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Some Closed Varieties of Semigroups

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Abstract In this paper, we have shown that the varieties of semigroups defined by the identities $axy = xya^2$, $axy = y^2ax$, $axy = yaxa$, $axy = yxya$, $axy = axaya$, $axy = yayxy$ and $axy = xaxyx$ respectively are closed in itself.

1 Introduction and Preliminaries

Let U be a subsemigroup of a semigroup S. Following Isbell [9], we say that U dominates an element d of S if for every semigroup T and for all homomorphisms $\beta, \gamma : S \longrightarrow T$ and $u\beta$=$u\gamma$ for every u in U implies $d\beta$=$d\gamma$. The set of all elements of S dominated by U is called dominion of U in S and we denote it by $Dom(U, S)$. It can be easily verified that $Dom(U, S)$ is a subsemigroup of S containing U. A subsemigroup U of a semigroup S is called closed if $Dom(U, S) = U$. A semigroup is called absolutely closed if it is closed in every containing semigroup. Let $\mathcal{C}$ be a class of semigroups. A semigroup U is said to be $\mathcal{C}$-closed if $Dom(U, S) = U$ for all $S \in \mathcal{C}$ such that $U \subseteq S$. Let $\mathcal{B}$ and $\mathcal{C}$ be the classes of semigroups such that $\mathcal{B}$ is a subclass of $\mathcal{C}$. We say that $\mathcal{B}$ is $\mathcal{C}$-closed if every member of $\mathcal{B}$ is $\mathcal{C}$-closed. A class $\mathcal{C}$ of semigroups is said to be closed if $Dom(U, S) = U$ for all $U, S \in \mathcal{C}$ with U as a subsemigroup of S. Let $\mathcal{A}$ and $\mathcal{D}$ be two categories of semigroups with $\mathcal{A}$ as a subcategory of $\mathcal{D}$. Then it can be easily verified that a semigroup U is $\mathcal{A}$-closed if it is $\mathcal{D}$-closed.

A (semigroup)amalgam $\mathcal{A}$ = $[S_i : i \in I; U ; \phi_i : i \in I]$ consists of a semigroup U (called the core of the amalgam), a family $S_i : i \in I$ of semigroups disjoint from each other and from U, and a family $\phi_i : U \to S_i (i \in I)$ of monomorphisms. We shall simplify the notation to $\mathcal{U}$ = $[S_i; U ; \phi_i]$ or to $\mathcal{U}$ = $[S_i; U]$ when the context allows. We shall say that the amalgam $\mathcal{A}$ is embedded in a semigroup T if there exist a monomorphism $\lambda : U \to T$ and, for each $i \in I$, a monomorphism $\lambda_i : S_i \to T$ such that

(a) $\phi_i\lambda_i$=λ for each $i \in I$;
(b) $S_i\lambda_i \cap S_j\lambda_j$ = $U\lambda$ for all $i, j \in I$ such that $i \neq j$.

A semigroup amalgam $\mathcal{U}$ = $[\ S,\ S'\ ;\ U\ ;\ i,\ \alpha \mid U\ ]$ consisting of a semigroup S, a subsemigroup U of S, an isomorphic copy S' of S, where α: $S \to S'$ is an isomorphism and i is the inclusion mapping of U into S, is called a special semigroup amalgam. A class $\mathcal{C}$ of semigroups is said to have the special amalgamation property if every special semigroup amalgam in $\mathcal{C}$ is embeddable in $\mathcal{C}$.

Theorem 1.1. *[8, Theorem VII.2.3]). Let U be a subsemigroup of a semigroup S, S' be a semigroup disjoint from S and let α: $S \to S'$ be an isomorphism. Let P = $S *_U S'$, be the free product of the amalgam*

$$\mathcal{U} = [S, S';\ U\ ; i, \alpha \mid U]$$

where i is the inclusion mapping of U into S, and let μ, μ' be the natural monomorphisms from

S, S' respectively into P. Then

$$(S\mu \,\cap\, S'\mu')\mu^{-1} = Dom(U,S).$$

From the above result, it follows that a special semigroup amalgam $[S,\, S'\,;\, U\,;\, i,\, \alpha \mid U]$ is embeddable in a semigroup if and only if $Dom(U,S) = U$. Therefore, the above amalgam with core U is embeddable in a semigroup if and only if U is closed in S.

