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On two dimensional Chlodowsky-Szász operators via Sheffer Polynomials

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Abstract. In this paper, we define a new sequence of positive linear operators in two variables by combining Chlodowsky operators and Szász type operators via Sheffer polynomials. Further, we calculate basic estimates. In the next section, we investigate their rapidity of convergence and order of approximation. Morever, Local approximation results and global approximation results are studied in various functional spaces.

1 Introduction

Operators Theory is a fascinating field of research for the last two decades due to the advent of computer. It plays a significant role in pure and applied , viz, fixed point theory, numerical analysis. In computational aspects mathematics and shape of geometric objects, CAGD(Computer-aided Geometric design) plays an interesting with the mathematical description. It focuses on Mathematics that is compatible with computers in shape designing. To investigate the behaviour of parametric surfaces and curves, control nets and control points play a significant role respectively. CAGD is widely used as an applications in Applied Mathematics and industries. It has several applications in other branches of sciences, e.g., Approximation theory, Computer graphics, data structures, Numerical analysis, Computer algebra etc. The first sequences of positive linear operator are developed by Bernstein [1] in 1912 to present the simplest and easiest proof of fundamental theorem of approximation theory named as Weierstrass approximation theorem. Later, these sequences are termed as classical Bernstein operators defined as:

$$B_n(f;x) = \sum_{\nu=0}^{n} p_{n,\nu}(x) f\left(\frac{\nu}{n}\right),\ n \in \mathbb{N}, \tag{1.1}$$

where $p_{n,\nu}(x) = \binom{n}{\nu} x^\nu (1-x)^{n-\nu}$, $x \in [0,1]$ and $f \in C[0,1]$ (spaces of all continuous function in $[0,1]$).

The basis $p_{n,\nu}(x) = \binom{n}{\nu} x^\nu (1-x)^{n-\nu}$ of Bernstein polynomials (1.1) has significant role in preserving the shape of the surfaces or curves. Graphic design programs, viz, photoshop inkspaces and Adobe's illustrator deals with Bernstein polynomials in the form of Bèzier curves. To Preserve the shape of the parametric surface or curve, it depends on basis $p_{n,\nu}(x) = \binom{n}{\nu} x^\nu (1-x)^{n-\nu}$ which is used to construct the curves. Several modifications have been studied for the operators defined in (1.1).

In 1937, Chlodowsky [2] presented a generalization of Bernstein polynomials which is well known as Bernstein-Chlodowsky polynomials on the interval $[0, c_n]$, where $c_n \to \infty$ as $n \to \infty$ as:

$$B_n(f;x) = \sum_{k=0}^{n} P_{n,k}\left(\frac{x}{c_n}\right) f\left(\frac{k}{n} c_n\right), \qquad n \in N, \tag{1.2}$$

where, $P_{n,k}\left(\frac{x}{c_n}\right) = \binom{n}{k}\left(\frac{x}{c_n}\right)^k \left(1 - \frac{x}{c_n}\right)^{n-k}$, $x \in [0, c_n]$ and $\lim\limits_{n\to\infty} \frac{c_n}{n} = 0$.

In 1974, Ismail [3] introduced a new form of Szász operators including Sheffer polynomials as follows:

$$T_n(f;x)=\frac{e^{-nxH(1)}}{A(1)}\sum_{k=0}^{\infty}p_k(nx)f\left(\frac{k}{n}\right), \tag{1.3}$$

where $x\geq 0$, $p_k(nx)\geq 0$, $A(1)\neq 0$, $H'(1)=1$ and $n\in\mathbb{N}$. In equation (1.3) $p_k(nx)$ are Sheffer polynomials given by the generating functions:

$$A(u)e^{xH(u)}=\sum_{k=0}^{\infty}p_k(x)u^k, \tag{1.4}$$

where

$$A(z)=\sum_{k=0}^{\infty}a_k z^k\ (a_0\neq 0) \tag{1.5}$$

and

$$H(z)=\sum_{k=0}^{\infty}h_k z^k\ (h_1\neq 0) \tag{1.6}$$

are the analytic functions in the disc $\mid z\mid< R\ (R>1)$. Relation (1.4) is valid for $\mid u\mid< R$ and the power series given by (1.6) converges for $\mid z\mid< R$, $R>1$. Various current literature are available in this direction (see [**?**]n1)-[**?**]n5))

The aim of this paper is to present a new sequence of operators for functions of two variables as:

$$L_{n_1,n_2}(f;x_1,x_2)=\frac{e^{-n_1x_1H(1)}}{A(1)}\sum_{k_1=0}^{n_1}\sum_{k_2=0}^{\infty}P_{n_1,k_1}\left(\frac{x_1}{c_{n_1}}\right)p_k(n_2x_2)f\left(\frac{k_1}{n_1}c_{n_1},\frac{k_2}{n_2}\right) \tag{1.7}$$

where $(x_1,x_2)\in I\times J, i.e. I=[0,c_n] and J=[0,\infty)$.

In subsequent sections,we investigate basic lemmas, rate of convergence and order of approximation for the class of continuous functions of two variables. In the last, local and global approximation results are obtained in different functional spaces.

2 Basic Estimates

Here, we recall some Lemmas from [3] and [2] as follows:

Lemma 2.1. *[3] Let $e_i(t)=t^i$, $i,j=0,1,2$ be the test functions. Then, we have*

$$\begin{aligned}T_n(1;x)&=1,\\T_n(t^1;x)&=x+\frac{A'(1)}{nA(1)},\\T_n(t^2;x)&=x_2^2+\left(\frac{2A'(1)}{A(1)}+H''(1)+1\right)\frac{x_2}{n_2}+\frac{A'(1)+A''(1)}{n_2^2A(1)}.\end{aligned}$$

Lemma 2.2. *[2] Let $e_i(t)=t^i$, $i,j=0,1,2$ be the test functions. Then, we have*

$$\begin{aligned}B_n(1;x)&=1,\\B_n(t;x)&=x_1,\\B_n(t^2;x)&=\left(1-\frac{1}{n_1}\right)x_1^2+\frac{c_{n_1}}{n_1}x_1.\end{aligned}$$

