama [42] and Rashid [34]. The Riesz idempotent E of T with respect to μ is typically not necessarily self-adjoint, even if T is a paranormal operator.

Theorem 8.1. *Let $T \in \mathcal{B}(\mathcal{H})$. If $T \in \mathbb{QP}(n, k)$, then T is isoloid.*

Proof. Assume that T has the representation specified by the Lemma 6.1 and Proposition 5.7. Let z represent an isolated point in $\sigma(T)$. Then z is an isolated point in $\sigma(T_1)$ or $z = 0$ because $\sigma(T) = \sigma(T_1) \cup \{0\}$. Lemma 6.1 and Proposition 5.7 states that if z is an isolated point in $\sigma(T_1)$, then z is a point in $\sigma_p(T_1)$. Assume that $z = 0$ and that $z \notin \sigma(T_1)$. Since $\ker(T_3) \neq 0$ and $T_3^n = 0$. Then for $x \in \ker(T_3)$, $-T_1^{-1}T_2x \oplus x \in \ker(T)$. Thus, the proof is obtained. ■

Theorem 8.2. *Let $T \in \mathbb{QP}(n,k)$. Then T is polaroid. Let λ be an isolated point of $\sigma(T)$ and E be Riesz idempotent for λ. Then $E\mathcal{H} = \ker(T-\lambda)$ if $\lambda \neq 0$ and $E\mathcal{H} = \ker(T^{n+1})$ if $\lambda = 0$.*

Proof. Since $E\mathcal{H}$ is an invariant subspace of T and $\sigma(T|_{E\mathcal{H}}) = \{\lambda\}$, we have $T|_{E\mathcal{H}} = \lambda$ if $\lambda \neq 0$ and $(T|_{E\mathcal{H}})^{k+1} = 0$ if $\lambda = 0$ by Theorem 6.10 and Proposition 5.7. Hence $E\mathcal{H} \subset \ker(T|_{E\mathcal{H}} - \lambda) \subset \ker(T-\lambda)$ if $\lambda \neq 0$ and $E\mathcal{H} \subset \ker(T|_{E\mathcal{H}})^{k+1} \subset \ker T^{k+1}$ if $\lambda = 0$. Since $\ker(T-\lambda) \subset E\mathcal{H}$ is always true, $E\mathcal{H} = \ker(T-\lambda)$ if $\lambda \neq 0$. And if $\lambda = 0$ then $\ker T^{k+1} \subset E\mathcal{H}$ also holds. Hence, $E\mathcal{H} = \ker T^{k+1}$ by Lemma 5.2 of [44]. Hence

$$T = \begin{pmatrix} T_1 & 0 \\ 0 & T_2 \end{pmatrix}$$

where $\sigma(T_1) = \sigma(T|E\mathcal{H}) = \{\lambda\}$ and $\sigma(T_2) = \sigma(T) \setminus \{\lambda\}$. Then $T_1 - \lambda$ is nilpotent and $T_2 - \lambda$ is invertible. Hence $T - \lambda$ has finite ascent and descent. Hence T is polaroid. ■

Theorem 8.3. *Let $T \in \mathbb{QP}(n,k)$ and μ be a non-zero isolated point of $\sigma(T)$. Then the Riesz idempotent E for μ is self-adjoint and*

$$E\mathcal{H} = \ker(T-\mu) = \ker(T-\mu)^*.$$

Proof. If $T \in \mathbb{QP}(n,k)$, then μ is an eigenvalue of T and $E\mathcal{H} = \ker(T-\mu)$ by Theorem 8.1. Since $\ker(T-\mu) \subseteq \ker(T-\mu)^*$ by Theorem 6.3, it suffices to show that $\ker(T-\mu)^* \subseteq \ker(T-\mu)$. Since $\ker(T-\mu)$ is a reducing subspace of T by Theorem 6.3 and the restriction of a $\mathbb{QP}(n,k)$ operator to its reducing subspace is also a $\mathbb{QP}(n,k)$ operator by Lemma 6.2, T can be written as

$$T = \mu \oplus T_1 \text{ on } \mathcal{H} = \ker(T-\mu) \oplus \ker(T-\mu)^\perp,$$

where T_1 is a n-perinormal with $\ker(T_1 - \mu) = \{0\}$. Since $\mu \in \sigma(T) = \sigma(T_1) \cup \{\mu\}$ is isolated, only two cases occur: either $\mu \notin \sigma(T_1)$, or μ is an isolated of $\sigma(T_1)$ and this contradicts the fact that $\ker(T_1 - \mu) = \{0\}$. Since T_1 is invertible as an operator on $\ker(T-\mu)^\perp$, we have $\ker(T-\mu) = \ker(T-\mu)^*$.