The following result provided by Isbell [9], known as Isbell's zigzag theorem, is a most useful characterization of semigroup dominions and is of basic importance to our investigations.

Theorem 1.2. *([9], Theorem 2.3) Let U be a subsemigroup of a semigroup S and let $d \in S$. Then $d \in Dom(U,S)$ if and only if $d \in U$ or there exists a series of factorizations of d as follows:*

$$d = a_0t_1 = y_1a_1t_1 = y_1a_2t_2 = y_2a_3t_2 = \cdots = y_ma_{2m-1}t_m = y_ma_{2m} \tag{1.1}$$

where $m \geq 1,\ a_i \in U\ (i = 0, 1, \ldots, 2m),\ y_i, t_i \in S\ (i = 1, 2, \ldots, m)$, and

$$a_0 = y_1a_1, \qquad a_{2m-1}t_m = a_{2m},$$
$$a_{2i-1}t_i = a_{2i}t_{i+1}, \qquad y_ia_{2i} = y_{i+1}a_{2i+1} \qquad (1 \leq i \leq m-1).$$

Such a series of factorization is called a $zigzag$ in S over U with value d, length m and spine $a_0, a_1, \ldots, a_{2m}$.

The following result is from Khan [10] and is also necessary for our investigations.

Theorem 1.3. *([10], Result 3) Let U and S be semigroups with U as a subsemigroup of S. Take any $d \in S\backslash U$ such that $d \in Dom(U,S)$. If (1.1) is a zigzag of shortest possible length m over U with value d, then $t_j, y_j \in S\backslash U$ for all $j = 1, 2, \ldots, m$.*

Semigroup theoretic notations and conventions of Clifford and Preston [6] and Howie [8] will be used throughout without explicit mention.

2 Closedness and Varieties of semigroups

In general varieties of bands containing the varieties of rectangular and normal bands are not absolutely closed as Higgins [7, Chapter 4] had given examples of a rectangular band and a normal band that were not absolutely closed. Therefore, for the varieties of semigroups, it is worthwhile to find largest subvarieties of the variety of all semigroups in which these varieties are closed. As a first step in this direction, one attempts to find those varieties of semigroups that are closed in itself. Encouraged by the fact that Scheiblich [11] had shown that the variety of all normal bands was closed, Alam and Khan in [3, 4, 5] had shown that the variety of left [right] regular bands, left [right] quasi-normal bands and left [right] semi-normal bands were closed. In [2], Ahanger and Shah had proved a stronger fact that the variety of left [right] regular bands was closed in the variety of all bands and, recently, Abbas and Ashraf [1] had shown that a variety of left [right] normal bands was closed in some containing homotypical varieties (varieties admitting an identity containing same variables on both sides) of semigroups.

In this section, we have shown that some varieties of semigroups are closed in itself and, as an application of these results, we conclude that the varieties of semigroups defined by the identities $axy = xya^2$, $axy = y^2ax$, $axy = yaxa$, $axy = yxya$, $axy = axaya$, $axy = yayxy$ and $axy = xaxyx$ respectively have special amalgamation property.

Finally, the results in this section raise open problem of whether the varieties considered in the paper are absolutely closed; if not, of finding out largest varieties of semigroups in which these varieties are closed.

Theorem 2.1. *The variety of semigroups defined by the identity $axy = xya^2$ is closed.*

Proof. Take any $U, S \in \mathcal{V}$ with U a subsemigroup of S and let $d \in Dom(U,S)\backslash U$. Suppose that d has zigzag of type (1.1) in S over U with value d of shortest possible length m.

Lemma 2.2.

$$(\prod_{i=0}^{m-1} a_{2i})t_m = (\prod_{i=0}^{m} a_{2i}).$$

Proof.