Proof. In the light of (1.7) and linearity property, we find that

$$\begin{aligned} L_{n_1,n_2}(e_{0,0};x_1,x_2) &= L_{n_1}(e_0;x_1)L_{n_2}(e_0;x_2), \\ L_{n_1,n_2}(e_{1,0};x_1,x_2) &= L_{n_1}(e_1;x_1)L_{n_2}(e_0;x_2), \\ L_{n_1,n_2}(e_{2,0};x_1,x_2) &= L_{n_1}(e_2;x_1)L_{n_2}(e_0;x_2), \\ L_{n_1,n_2}(e_{0,1};x_1,x_2) &= L_{n_1}(e_0;x_1)L_{n_2}(e_1;x_2), \\ L_{n_1,n_2}(e_{0,2};x_1,x_2) &= L_{n_1}(e_0;x_1)L_{n_2}(e_2;x_2). \end{aligned}$$

using these equalities, we can easily prove Lemma 2.2. □

Lemma 2.3. *For the operators constructed in* (1.7)*,we get*

$$\begin{aligned} L_{n_1,n_2}((t_1-x_1)^2;x_1,x_2) &= \frac{c_{n_1}}{n_1}x_1 - \frac{x_1^2}{n_1}, \\ L_{n_1,n_2}((t_1-x_1)^2;x_1,x_2) &= 2\left(\frac{A'(1)}{A(1)} + H''(1) + 1\right)\frac{x_2}{n_2} + \frac{A'(1)+A''(1)}{n_2^2 A(1)}. \end{aligned}$$

Proof. In view of Lemma 2.2 and linearity property, it is easy to prove Lemma 2.3. □

3 Rate of Conergence and Order of Approximation

Definition 3.1. Let $X, Y \subset \mathbb{R}$ be any two given intervals and the set $B(X\times Y) = \{f : X\times Y \to \mathbb{R} | f \text{ is bounded on } X\times Y\}$. For $f \in B(X\times Y)$, let the function $\omega_{total}(f;\cdot,*) : [0,\infty)\times[0,\infty) \to \mathbb{R}$, defined for any $(\delta_1,\delta_2) \in [0,\infty)\times[0,\infty)$ by

$$\omega_{total}(f;\delta_1,\delta_2) = sup\{|f(x,y) - f(x',y')| : (x,y),(x',y') \in [0,\infty)\times[0,\infty), |x-x'| \le \delta_1, |y-y'| \le \delta_2\},$$

is called the first order modulus of smoothness of the function f or the total modulus of continuity of the function f.

Here, we recall the following result due to Volkov [4]:

Theorem 3.2. *Let I and J be compact intervals of the real line. Let $L_{n_1,n_2} : C(I\times J) \to C(I\times J), (n_1,n_2) \in \mathbb{N}\times\mathbb{N}$ be linear positive operators. If*

$$\begin{aligned} \lim_{n_1,n_2\to\infty} L_{n_1,n_2}(e_{ij}) &= e_{ij}, (i,j) \in \{(0,0),(1,0),(0,1)\}, \\ \lim_{n_1,n_2\to\infty} L_{n_1,n_2}(e_{20}+e_{02}) &= e_{20}+e_{02}, \end{aligned}$$

uniformly on $I\times J$, then the sequence $(L_{n_1,n_2}f)$ converges to f uniformly on $I\times J$ for any $f \in C(I\times J)$.

Theorem 3.3. *Let $e_{ij}(x_1,x_2) = x_1^i x_2^j (0 \le i+j \le 2, i,j \in \mathbb{N})$ be the test functions defined on $I\times J$. If*

$$\lim_{n_1,n_2\to\infty}(L_{n_1,n_2}e_{ij})(x_1,x_2) = e_{ij}(x_1,x_2) and \lim_{n_1,n_2\to\infty} L_{n_1,n_2}(e_{20}+e_{02}) = e_{20}+e_{02}$$

uniformly on $I\times J$, then

$$\lim_{n_1,n_2\to\infty}(L_{n_1,n_2}f)(x_1,x_2) = f(x_1,x_2)$$

uniformly for any $f \in C(I\times J)$.

Proof. Using the Theorem 3.2 and Lemma 2.2, Theorem 3.3 can easily be proved. □

Theorem 3.4. *[5] Let $L : C([0,\infty)\times[0,\infty)) \to B([0,\infty)\times[0,\infty))$ be a linear positive operator. For any $f \in C(X \times Y)$, any $(x,y) \in X \times Y$ and any $\delta_1, \delta_2 > 0$, the following inequality*

$$\begin{aligned} |(Lf)(x,y) - f(x,y)| &\leq |Le_{0,0}(x,y) - 1||f(x,y)| + \Big[Le_{0,0}(x,y) + \delta_1^{-1}\sqrt{Le_{0,0}(x,y)(L(\cdot - x^2))^2(x,y)} \\ &+ \delta_2^{-1}\sqrt{Le_{0,0}(x,y)(L(* - y^2))^2(x,y)} + \delta_1^{-1}\delta_2^{-1}\sqrt{(Le_{0,0})^2(x,y)(L(\cdot - x^2))^2(x,y)(\\ &\times \omega_{total}(f;\delta_1,\delta_2), \end{aligned}$$

holds.

Theorem 3.5. *Let $f \in C(I \times J)$ and $(x_1,x_2) \in I \times J$. Then, for $n_1, n_2 \in \mathbb{N}$ and for any $\delta_1, \delta_2 > 0$, we have*

$$|(L_{n_1,n_2}f)(x_1,x_2) - f(x_1,x_2)| \leq 4\omega_{total}(f;\delta_1,\delta_2),$$

where $\delta_{n_1,n_2} = \sqrt{L_{n_1,n_2}((t_1-x_1)^2;x_1,x_2)(((t_i-x_i)^2);x_1,x_2)}$.