Next, we show that E is self-adjoint. Since

$$E\mathcal{H} = \ker(T-\mu) = \ker(T-\mu)^*,$$

we have

$$((z-T)^*)^{-1}E = \overline{(z-\mu)^{-1}}E.$$

Therefore

$$\begin{aligned} E^*E &= -\frac{1}{2\pi i}\int_{\partial D} ((z-T)^*)^{-1}E\,d\bar{z} = -\frac{1}{2\pi i}\int_{\partial D} \overline{(z-T)^{-1}}E\,d\bar{z} \\ &= \overline{\left(\frac{1}{2\pi i}\int_{\partial D}(z-T)^{-1}\,dz\right)}E = E. \end{aligned}$$

This achieves the proof. ■

9 Tensor Product

Let's use the Hilbert spaces' symbols $\mathcal{H}$ and $\mathcal{K}$. $\mathcal{H} \otimes \mathcal{K}$ signifies the tensor product on the product space $T \otimes S$ for the non-zero operators $T \in \mathcal{B}(\mathcal{H})$ and $S \in \mathcal{B}(\mathcal{K})$ that are specified. In terms of tensor products, the normaloid property is invariant [36]. According to [12, 38], $T \otimes S$ is normal if and only if T and S are normal. There are paranormal operators T and S such that $T \otimes S$ is not paranormal [1]. I.H. Kim shown in [23] that for non-zero $T \in \mathcal{B}(\mathcal{H})$ and $S \in \mathcal{B}(\mathcal{K})$, $T \otimes S$ is log-hyponormal if and only if T and S are log-hyponormal. In in [23], [22], [20], [24] and [33], respectively, this finding was extended to p-quasihyponormal operators, class A operators, quasi-class A, quasi-class (A, k) operators, and class A_k operators. In this section, we prove an analogous result for class (U, n) operators.

Remark 9.1. Let $T \in \mathcal{B}(\mathcal{H})$ and $S \in \mathcal{B}(\mathcal{K})$ be non-zero operators, then we have

(i) $(T \otimes S)^*(T \otimes S) = T^*T \otimes S^*S$

(ii) $|T \otimes S|^t = |T|^t \otimes |S|^t$ for any positive real t.

Lemma 9.2. *([38]) Let $T_1, T_2 \in \mathcal{B}(\mathcal{H})$, $S_1, S_2 \in \mathcal{B}(\mathcal{K})$ be non-negative operators. If T_1 and S_1 are non-zero, then the following assertions are equivalent:*

(a) $T_1 \otimes S_1 \leq T_2 \otimes S_2$

(b) there exists $c > 0$ such that $T_1 \leq cT_2$ and $S_1 \leq c^{-1}S_2$.

Theorem 9.3. *([45]) Let $T \in \mathcal{B}(\mathcal{H})$ and $S \in \mathcal{B}(\mathcal{K})$ be non-zero operators. Then $T \otimes S \in \mathcal{B}(\mathcal{H} \otimes \mathcal{K})$ is a class (M, n) operator if and only if T and S are class (M, n) operators.*

Theorem 9.4. *Let $T \in \mathcal{B}(\mathcal{H})$ and $S \in \mathcal{B}(\mathcal{K})$ be non-zero operators. Then $T \otimes S \in \mathcal{B}(\mathcal{H} \otimes \mathcal{K})$ is a class (U, n) operator if and only if T and S are class (U, n) operators.*