$(\prod_{i=0}^{m-1} a_{2i})t_m$

$= a_0a_2a_4\cdots a_{2m-4}a_{2m-2}t_m$

$= y_1(a_1a_2(a_4\cdots a_{2m-4}a_{2m-2}))t_m$ (by zigzag equations)

$= y_1(a_2(a_4\cdots a_{2m-4}a_{2m-2})a_1a_1)t_m$ (since S satisfies the identity $axy = xyaa$)

$= y_2a_3a_4\cdots a_{2m-4}a_{2m-2}a_1a_1t_m$ (by zigzag equations)

$= y_2(a_3(a_4\cdots a_{2m-4}a_{2m-2})a_1a_1)t_m$

$= y_2(a_1a_3(a_4a_6\cdots a_{2m-4}a_{2m-2}))t_m$ (since S satisfies the identity $axy = xyaa$)

$\vdots$

$= y_{m-1}((a_1a_3\cdots a_{2m-3})a_{2m-2}t_m)$

$= y_{m-1}(a_{2m-2}t_m(a_1a_3\cdots a_{2m-3})(a_1a_3\cdots a_{2m-3}))$
(since S satisfies the identity $axy = xyaa$)

$= y_ma_{2m-1}t_ma_1a_3\cdots a_{2m-3}a_1a_3\cdots a_{2m-3}$ (by zigzag equations)

$= y_m(a_{2m-1}t_m(a_1a_3\cdots a_{2m-3}))a_1a_3\cdots a_{2m-3}$

$= y_m(t_m(a_1a_3\cdots a_{2m-3})a_{2m-1}a_{2m-1})a_1a_3\cdots a_{2m-3}$
(since S satisfies the identity $axy = xyaa$)

$= y_mt_ma_1a_3\cdots a_{2m-3}(a_{2m-1}a_{2m-1}(a_1a_3\cdots a_{2m-3}))$

$= y_mt_ma_1a_3\cdots a_{2m-3}(a_{2m-1}(a_1a_3\cdots a_{2m-3})a_{2m-1}a_{2m-1})$
(since S satisfies the identity $axy = xyaa$)

$= y_mt_ma_1a_3\cdots a_{2m-3}(a_{2m-1}(a_1a_3\cdots a_{2m-3})a_{2m-1})a_{2m-1}$

$= y_mt_ma_1a_3\cdots a_{2m-3}((a_1a_3\cdots a_{2m-3})a_{2m-1}a_{2m-1}a_{2m-1})a_{2m-1}$
(since S satisfies the identity $axy = xyaa$)

$= y_m(t_m(a_1a_3\cdots a_{2m-3}a_1a_3\cdots a_{2m-3})a_{2m-1}a_{2m-1})a_{2m-1}a_{2m-1}$

$= y_m(a_{2m-1}t_m(a_1a_3\cdots a_{2m-3}a_1a_3\cdots a_{2m-3}))a_{2m-1}a_{2m-1}$
(since S satisfies the identity $axy = xyaa$)

$= y_ma_{2m-1}(t_m(a_1a_3\cdots a_{2m-3}a_1a_3\cdots a_{2m-3})a_{2m-1}a_{2m-1})$

$= y_ma_{2m-1}(a_{2m-1}t_m(a_1a_3\cdots a_{2m-3}a_1a_3\cdots a_{2m-3}))$
(since S satisfies the identity $axy = xyaa$)

$= y_{m-1}a_{2m-2}a_{2m-1}t_ma_1a_3\cdots a_{2m-3}a_1a_3\cdots a_{2m-3}$ (by zigzag equations)

$= y_{m-1}(a_{2m-2}(a_{2m-1}t_m)(a_1a_3\cdots a_{2m-3})(a_1a_3\cdots a_{2m-3}))$

$= y_{m-1}a_1a_3\cdots a_{2m-5}a_{2m-3}a_{2m-2}a_{2m}$ (since S satisfies the identity $axy = xyaa$ and by zigzag equations)