Proof. In the direction of 3.4 and Lemma 2.3 , we can arrive at the proof of the Theorem 3.5. □

4 Local and Global Approximation Results

Let ϕ be weight function such that $\phi(x_1,x_2) = 1 + x_1^2 + x_2^2$ and satisfying $B_\phi(\mathcal{I}\times\mathcal{J}) = \{g :| g(x_1,x_2) |\leq C_g\phi(x_1,x_2),\ C_g > 0\}$, where $B_\phi(\mathcal{I}\times\mathcal{J})$ is the set of all bounded function on $\mathcal{I}\times\mathcal{J} = [0,c_n]) \times [0,\infty)$. Suppose $C^{(m)}(\mathcal{I}\times\mathcal{J})$ be the m-times continuously differentiable functions defined on $\mathcal{I}\times\mathcal{J} = \{(x_1,x_2) \in \mathcal{I}\times\mathcal{J} : x_1,x_2 \in [0,c_n]\times[0,\infty)\}$. The equipped norm on B_ϕ given by $\| g \|_\phi = \sup_{x_1,x_2\in\mathcal{I}\times\mathcal{J}} \frac{|g(x_1,x_2)|}{\phi(x_1,x_2)}$. Moreover, we have some classes of function as follows:

$$C_\phi^m(\mathcal{I}\times\mathcal{J}) = \{g : g \in C_\phi(\mathcal{I}\times\mathcal{J});\quad \text{such that} \lim_{(x_1,x_2)\to\infty}\frac{g(x_1,x_2)}{\phi(x_1,x_2)} = K_g < \infty\},$$

$$C_\phi^0(\mathcal{I}\times\mathcal{J}) = \{h : h \in C_\phi^m(\mathcal{I}\times\mathcal{J});\quad \text{such that} \lim_{(x_1,x_2)\to\infty}\frac{g(x_1,x_2)}{\phi(x_1,x_2)} = 0\},$$

$$C_\phi(\mathcal{I}\times\mathcal{J}) = \{g : g \in B_\phi \cap C_\phi(\mathcal{I}\times\mathcal{J})\}.$$

Let $\omega_\phi(g;\delta_1,\delta_2)$ be the weighted modulus of smoothness for all $g \in C_\phi^0(\mathcal{I}\times\mathcal{J})$ and $\delta_1,\delta_2 > 0$, defined by

$$\omega_\phi(g;\delta_1,\delta_2) = \sup_{(x_1,x_2)\in[0,c_n]\times[0,\infty)}\ \sup_{0\leq|\theta_1|\leq\delta_1,\ 0\leq|\theta_2|\leq\delta_2} \frac{|\ g(x_1+\theta_1,x_2+\theta_2) - g(x_1,x_2)\ |}{\phi(x_1,x_2)\phi(\theta_1,\theta_2)}. \tag{4.1}$$

For any $\eta_1,\eta_2 > 0$ one has

$$\omega_\phi(g;\eta_1\delta_1,\eta_2\delta_2) \leq 4(1+\eta_1)(1+\eta_2)(1+\delta_1^2)(1+\delta_2^2)\omega_\phi(g;\delta_1,\delta_2), \tag{4.2}$$

$$\begin{aligned} |\ g(t,s) - g(x_1,x_2)\ | &\leq \phi(x_1,x_2)\phi(|\ t-x_1\ |,|\ s-x_2\ |)\,\omega_\phi(g;|\ t-x_1\ |,|\ s-x_2\ |) \\ &\leq (1+x_1^2+x_2^2)(1+(t-x_1)^2)(1+(s-x_2)^2)\omega_\phi(g;|\ t-x_1\ |,|\ s-x_2\ |). \end{aligned}$$

Theorem 4.1. *For the operator $L_{n_1,n_2}(f;x_1,x_2)$ defined by (1.7), we have*

$$\frac{|\ L_{n_1,n_2}(f;x_1,x_2)(g;x_1,x_2) - g(x_1,x_2)\ |}{(1+x_1^2+x_2^2)} \leq \Psi_{x_1,x_2}\Big(1+o\left(n_1^{-1}\right)\Big)\Big(1+o\left(n_2^{-1}\right)\Big)\omega_\phi\Big(g;o\left(n_1^{-\frac{1}{2}}\right),\left(n_2^{-}\right.$$

where $\Psi_{n_1,n_2} = \Big(1+(x_1+1)+C_1(x_1+1)^2+\sqrt{C_3}(x_1+1)^3\Big)\Big(1+(x_2+1)+C_2(x_2+1)^2+\sqrt{C_4}(x_2+1)^3\Big)$ and $C_1,\ C_2,\ C_3,\ C_4 > 0$ and $g \in C^0(I\times J)$.

Proof. For all $\delta_{n_1}, \delta_{n_2} > 0$ we have

$$\begin{aligned}
\mid g(t_1,t_2)-g(x_1,x_2)\mid &\leq 4(1+x_1^2+x_2^2)\left(1+(t_1-x_1)^2\right)\left(1+(s_2-x_2)^2\right)\\
&\times \left(1+\frac{\mid t_1-x_1\mid}{\delta_{n_1}}\right)\left(1+\frac{\mid s_1-x_2\mid}{\delta_{n_2}}\right)(1+\delta_{n_1}^2)(1+\delta_{n_2}^2)\omega_\phi\left(g;\delta_{n_1},\delta_{n_2}\right)\\
&= 4(1+x_1^2+x_2^2)(1+\delta_{n_1}^2)(1+\delta_{n_2}^2)\\
&\times \left(1+\frac{\mid t_1-x_1\mid}{\delta_{n_1}}+(t_1-x_1)^2+\frac{1}{\delta_{n_1}}\mid t-x\mid(t_1-x_1)^2\right)\\
&\times \left(1+\frac{\mid t_2-x_2\mid}{\delta_{n_2}}+(t_2-x_2)^2+\frac{\mid t_2-x_2\mid}{\delta_{n_2}}(t_2-x_2)^2\right)\omega_\varphi\left(g;\delta_{n_1},\delta_{n_2}\right).
\end{aligned}$$