Proof. It is clear that $T \otimes S$ is a class (U, n) operator if and only if

$$\begin{aligned} & |(T \otimes S)^n|^{\frac{4}{n}} \geq |T \otimes S|^4 \\ \iff & |T^n \otimes S^n|^{\frac{4}{n}} \geq |T|^4 \otimes |S|^4 \\ \iff & |T^n|^{\frac{4}{n}} \otimes |S^n|^{\frac{4}{n}} \geq |T|^4 \otimes |S|^4 \\ \iff & (|T^n|^{\frac{4}{n}} - |T|^4) \otimes |S^n|^{\frac{4}{n}} + |T|^4 \otimes (|S^n|^{\frac{4}{n}} - |S|^4) \geq 0 \end{aligned}$$

Therefore, the sufficiency is clear.
Conversely, suppose that $T \otimes S$ is a class (U, n) . Let $x \in \mathcal{H}$ and $y \in \mathcal{K}$ be arbitrary. Then we have

$$\left\langle (|T^n|^{\frac{4}{n}} - |T|^4)x, x \right\rangle \left\langle |S^n|^{\frac{4}{n}} y, y \right\rangle + \langle |T|^4 x, x \rangle \left\langle (|S^n|^{\frac{4}{n}} - |S|^4)y, y \right\rangle \geq 0 \tag{9.1}$$

Suppose on the contrary that T is not a class (U, n) operator; then there exists $x_0 \in \mathcal{H}$ such that

$$\begin{cases} \left\langle (|T^n|^{\frac{4}{n}} - |T|^4)x_0, x_0 \right\rangle = \alpha < 0 \\ \langle |T|^4 x_0, x_0 \rangle = \beta > 0 \end{cases} \tag{9.2}$$

From (9.1), we have

$$\alpha\left\langle|S^n|^{\frac{4}{n}}y,y\right\rangle+\beta\left\langle(|S^n|^{\frac{4}{n}}-|S|^4)y,y\right\rangle\geq 0 \tag{9.3}$$

for all $y\in\mathcal{K}$; that is,

$$(\alpha+\beta)\left\langle|S^n|^{\frac{4}{n}}y,y\right\rangle\geq\beta\left\langle|S|^4y,y\right\rangle \tag{9.4}$$

for all $y\in\mathcal{K}$. Therefore, S is a class (U,n) operator. So, we have

$$(\alpha+\beta)\left\||S^n|^{\frac{2}{n}}y\right\|^2\geq\beta\left\||S|^2y\right\|^2 \tag{9.5}$$

for all $y\in\mathcal{K}$ by (9.4). By (9.5), we have

$$(\alpha+\beta)\left\||S^n|^{\frac{2}{n}}\right\|^2\geq\beta\left\||S|^2\right\|^2. \tag{9.6}$$

Since self-adjoint operators are normaloid, we have

$$\begin{aligned}(\alpha+\beta)\|S^n\|^{\frac{4}{n}} &= (\alpha+\beta)\||S^n|\|^{\frac{4}{n}}=(\alpha+\beta)\left\||S^n|^2\right\|^{\frac{2}{n}}\\ &\geq \beta\left\||S|^2\right\|^2=\beta\||S|\|^4=\beta\|S\|^4. \end{aligned}\tag{9.7}$$

Hence

$$\beta\|S\|^4\leq(\alpha+\beta)\|S^n\|^{\frac{4}{n}}\leq(\alpha+\beta)\|S\|^4.$$

This implies that $S=0$. This contradicts the assumption $S\neq 0$. Hence T must be a class (U,n) operator. A similar argument shows that S is also a class (U,n) operator. ■

Theorem 9.5. *Let $T\in\mathcal{B}(\mathcal{H})$ and $S\in\mathcal{B}(\mathcal{K})$ be non-zero operators. Then $T\otimes S\in\mathbb{QP}(n,k)$ if and only if one of the following holds:*

(i) T and S are in $\mathbb{QP}(n,k)$.

(ii) $T^{k+1}=0$ or $S^{k+1}=0$.