$= y_{m-1}((a_1a_3\cdots a_{2m-5})a_{2m-3}a_{2m-2})a_{2m}$
$= y_{m-1}(a_{2m-3}a_{2m-2}(a_1a_3\cdots a_{2m-5})(a_1a_3\cdots a_{2m-5}))a_{2m}$
(since S satisfies the identity $axy = xyaa$)
$= y_{m-2}a_{2m-4}a_{2m-2}a_1a_3\cdots a_{2m-5}a_1a_3\cdots a_{2m-5}a_{2m}$ (by zigzag equations)
$= y_{m-2}(a_{2m-4}a_{2m-2}(a_1a_3\cdots a_{2m-5})(a_1a_3\cdots a_{2m-5}))a_{2m}$
$= y_{m-2}((a_1a_3\cdots a_{2m-5})a_{2m-4}a_{2m-2})a_{2m}$ (since S satisfies the identity $axy = xyaa$)
$\vdots$
$= y_1a_1a_2a_4a_6\cdots a_{2m-4}a_{2m-2}a_{2m}$
$= a_0a_2a_4a_6\cdots a_{2m-4}a_{2m-2}a_{2m}$ (by zigzag equations)
$= (\prod_{i=0}^{m} a_{2i}),$

as required. □

Now

$d = a_0t_1$ (by zigzag equations)
$= y_1a_1t_1$(by zigzag equations)
$= a_1t_1y_1y_1$ (since S satisfies the identity $axy = xyaa$)
$= (a_1t_1y_1)y_1$
$= (t_1y_1a_1a_1)y_1$ (since S satisfies the identity $axy = xyaa$)
$= t_1y_1(a_1a_1y_1)$
$= t_1y_1(a_1y_1a_1a_1)$ (since S satisfies the identity $axy = xyaa$)
$= ((t_1y_1)(a_1y_1)a_1a_1)$
$= (a_1(t_1y_1)(a_1y_1))$ (since S satisfies the identity $axy = xyaa$)
$= a_2t_2y_1a_1y_1$ (by zigzag equations)
$= (a_2t_2y_1)a_1y_1$
$= (t_2y_1a_2a_2)a_1y_1$ (since S satisfies the identity $axy = xyaa$)
$= t_2y_2a_3a_2a_1y_1$ (by zigzag equations)
$= t_2y_2(a_3a_2a_1)y_1$
$= t_2y_2(a_2a_1a_3a_3)y_1$ (since S satisfies the identity $axy = xyaa$)
$= t_2y_2a_2a_1(a_3a_3y_1)$
$= t_2y_2a_2a_1(a_3y_1a_3a_3)$ (since S satisfies the identity $axy = xyaa$)
$= ((t_2y_2a_2a_1)(a_3y_1)a_3a_3)$
$= (a_3(t_2y_2a_2a_1)(a_3y_1))$ (since S satisfies the identity $axy = xyaa$)
$= a_3t_2y_2a_2(a_1a_3y_1)$
$= a_3t_2y_2a_2(a_3y_1a_1a_1)$ (since S satisfies the identity $axy = xyaa$)
$= a_3t_2y_2a_2a_3a_0a_1$ (by zigzag equations)
$= a_3t_2y_2(a_2a_3(a_0a_1))$
$= a_3t_2y_2(a_3(a_0a_1)a_2a_2)$ (since S satisfies the identity $axy = xyaa$)
$= a_3t_2y_1a_2a_0a_1a_2a_2$ (by zigzag equations)

$$= a_3t_2y_1((a_2a_0)a_1(a_2a_2))$$
$$= a_3t_2y_1(a_1(a_2a_2)(a_2a_0)(a_2a_0)) \text{ (since } S \text{ satisfies the identity } axy = xyaa)$$
$$= a_3t_2a_0a_2a_2a_2a_0a_2a_0 \text{ (by zigzag equations)}$$
$$= a_3t_2a_0(a_2a_2(a_2a_0)(a_2a_0))$$
$$= a_3t_2a_0((a_2a_0)a_2a_2) \text{ (since } S \text{ satisfies the identity } axy = xyaa)$$
$$= a_3t_2a_0(a_2a_0a_2a_2)$$
$$= a_3t_2a_0(a_2a_2a_0) \text{ (since } S \text{ satisfies the identity } axy = xyaa)$$
$$= ((a_3t_2)a_0a_2a_2)a_0$$
$$= (a_2(a_3t_2)a_0)a_0 \text{ (since } S \text{ satisfies the identity } axy = xyaa)$$
$$= (a_2(a_3t_2)a_0a_0)$$
$$= a_0a_2a_3t_2 \text{ (since } S \text{ satisfies the identity } axy = xyaa)$$
$$= (\prod_{i=0}^{1} a_{2i})(a_3t_2)$$
$$\vdots$$
$$= (\prod_{i=0}^{m-2} a_{2i})(a_{2m-3}t_{m-1})$$
$$= a_0a_2a_4 \cdots a_{2m-4}a_{2m-2}t_m \text{ (by zigzag equations)}$$
$$= a_0a_2a_4 \cdots a_{2m-4}a_{2m-2}a_{2m} \text{ (by Lemma 2.2)}$$
$$= (\prod_{i=0}^{m} a_{2i})$$
$$\in U$$