Apply operator $L_{n_1,n_2}(f;x_1,x_2)$ given by (1.7) both the sides,

$$\begin{aligned}
\mid L_{n_1,n_2}(f;x_1,x_2)(g;x_1,x_2)-g(x_1,x_2)\mid &\leq L_{n_1,n_2}(f;x_1,x_2)\left(\mid g(.,.)-g(x_1,x_2)\mid;x_1,x_2\right)4(1+x_1^2+x_2^2)\\
&\times L_{n_1,n_2}(f;x_1,x_2)\left(1+\frac{\mid t_1-x_1\mid}{\delta_{n_1}}+(t_1-x_1)^2+\frac{1}{\delta_{n_1}}\mid t_1-x\right.\\
&\times L_{n_1,n_2}(f;x_1,x_2)\left(1+\frac{\mid t_2-x_2\mid}{\delta_{n_2}}+(t_2-x_2)^2+\frac{\mid s_2-x_2\mid}{\delta_{n_2}}(\right.\\
&\times (1+\delta_{n_1}^2)(1+\delta_{n_2}^2)\omega_\phi\left(g;\delta_{n_1},\delta_{n_2}\right)\\
&= 4(1+x_1^2+x_2^2)(1+\delta_{n_1}^2)(1+\delta_{n_2}^2)\omega_\phi\left(g;\delta_{n_1},\delta_{n_2}\right)\\
&\times \left(1+\frac{1}{\delta_{n_1}}L_{n_1,n_2}(f;x_1,x_2)(\mid t_1-x\mid;x_1,x_2)+L_{n_1,n_2}(f;x_1,x\right.\\
&+ \left.\frac{1}{\delta_{n_1}}L_{n_1,n_2}(f;x_1,x_2)(\mid t_1-x_1\mid(t_1-x_1)^2;x_1,x_2\right)\\
&\times \left(1+\frac{1}{\delta_{n_2}}L_{n_1,n_2}(f;x_1,x_2)(\mid t_2-x_2\mid;x_1,x_2)+L_{n_1,n_2}(f;x_1,\right.\\
&+ \left.\frac{1}{\delta_{n_2}}L_{n_1,n_2}(f;x_1,x_2)(\mid t_2-x_2\mid(t_2-x_2)^2;x_1,x_2\right);
\end{aligned}$$

$$\begin{aligned}
\mid L_{n_1,n_2}(f;x_1,x_2)(g;x_1,x_2)-g(x_1,x_2)\mid &\leq 4(1+x_1^2+x_2^2)(1+\delta_{n_1}^2)(1+\delta_{n_2}^2)\omega_\phi\left(g;\delta_{n_1},\delta_{n_2}\right)\\
&\times \left[1+\frac{1}{\delta_{n_1}}\sqrt{L_{n_1,n_2}(f;x_1,x_2)((t_1-x_1)^2;x_1,x_2)}+L_{n_1,n_2}(f;x\right.\\
&+ \frac{1}{\delta_{n_1}}\sqrt{L_{n_1,n_2}(f;x_1,x_2)((t_1-x_1)^2;x_1,x_2)}\sqrt{L_{n_1,n_2}(f;x_1,x_2)(}\\
&\times \left[1+\frac{1}{\delta_{n_2}}\sqrt{L_{n_1,n_2}(f;x_1,x_2)((t_2-x_2)^2;x_1,x_2)}+L_{n_1,n_2}(f;x\right.\\
&+ \frac{1}{\delta_{n_2}}\sqrt{L_{n_1,n_2}(f;x_1,x_2)((t_2-x_2)^2;x_1,x_2)}\sqrt{L_{n_1,n_2}(f;x_1,x_2)(}
\end{aligned}$$

If we choose $\delta_{n_1}=o\left(n_1^{-\frac{1}{2}}\right)$ and $\delta_{n_2}=o\left(n_2^{-\frac{1}{2}}\right)$, then

$$\begin{aligned}
\mid L_{n_1,n_2}(f;x_1,x_2)(g;x_1,x_2)-g(x_1,x_2)\mid &\leq 4(1+x_1^2+x_2^2)(1+\delta_{n_1}^2)(1+\delta_{n_2}^2)\omega_{\phi}\left(g;\delta_{n_1},\delta_{n_2}\right)\\
&\times \left[1+\frac{1}{\delta_{n_1}}\sqrt{o\left(\frac{1}{n_1}\right)(x_1+1)^2+o\left(\frac{1}{n_1}\right)(x_1+1)^2}\right.\\
&+ \left.\frac{1}{\delta_{n_1}}\sqrt{o\left(\frac{1}{n_1}\right)(x_1+1)^2}\sqrt{o\left(\frac{1}{n_1}\right)(x_1+1)^4}\right]\\
&\times \left[1+\frac{1}{\delta_{n_2}}\sqrt{o\left(\frac{1}{n_2}\right)(x_2+1)^2+o\left(\frac{1}{n_2}\right)(x_2+1)^2}\right.\\
&+ \left.\frac{1}{\delta_{n_2}}\sqrt{o\left(\frac{1}{n_2}\right)(x_2+1)^2}\sqrt{o\left(\frac{1}{n_2}\right)(x_2+1)^4}\right]\\
&\leq 4(1+x_1^2+x_2^2)(1+\delta_{n_1}^2)(1+\delta_{n_2}^2)\omega_{\varphi}\left(g;\delta_{n_1},\delta_{n_2}\right)\\
&\times \left[1+(x_1+1)+C_1(x_1+1)^2+\sqrt{C_2}(x_1+1)^3\right]\left[1+(x_2+1\right.\\
&+ \left.C_3(x_2+1)^2+\sqrt{C_4}(x_2+1)^3\right].
\end{aligned}$$

□

Which complete the proof of the theorem 4.1

Lemma 4.2 ([6, 7]). *For the positive sequence of operators $\{L_{n_1,n_2}(f;x_1,x_2)\}_{n_1,n_2\geq 1}$, which acting $C_\phi\to B_\phi$ defined earlier then there exists some positive K such that*

$$\| L_{n_1,n_2}(f;x_1,x_2)(\phi;x_1,x_2)\|_\phi\leq K.$$

Theorem 4.3 ([6, 7]). *or the positive sequence of operators $\{L_{n_1,n_2}(f;x_1,x_2)\}_{n_1,n_2\geq 1}$ acting $C_\phi\to B_\phi$ defined earlier satisfying the following conditions*

$$\begin{aligned}
(1)\quad \lim n_1,n_2\to\infty \| L_{n_1,n_2}(f;x_1,x_2)(1;x_1,x_2)-1\|_\phi &= 0;\\
(2)\quad \lim_{n_1,n_2\to\infty}\| L_{n_1,n_2}(f;x_1,x_2)(t_1;x_1,x_2)-x_1\|_\varphi &= 0;\\
(3)\quad \lim_{n_1,n_2\to\infty}\| L_{n_1,n_2}(f;x_1,x_2)(t_2;x_1,x_2)-x_2\|_\varphi &= 0;\\
(4)\quad \lim_{n_1,n_2\to\infty}\| L_{n_1,n_2}(f;x_1,x_2)((t_1^2+t_2^2);x_1.x_2)-(x_1^2+x_2^2)\|_\varphi &= 0.
\end{aligned}$$