Proof. By simple calculation we have

$$\begin{aligned}T\otimes S\in\mathbb{QP}(n,k)&\Leftrightarrow(T\otimes S)^{*k}\left(|(T\otimes S)^n|^2-|T\otimes S|^{2n}\right)(T\otimes S)^k\geq 0\\ &\Leftrightarrow T^{*k}(|T^n|^2-|T|^{2n})T^k\otimes S^{*k}|S^n|^2S^k+T^{*k}|T|^{2n}T^k\otimes S^{*k}(|S^n|^2-|S|^{2n})S^k\geq 0.\end{aligned}$$

Thus the sufficiency is easily proved because $T^{*k}|T|^{2n}T^k=0$ if $T^{k+1}=0$. Conversely, suppose that $T\otimes S\in\mathbb{QP}(n,k)$. Then for $x,y\in\mathcal{H}$ we have

$$\begin{aligned}&\left\langle T^{k*}(|T^n|^2-|T|^{2n})T^kx,x\right\rangle\left\langle S^{k*}|S^n|^2S^ky,y\right\rangle\\ &\qquad+\left\langle T^{k*}|T|^{2n}T^kx,x\right\rangle\left\langle S^{k*}(|S^n|^2-|S|^{2n})S^ky,y\right\rangle\geq 0.\end{aligned}\tag{9.8}$$

It suffices to show that if the statement (ii) does not hold, the statement (i) holds.Thus, assume to the contrary that neither of T^{k+1} and S^{k+1} is the zero operator, and T is not in $\mathbb{QP}(n,k)$. Then there exists $x_0\in\mathcal{H}$ such that

$$\left\langle T^{k*}(|T^n|^2-|T|^2)T^kx_0,x_0\right\rangle:=\alpha<0\quad\text{and}\quad\left\langle T^{k*}|T|^{2n}T^kx_0,x_0\right\rangle:=\beta>0.$$

From (9.8) we have

$$(\alpha+\beta)\left\langle S^{k*}|S^n|^2 S^k y, y\right\rangle \geq \beta\left\langle S^{k*}|S|^{2n} S^k y, y\right\rangle. \tag{9.9}$$

Thus $S \in \mathbb{QP}(n,k)$. By Hölder McCarthy Inequality, we have

$$\left\langle S^{k*}|S^n|^2 S^k y, y\right\rangle = \left\|S^{n+k}y\right\|^2$$

and

$$\left\langle S^{k*}|S|^{2n} S^k y, y\right\rangle \geq \left\langle |S|^2 S^k y, S^k y\right\rangle^n \left\|S^k y\right\|^{2(1-n)} = \left\|S^{k+1}y\right\|^{2n} \left\|S^k y\right\|^{2(1-n)}.$$

Therefore, we have

$$(\alpha+\beta)\left\|S^{n+k}y\right\|^2 \geq \beta\left\|S^{k+1}y\right\|^{2n}\left\|S^k y\right\|^{2(1-n)}. \tag{9.10}$$

Since $S \in \mathbb{QP}(n,k)$, from Lemma 6.1 we have a decomposition of S as the following:

$$S = \begin{bmatrix} S_1 & S_2 \\ 0 & S_3 \end{bmatrix} \quad \text{on} \quad \mathcal{H} = \overline{\Re(S^k)} \oplus \ker(S^{*k}), \quad \text{where } S_1 \text{ is a class } (U,n).$$

By (9.10) and Lemma 6.2 we have

$$(\alpha+\beta)\left\|S_1^n \xi\right\|^2 \geq \beta\left\|S_1\xi\right\|^{2n} \quad \text{for all } \xi \in \overline{\Re(S^k)}. \tag{9.11}$$

So, we have

$$(\alpha+\beta)\left\|S_1\right\|^4 \geq \beta\left\|S_1\right\|^4,$$

where equality holds since S_1 is normaloid by Proposition 5.7.
This implies that $S_1 = 0$. Since $S^{k+1}y = S_1 S^k y = 0$ for all $y \in \mathcal{K}$, we have $S^{k+1} = 0$. This contradicts the assumption $S^{k+1} \neq 0$. Hence T must be a (n,k)-quasiperinormal operator. A similar argument shows that S is also a (n,k)-quasiperinormal operator. The proof is complete. ■

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