$\Rightarrow dom(U, S) = U$.
Thus the proof of the theorem is completed. □

Dually, we have the following result:

Theorem 2.3. *The variety of semigroups defined by the identity $axy = y^2ax$ is closed.*

Theorem 2.4. *The variety of semigroups defined by the identity $axy = yaxa$ is closed.*

Proof. Take any $U, S \in \mathcal{V}$ with U a subsemigroup of S and let $d \in Dom(U, S)\setminus U$. Suppose that d has zigzag of type (1.1) in S over U with value d of shortest possible length m.

Lemma 2.5.

$$(\prod_{i=0}^{m-1} a_{2i})t_m = (\prod_{i=0}^{m} a_{2i}).$$

Proof.

$$(\prod_{i=0}^{m-1} a_{2i})t_m$$
$$= a_0a_2a_4 \cdots a_{2m-4}a_{2m-2}t_m$$
$$= (y_1a_1(a_2a_4 \cdots a_{2m-4}a_{2m-2}))t_m \text{ (by zigzag equations)}$$
$$= ((a_2a_4 \cdots a_{2m-4}a_{2m-2})y_1a_1y_1)t_m \text{ (since } S \text{ satisfies the identity } axy = yaxa)$$

$= (a_2(a_4 \cdots a_{2m-4}a_{2m-2}y_1)(a_1y_1))t_m$
$= ((a_1y_1)a_2(a_4 \cdots a_{2m-4}a_{2m-2}y_1)a_2)t_m$ (since S satisfies the identity $axy = yaxa$)
$= a_1y_2a_3a_4 \cdots a_{2m-4}a_{2m-2}y_1a_2t_m$ (by zigzag equations)
$= a_1(y_2a_3(a_4 \cdots a_{2m-4}a_{2m-2}y_1))a_2t_m$
$= a_1((a_4 \cdots a_{2m-4}a_{2m-2}y_1)y_2a_3y_2)a_2t_m$ (since S satisfies the identity $axy = yaxa$)
$= a_1(a_4(a_6 \cdots a_{2m-4}a_{2m-2}y_1y_2)(a_3y_2))a_2t_m$
$= a_1((a_3y_2)a_4(a_6 \cdots a_{2m-4}a_{2m-2}y_1y_2)a_4)a_2t_m$ (since S satisfies the identity $axy = yaxa$)