Then for all $g\in C_\phi^0$,

$$\lim_{n_1,n_2\to\infty}\| L_{n_1,n_2}(f;x_1,x_2)g-g\|_\phi=0$$

and there exists another function $h\in C_\phi\setminus C_\phi^0$, such that

$$\lim_{n_1,n_2\to\infty}\| L_{n_1,n_2}(f;x_1,x_2)h-h\|_\phi\geq 1.$$

Theorem 4.4. *If $g\in C_\phi^0\left(\mathcal{I}\times\mathcal{J}\right)$, then we have*

$$\lim_{n_1,n_2\to\infty}\| L_{n_1,n_2}(f;x_1,x_2)(g)-g\|_\phi=0.$$

Proof. We know that,

$$\begin{aligned}\| L_{n_1,n_2}(f;x_1,x_2)(\phi;x_1,x_2)\|_\phi &= \sup_{(x_1,x_2)\in\mathcal{I}\times\mathcal{J}} \frac{| L^{\mu,\nu}_{n_1,n_2}(1+x_1^2+x_2^2;x_1,x_2)|}{1+x_1^2+x_2^2}\\
&= 1+\sup_{(x_1,x_2)\in\mathcal{I}\times\mathcal{J}}\left[\frac{1}{1+x_1^2+x_2^2}\right]\Bigg|\Bigg(L_{n_1,n_2}(f;x_1,x_2)(x_1^2;x_1,x_2)+L_{n_1,n_2}(\\
&= 1+\sup_{(x_1,x_2)\in\mathcal{I}^2}\frac{x_1^2}{1+x_1^2+x_2^2}+\left|\frac{1}{n_1}\left(4+2\lambda+2\mu\frac{e_\mu(-n_1x_1)}{e_\mu(n_1x_1)}\right)\right|\sup_{(x_1,x_2)\in\mathcal{I}\times}\\
&+\left|\frac{(\lambda+1)(\lambda+2)}{n_1^2}\right|+\sup_{(x_1,x_2)\in\mathcal{I}\times\mathcal{J}}\frac{x_2^2}{1+x_1^2+x_2^2}+\left|\frac{(\lambda+1)(\lambda+2)}{n_2^2}\right|\\
&+\left|\frac{1}{n_2}\left(4+2\lambda+2\nu\frac{e_\nu(-n_2x_2)}{e_\nu(n_2x_2)}\right)\right|\sup_{(x_1,x_2)\in\mathcal{I}\times\mathcal{J}}\frac{x_2}{1+x_1^2+x_2^2}\\
&\le 1+\left|\frac{4+2\lambda+2\mu}{n_1}\right|+\left|\frac{4+2\lambda+2\nu}{n_2}\right|+\left|\frac{(\lambda+1)(\lambda+2)}{n_1^2}\right|+\left|\frac{(\lambda+1)(\lambda}{n_2^2}\right.\end{aligned}$$

Now for all $n_1,n_2\neq 0$, there exists a positive constant K such that

$$\| L_{n_1,n_2}(f;x_1,x_2)(\phi;x_1,x_2)\|_\phi\le K.$$

Therefore, in order to prove Theorem 4.4 it is sufficient from the Theorem 4.1. Thus we led to prove of Theorem 4.4. For any $g\in C(\mathcal{I}\times\mathcal{J})$ and $\delta>0$ modulus of continuity of order second is given by

$$\omega\,(g;\delta_{n_1},\delta_{n_2})=\sup\{| g(t_1,t_2)-g(x_1,x_2)|:(t_1,t_2),(x_1,x_2)\in\mathcal{I}\times\mathcal{J}\}$$

with $| t_1-x_1|\le\delta_{n_1}$, $| t_2-x_2|\le\delta_{n_2}$ with the partial modulus of continuity defined as:

$$\omega_1\,(g;\delta)=\sup_{0\le x_2\le 1}\sup_{|x_1-x_2|\le\delta}\{| g(x_1,x_2)-g(x_1,x_2)|\},$$

$$\omega_2\,(g;\delta)=\sup_{0\le x_1\le 1}\sup_{|y_1-y_2|\le\delta}\{| g(x,y_1)-g(x,y_2)|\}.$$

□

Theorem 4.5. *For any $g\in C(\mathcal{I}\times\mathcal{J})$ we have*

$$| L_{n_1,n_2}(f;x_1,x_2)(g;x_1,x_2)-g(x_1,x_2)|\le 2\Big(\omega_1(g;\delta_{x_1,n_1})+\omega_2(g;\delta_{x_2,n_2})\Big).$$

Proof. In order to give the prove of Theorem 4.5, in general we use well-known Cauchy-Schwarz inequality. Thus we see that

$$\begin{aligned}| L_{n_1,n_2}(f;x_1,x_2)(g;x_1,x_2)-g(x_1,x_2)| &\le L_{n_1,n_2}(f;x_1,x_2)\,(| g(t_1,t_2)-g(x_1,x_2)|;x_1,x_2)\\
&\le L_{n_1,n_2}(f;x_1,x_2)\,(| g(t_1,t_2)-g(x_1,t_2)|;x_1,x_2)\\
&+ L_{n_1,n_2}(f;x_1,x_2)\,(| g(x_1,t_1)-g(x_1,x_2)|;x_1,x_2)\\
&\le L_{n_1,n_2}(f;x_1,x_2)\,(\omega_1(g;| t_1-x_1|);x_1,x_2)+L_{n_1,n_2}(f;x_1,x_2)\\
&\le \omega_1(g;\delta_{n_1})\left(1+\delta_{n_1}^{-1}L^{\mu,\nu}_{n_1,n_2}(| t_1-x_1|;x_1,x_2)\right)\\
&+ \omega_2(g;\delta_{n_2})\left(1+\delta_{n_2}^{-1}L_{n_1,n_2}(f;x_1,x_2)(| t_2-x_2|;x_1,x_2)\right)\\
&\le \omega_1(g;\delta_{n_1})\left(1+\frac{1}{\delta_{n_1}}\sqrt{L_{n_1,n_2}(f;x_1,x_2)((t_1-x_1)^2;x_1,x_2)}\right)\\
&+ \omega_2(g;\delta_{n_2})\left(1+\frac{1}{\delta_{n_2}}\sqrt{L_{n_1,n_2}(f;x_1,x_2)((t_2-x_2)^2;x_1,x_2)}\right).\end{aligned}$$