$\vdots$

$= a_1a_3 \cdots a_{2m-3}y_{m-1}a_{2m-2}y_1y_2 \cdots y_{m-1}a_{2m-2} \cdots a_2t_m$
$= a_1a_3 \cdots a_{2m-3}y_ma_{2m-1}y_1y_2 \cdots y_{m-1}a_{2m-2} \cdots a_2t_m$ (by zigzag equations)
$= a_1a_3 \cdots a_{2m-3}y_m(a_{2m-1}(y_1y_2 \cdots y_{m-1})(a_{2m-2} \cdots a_2))t_m$
$= a_1a_3 \cdots a_{2m-3}y_m((a_{2m-2} \cdots a_2)a_{2m-1}(y_1y_2 \cdots y_{m-1})a_{2m-1})t_m$
(since S satisfies the identity $axy = yaxa$)
$= a_1a_3 \cdots a_{2m-3}y_ma_{2m-2} \cdots a_2(a_{2m-1}(y_1y_2 \cdots y_{m-1})a_{2m-1})t_m$
$= a_1a_3 \cdots a_{2m-3}y_ma_{2m-2} \cdots a_2(a_{2m-1}a_{2m-1}(y_1y_2 \cdots y_{m-1})a_{2m-1})t_m$
(since S satisfies the identity $axy = yaxa$)
$= a_1a_3 \cdots a_{2m-3}y_m((a_{2m-2} \cdots a_2a_{2m-1})a_{2m-1}(y_1y_2 \cdots y_{m-1})a_{2m-1})t_m$
$= a_1a_3 \cdots a_{2m-3}y_m(a_{2m-1}(y_1y_2 \cdots y_{m-1})(a_{2m-2} \cdots a_2a_{2m-1}))t_m$
(since S satisfies the identity $axy = yaxa$)
$= a_1a_3 \cdots a_{2m-3}y_{m-1}a_{2m-2}y_1y_2 \cdots y_{m-1}a_{2m-2} \cdots a_2a_{2m}$ (by zigzag equations)
$= a_1a_3 \cdots a_{2m-5}((a_{2m-3}y_{m-1})a_{2m-2}(y_1y_2 \cdots y_{m-1})a_{2m-2})a_{2m-4} \cdots a_2a_{2m}$
$= a_1a_3 \cdots a_{2m-5}(a_{2m-2}(y_1y_2 \cdots y_{m-1})(a_{2m-3}y_{m-1}))a_{2m-4} \cdots a_2a_{2m}$
(since S satisfies the identity $axy = yaxa$)
$= a_1a_3 \cdots a_{2m-5}((a_{2m-2}y_1y_2 \cdots y_{m-2})y_{m-1}a_{2m-3}y_{m-1})a_{2m-4} \cdots a_2a_{2m}$
$= a_1a_3 \cdots a_{2m-5}y_{m-2}a_{2m-4}a_{2m-2}y_1y_2 \cdots y_{m-2}a_{2m-4} \cdots a_2a_{2m}$
(since S satisfies the identity $axy = yaxa$ and by zigzag equations)

$\vdots$

$= ((a_1y_1)a_2(a_4a_6 \cdots a_{2m-4}a_{2m-2}y_1)a_2)a_{2m}$
$= (a_2(a_4a_6 \cdots a_{2m-4}a_{2m-2}y_1)(a_1y_1))a_{2m}$ (since S satisfies the identity $axy = yaxa$)
$= ((a_2a_4a_6 \cdots a_{2m-4}a_{2m-2})y_1a_1y_1)a_{2m}$
$= (y_1a_1(a_2a_4a_6 \cdots a_{2m-4}a_{2m-2}))a_{2m}$ (since S satisfies the identity $axy = yaxa$)
$= a_0a_2a_4a_6 \cdots a_{2m-4}a_{2m-2}a_{2m}$ (by zigzag equations)
$= (\prod_{i=0}^{m} a_{2i}),$