If we choose $\delta_{n_1}^2 = \delta_{n_1,x_1}^2 = L_{n_1,n_2}(f;x_1,x_2)((t_1-x_1)^2;x_1,x_2)$ and $\delta_{n_2}^2 = \delta_{n_2,x_2}^2 = L_{n_1,n_2}(f;x_1,x_2)((t_2-x_2)^2;x_1,x_2)$, then we easily to reach our desired results.
Here we find convergence in terms of the Lipschitz class for bivariate function. For $M > 0$ and $\rho_1,\rho_2 \in [0,1]$, Lipschitz maximal function space on $E \times E \subset \mathcal{I} \times \mathcal{J}$ defined by

$$\begin{aligned}\mathcal{L}_{\rho_1,\rho_2}(E\times E) &= \Big\{ g : \sup(1+t_1)^{\rho_1}(1+t_2)^{\rho_2}\left(g_{\rho_1,\rho_2}(t_1,t_2) - g_{\rho_1,\rho_2}(x_1,x_2)\right) \\ &\leq M\frac{1}{(1+x_1)^{\rho_1}}\frac{1}{(1+x_2)^{\rho_2}}\Big\},\end{aligned}$$

where g is continuous and bounded on $\mathcal{I}\times\mathcal{J}$, and

$$g_{\rho_1,\rho_2}(t_1,t_2) - g_{\rho_1,\rho_2}(x_1,x_2) = \frac{\mid g(t_1,t_2) - g(x_1,x_2)\mid}{\mid t_1 - x_1\mid^{\rho_1}\mid t_2 - x_2\mid^{\rho_2}};\quad (t_1,t_2),(x_1,x_2)\in\mathcal{I}\times\mathcal{J}. \tag{4.3}$$

□

Theorem 4.6. *Let $g \in \mathcal{L}_{\rho_1,\rho_2}(E\times E)$, then for any $\rho_1,\rho_2 \in [0,1]$, there exists $M > 0$ such that*

$$\begin{aligned}\mid L_{n_1,n_2}(f;x_1,x_2)(g;x_1,x_2) - g(x_1,x_2)\mid &\leq M\Big\{\Big((d(x_1,E))^{\rho_1} + \left(\delta_{n_1,x_1}^2\right)^{\frac{\rho_1}{2}}\Big)\Big((d(x_2,E))^{\rho_2} + \left(\delta_{n_2,x_2}^2\right)^{\frac{\rho_2}{2}} \\ &+ (d(x_1,E))^{\rho_1}(d(x_2,E))^{\rho_2}\Big\},\end{aligned}$$

where δ_{n_1,x_1} and δ_{n_2,x_2} defined by Theorem 4.6.

Proof. Take $\mid x_1 - x_0 \mid = d(x_1,E)$ and $\mid x_2 - x_0' \mid = d(x_2,E)$. For any $(x_1,x_2)\in\mathcal{I}\times\mathcal{J}$, and $(x_0,x_0')\in E\times E$ we let $d(x_1,E) = \inf\{\mid x_1 - x_2 \mid : x_2 \in E\}$. Thus we can write here

$$\mid g(t_1,t_2) - g(x_1,x_2)\mid \leq M \mid g(t_1,t_2) - g(x_0,x_0')\mid + \mid g(x_0,x_0') - g(x_1,x_2)\mid. \tag{4.4}$$

Apply $L_{n_1,n_2}(f;x_1,x_2)$, we obtain

$$\begin{aligned}\mid L_{n_1,n_2}(f;x_1,x_2)(g;x_1,x_2) - g(x_1,x_2)\mid &\leq L_{n_1,n_2}(f;x_1,x_2)\left(\mid g(x_1,x_2) - g(x_0,x_0')\mid + \mid g(x_0,x_0') - g(x\right. \\ &\leq ML_{n_1,n_2}(f;x_1,x_2)\left(\mid t_1 - x_0\mid^{\rho_1}\mid t_2 - x_0'\mid^{\rho_2};x_1,x_2\right) \\ &+ M\mid x - x_0\mid^{\rho_1}\mid x_2 - x_0'\mid^{\rho_2}.\end{aligned}$$

For all $A,B \geq 0$ and $\rho\in[0,1]$ we know inequality $(A+B)^\rho \leq A^\rho + B^\rho$, thus

$$\mid t_1 - x_0\mid^{\rho_1} \leq \mid t_1 - x_1\mid^{\rho_1} + \mid x - x_0\mid^{\rho_1},$$

$$\mid t_2 - x_0'\mid^{\rho_1} \leq \mid t_2 - x_2\mid^{\rho_2} + \mid x_2 - x_0'\mid^{\rho_2}.$$

Therefore

$$\begin{aligned}\mid L_{n_1,n_2}(f;x_1,x_2)(g;x_1,x_2) - g(x_1,x_2)\mid &\leq ML_{n_1,n_2}(f;x_1,x_2)\left(\mid t_1 - x_1\mid^{\rho_1}\mid t_2 - x_2\mid^{\rho_2};x_1,x_2\right) \\ &+ M\mid x_1 - x_0\mid^{\rho_1} L_{n_1,n_2}(f;x_1,x_2)\left(\mid t_2 - x_2\mid^{\rho_2};x_1,x_2\right) \\ &+ M\mid x_2 - x_0'\mid^{\rho_2} L_{n_1,n_2}(f;x_1,x_2)\left(\mid t_1 - x_1\mid^{\rho_1};x_1,x_2\right) \\ &+ 2M\mid x_1 - x_0\mid^{\rho_1}\mid x_2 - x_0'\mid^{\rho_2} L_{n_1,n_2}(f;x_1,x_2)\left(\alpha_{0,0};x_1,x_2\right).\end{aligned}$$

On apply Hölder inequality on $L_{n_1,n_2}^{\alpha_1,\alpha_2}$, we get

$$\begin{aligned}L_{n_1,n_2}(f;x_1,x_2)\left(\mid t_1 - x_1\mid^{\rho_1}\mid t_2 - x_2\mid^{\rho_2};x_1,x_2\right) &= \mathcal{U}_{n_1,k}^{\mu}\left(\mid t_1 - x_1\mid^{\rho_1};x_1,x_2\right)\mathcal{V}_{n_2,l}^{\nu}\left(\mid t_2 - x_2\mid^{\rho_2};x_1,x_2\right) \\ &\leq \left(L_{n_1,n_2}(f;x_1,x_2)(\mid t_1 - x_1\mid^2;x_1,x_2)\right)^{\frac{\rho_1}{2}}\left(L_{n_1,n_2}(f;x_1\right. \\ &\times \left(L_{n_1,n_2}(f;x_1,x_2)(\mid t_2 - x_2\mid^2;x_1,x_2)\right)^{\frac{\rho_2}{2}}\left(L_{n_1,n_2}(f;x_1\right.\end{aligned}$$

Thus we can obtain

$$\begin{aligned} | L_{n_1,n_2}(f;x_1,x_2)(g;x_1,x_2) - g(x_1,x_2) | &\leq M\left(\delta^2_{n_1,x_1}\right)^{\frac{\rho_1}{2}}\left(\delta^2_{n_2,x_2}\right)^{\frac{\rho_2}{2}} + 2M\left(d(x_1,E)\right)^{\rho_1}\left(d(x_2,E)\right)^{\rho_2} \\ &+ M\left(d(x_1,E)\right)^{\rho_1}\left(\delta^2_{n_2,x_2}\right)^{\frac{\rho_2}{2}} + L\left(d(x_2,E)\right)^{\rho_2}\left(\delta^2_{n_1,x_1}\right)^{\frac{\rho_1}{2}}. \end{aligned}$$

We have complete the proof. □

Theorem 4.7. *If $g \in C'(\mathcal{I}\times\mathcal{J})$, then for all $(x_1,x_2)\in\mathcal{I}\times\mathcal{J}$, operator $L_{n_1,n_2}(f;x_1,x_2)$ satisfying*

$$| L_{n_1,n_2}(f;x_1,x_2)(g;x_1,x_2) - g(x_1,x_2) | \leq \| g_{x_1} \|_{C(\mathcal{I}\times\mathcal{J})}\left(\delta^2_{n_1,x_1}\right)^{\frac{1}{2}} + \| g_{x_2} \|_{C(\mathcal{I}\times\mathcal{J})}\left(\delta^2_{n_2,x_2}\right)^{\frac{1}{2}},$$

where $\delta_{n,x}$ and $\delta_{m,y}$ are defined by Theorem 4.7.

Proof. Let $g \in C'(\mathcal{I}\times\mathcal{J})$, and for any fixed $(x_1,x_2)\in\mathcal{I}\times\mathcal{J}$ we have

$$g(t_1,t_2) - g(x_1,x_2) = \int_{x_1}^{t_1} g_r(r,t_2)dr + \int_{x_2}^{t_2} g_s(x_1,s)ds.$$

On apply $L_{n_1,n_2}(f;x_1,x_2)$

$$L_{n_1,n_2}(f;x_1,x_2)\left(g(t_1,t_2);x_1,x_2\right) - g(x_1,x_2) = L_{n_1,n_2}(f;x_1,x_2)\left(\int_{x_1}^{t_1} g_r(r,t_2)dr;x_1,x_2\right) + L_{n_1,n_2}(f;x_1,x_2)\Big(\tag{4.5}$$

From the sup-norm on $\mathcal{I}\times\mathcal{J}$ we can see that

$$\left| \int_{x_1}^{t_1} g_r(r,t_2)dr \right| \leq \int_{x_1}^{t_1} | g_r(r,t_2)dr | \leq \| g_{x_1} \|_{C(\mathcal{I}\times\mathcal{J})} | t_1 - x_1 | \tag{4.6}$$

$$\left| \int_{x_2}^{t_2} g_s(x_1,s)ds \right| \leq \int_{x_2}^{t_2} | g_\mu(x_1,s)ds | \leq \| g_{x_2} \|_{C(\mathcal{I}\times\mathcal{J})} | t_2 - x_2 |. \tag{4.7}$$

In the view of (4.4), (4.5) and (4.6) we can obtain

$$\begin{aligned} | L_{n_1,n_2}(f;x_1,x_2)\left(g(x_1,x_2);x_1,x_2\right) - g(x_1,x_2) | &\leq L_{n_1,n_2}(f;x_1,x_2)\left(\left|\int_{x_1}^{t_1} g_r(r,t_2)dr\right|;x_1,x_2\right) \\ &+ L_{n_1,n_2}(f;x_1,x_2)\left(\left|\int_{x_2}^{t_2} g_s(x_1,s)ds\right|;x_1,x_2\right) \\ &\leq \| g_{x_1} \|_{C(\mathcal{I}\times\mathcal{J})} L_{n_1,n_2}(f;x_1,x_2)\left(| t_1 - x_1 |;x_1,x_2\right) \\ &+ \| g_{x_2} \|_{C(\mathcal{I}\times\mathcal{J})} L_{n_1,n_2}(f;x_1,x_2)\left(| t_2 - x_2 |;x_1,x_2\right) \\ &\leq \| g_{x_1} \|_{C(\mathcal{I}\times\mathcal{J})}\left(L_{n_1,n_2}(f;x_1,x_2)((t_1-x_1)^2;x_1,x_2)L \right. \\ &+ \| g_{x_2} \|_{C(\mathcal{I}\times\mathcal{J})}\left(L_{n_1,n_2}(f;x_1,x_2)((t_2-x_2)^2;x_1,x_2)L \right. \\ &= \| g_{x_1} \|_{C(\mathcal{I}\times\mathcal{J})}\left(\delta^2_{n_1,x_1}\right)^{\frac{1}{2}} + \| g_{x_2} \|_{C(\mathcal{I}\times\mathcal{J})}\left(\delta^2_{n_2,x_2}\right)^{\frac{1}{2}}. \end{aligned}$$

□

Theorem 4.8. *For any $f \in C(\mathcal{I}\times\mathcal{J})$, if we define an auxiliary operator such that*

$$T^{\mu,\nu}_{n_1,n_2}(f;x_1,x_2) = L_{n_1,n_2}(f;x_1,x_2)(g;u_1,u_2) + f(u_1,u_2) - f\left(\mathcal{U}^{\mu}_{n_1,k}(e_{1,0};x_1,x_2), \mathcal{V}^{\nu}_{n_2,l}(e_{0,1};x_1,x_2)\right).$$

where, $\mathcal{U}^{\mu}_{n_1,k}(e_{1,0};x_1,x_2) = x_1 + \frac{\lambda+1}{n_1}$ and $\mathcal{V}^{\nu}_{n_2,l}(e_{0,1};x_1,x_2) = x_2 + \frac{\lambda+1}{n_2}$.
Then for all $g \in C'(\mathcal{I}\times\mathcal{J})$, $T^{\mu,\nu}_{n_1,n_2}$ satisfying

$$\begin{aligned} R^{\mu,\nu}_{n_1,n_2}(f;t_1,t_2) - f(x_1,x_2) &\leq \left\{ \delta^2_{n_1,u_1} + \delta^2_{n_2,u_2} + \left(\frac{1}{(n_1-1)}\left(n_1 + 2(\alpha_1-1)\right)u_1 + \frac{1}{(n_1-1)} - u_1\right)^2 \right. \\ &+ \left. \left(\frac{1}{(n_2-1)}\left(n_2 + 2(\alpha_2-1)\right)u_2 + \frac{1}{(n_2-1)} - u_2\right)^2 \right\} \| g \|_{C^2(\mathcal{I}\times\mathcal{J})}. \end{aligned}$$

Proof. In the light of operator $T^{\mu,\nu}_{n_1,n_2}(f;x_1,x_2)$, we obtain $T^{\mu,\nu}_{n_1,n_2}(1;x_1,x_2) = 1$, $T^{\mu,\nu}_{n_1,n_2}(t_1 - x_1;x_1,x_2) = 0$ and $T^{\mu,\nu}_{n_1,n_2}(t_2 - x_2;x_1,x_2) = 0$. For any $g \in C'(\mathcal{I}\times\mathcal{J})$ the Taylor series give us:

$$\begin{aligned} f(t_1,t_2) - f(x_1,x_2) &= \frac{\partial f(x_1,x_2)}{\partial x_1}(t_1-x_1) + \int_{u_1}^{t}(t_1-\alpha)\frac{\partial^2 f(\lambda,x_2)}{\partial\alpha^2}\mathrm{d}\alpha \\ &+ \frac{\partial f(x_1,x_2)}{\partial t_2}(t_2-x_2) + \int_{x_2}^{t_2}(t_2-\phi)\frac{\partial^2 f(x_!,\phi)}{\partial\phi^2}\mathrm{d}\phi. \end{aligned}$$

On apply $T^{\mu,\nu}_{n_1,n_2}$, we see that

$$\begin{aligned} & T^{\mu,\nu}_{n_1,n_2}\left(f(t_1,t_2);x_1,x_2\right) - T^{\mu,\nu}_{n_1,n_2}\left(f(x_1,x_2)\right. \\ &= T^{\mu,\nu}_{n_1,n_2}\left(\int_{x_1}^{t_1}(t_1-\alpha)\frac{\partial^2 f\alpha,x_2)}{\partial\alpha^2}\mathrm{d}\alpha;x_1,x_2\right) + T^{\mu,\nu}_{n_1,n_2}\left(\int_{n_2}^{t_2}(t_2-\phi)\frac{\partial^2 f(x_1,\phi)}{\partial\phi^2}\mathrm{d}\phi;x_1,x_2\right) \\ &= L_{n_1,n_2}(f;x_1,x_2)\left(\int_{x_1}^{t_1}(t_1-\alpha)\frac{\partial^2 f(\alpha,x_2)}{\partial\alpha^2}\mathrm{d}\alpha;x_1,x_2\right) + L_{n_1,n_2}(f;x_1,x_2)\left(\int_{x_2}^{t_2}(t_2-\phi)\frac{\partial^2 f(x_1,\phi)}{\partial\phi^2}\mathrm{d}\phi;\right. \\ &- \int_{x_1}^{x_1+\frac{\lambda+1}{n_1}}\left(x_1+\frac{1+\lambda}{n_1}-\alpha\right)\frac{\partial^2 f(\lambda,x_2)}{\partial\alpha^2}\mathrm{d}\alpha - \int_{x_2}^{x_2+\frac{\lambda+1}{n_2}}\left(x_2+\frac{1+\lambda}{n_2}-\phi\right)\frac{\partial^2 f(x_1,\phi)}{\partial\phi^2}\mathrm{d}\phi. \end{aligned}$$

From hypothesis we easily obtain

$$\left|\int_{x_1}^{t_1}(t_1-\alpha)\frac{\partial^2 f(\alpha,x_2)}{\partial\alpha^2}\mathrm{d}\alpha\right| \le \int_{x_1}^{t}\left|(t_1-\alpha)\frac{\partial^2 f(\alpha,x_2)}{\partial\alpha^2}\mathrm{d}\alpha\right| \le \| f \|_{C^2(\mathcal{I}\times\mathcal{J})}(t_1-x_1)^2,$$

$$\left|\int_{x_2}^{t_2}(t_2-\phi)\frac{\partial^2 f(x_1,\phi)}{\partial\phi^2}\mathrm{d}\phi\right| \le \int_{x_2}^{t_2}\left|(t_2-\phi)\frac{\partial^2 f(x_1,\phi)}{\partial\phi^2}\mathrm{d}\phi\right| \le \| f \|_{C^2(\mathcal{I}\times\mathcal{J})}(t_2-x_2)^2,$$

$$\left|\int_{x_1}^{x_1+\frac{\lambda+1}{n_1}}\left(x_1+\frac{\lambda+1}{n_1}-\alpha\right)\frac{\partial^2 f(\alpha,x_2)}{\partial\alpha^2}\mathrm{d}\alpha\right| \le \| f \|_{C^2(\mathcal{I}\times\mathcal{J})}\left(\frac{\lambda+1}{n_1}\right)^2$$

$$\left|\int_{x_2}^{x_2+\frac{\lambda+1}{n_2}}\left(x_2+\frac{\lambda+1}{n_2}-\phi\right)\frac{\partial^2 f(\phi,x_1)}{\partial\phi^2}\mathrm{d}\phi\right| \le \| f \|_{C^2(\mathcal{I}\times\mathcal{J})}\left(\frac{\lambda+1}{n_2}\right)^2$$

Thus,

$$\begin{aligned} \mid T^{\mu,\nu}_{n_1,n_2}(f;t_1,t_2) - f(x_1,x_2) \mid &\le \Big\{L_{n_1,n_2}(f;x_1,x_2)((t_1-x_1)^2;x_1,x_2) + L_{n_1,n_2}(f;x_1,x_2)((t_2-x_2)^2;x_1, \\ &+ \left(\frac{\lambda+1}{n_1}\right)^2 + \left(\frac{\lambda+1}{n_2}\right)^2\Big\} \| g \|_{C^2(\mathcal{I}\times\mathcal{J})}. \end{aligned}$$

□

We complete the proof of desired Theorem 4.8.

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