as required. □

Now

$$
\begin{aligned}
d &= a_0t_1 \text{ (by zigzag equations)}\\
&= y_1a_1t_1 \text{ (by zigzag equations)}\\
&= t_1y_1a_1y_1 \text{ (since } S \text{ satisfies the identity } axy = yaxa)\\
&= (t_1y_1a_1)y_1\\
&= (a_1t_1y_1t_1)y_1 \text{ (since } S \text{ satisfies the identity } axy = yaxa)\\
&= (a_1t_1y_1)t_1y_1\\
&= (y_1a_1t_1a_1)t_1y_1 \text{ (since } S \text{ satisfies the identity } axy = yaxa)\\
&= y_1a_1t_1a_2t_2y_1 \text{ (by zigzag equations)}\\
&= y_1(a_1t_1a_2)t_2y_1\\
&= y_1(a_2a_1t_1a_1)t_2y_1 \text{ (since } S \text{ satisfies the identity } axy = yaxa)\\
&= y_2a_3a_2t_2a_1t_2y_1 \text{ (by zigzag equations)}\\
&= y_2a_3(a_2t_2a_1t_2)y_1\\
&= y_2a_3(t_2a_1a_2)y_1 \text{ (since } S \text{ satisfies the identity } axy = yaxa)\\
&= y_2(a_3t_2a_1)a_2y_1\\
&= y_2(a_1a_3t_2a_3)a_2y_1 \text{ (since } S \text{ satisfies the identity } axy = yaxa)\\
&= y_2(a_1(a_3t_2)a_3)a_2y_1\\
&= y_2(a_3a_1(a_3t_2)a_1)a_2y_1 \text{ (since } S \text{ satisfies the identity } axy = yaxa)\\
&= y_1a_2a_1a_3t_2a_1a_2y_1 \text{ (by zigzag equations)}\\
&= y_1a_2a_1a_3t_2(a_1a_2y_1)\\
&= y_1a_2a_1a_3t_2(y_1a_1a_2a_1) \text{ (since } S \text{ satisfies the identity } axy = yaxa)\\
&= y_1(a_2a_1(a_3t_2))y_1a_1a_2a_1\\
&= y_1((a_3t_2)a_2a_1a_2)y_1a_1a_2a_1 \text{ (since } S \text{ satisfies the identity } axy = yaxa)\\
&= y_1a_3t_2(a_2a_1(a_2y_1)a_1)a_2a_1\\
&= y_1a_3t_2(a_1a_2y_1a_2)a_2a_1 \text{ (since } S \text{ satisfies the identity } axy = yaxa)\\
&= y_1a_3t_2(a_2y_1a_1)a_2a_1 \text{ (since } S \text{ satisfies the identity } axy = yaxa)\\
&= y_1a_3t_2a_2a_0a_2a_1 \text{ (by zigzag equations)}\\
&= y_1((a_3t_2)(a_2a_0a_2)a_1)\\
&= y_1(a_1(a_3t_2)(a_2a_0a_2)(a_3t_2)) \text{ (since } S \text{ satisfies the identity } axy = yaxa)\\
&= a_0a_3t_2a_2a_0a_2a_3t_2 \text{ (by zigzag equations)}\\
&= a_0a_3t_2a_2(a_0a_2(a_3t_2))\\
&= a_0a_3t_2a_2((a_3t_2)a_0a_2a_0) \text{ (since } S \text{ satisfies the identity } axy = yaxa)\\
&= a_0((a_3t_2)a_2(a_3t_2a_0)a_2)a_0\\
&= a_0(a_2(a_3t_2a_0)(a_3t_2))a_0 \text{ (since } S \text{ satisfies the identity } axy = yaxa)\\
&= a_0(a_2(a_3t_2a_0a_3t_2)a_0)\\
&= a_0(a_0a_2(a_3t_2a_0a_3t_2)a_2) \text{ (since } S \text{ satisfies the identity } axy = yaxa)\\
&= (a_0a_0(a_2a_3t_2)a_0)a_3t_2a_2\\
&= (a_0(a_2a_3t_2)a_0)a_3t_2a_2 \text{ (since } S \text{ satisfies the identity } axy = yaxa)\\
&= (a_0a_2(a_3t_2a_0a_3t_2)a_2)\\
&= (a_2(a_3t_2a_0a_3t_2)a_0) \text{ (since } S \text{ satisfies the identity } axy = yaxa)\\
&= (a_2(a_3t_2)a_0(a_3t_2))a_0\\
&= ((a_3t_2)a_0a_2)a_0 \text{ (since } S \text{ satisfies the identity } axy = yaxa)\\
&= ((a_3t_2)a_0a_2a_0)\\
&= a_0a_2a_3t_2 \text{ (since } S \text{ satisfies the identity } axy = yaxa)
\end{aligned}
$$

$$= (\prod_{i=0}^{1} a_{2i})(a_3 t_2)$$

$$\vdots$$

$$= (\prod_{i=0}^{m-2} a_{2i})(a_{2m-3} t_{m-1})$$

$$= a_0 a_2 a_4 \cdots a_{2m-4} a_{2m-2} t_m \text{ (by zigzag equations)}$$

$$= a_0 a_2 a_4 \cdots a_{2m-4} a_{2m-2} a_{2m} \text{ (by Lemma 2.5)}$$

$$= (\prod_{i=0}^{m} a_{2i})$$

$$\in U$$

$\Rightarrow dom(U, S) = U.$
Thus the proof of the theorem is completed. □

Dually, one may prove the following result:

Theorem 2.6. *The variety of semigroups defined by the identity* $axy = yxya$ *is closed.*

Theorem 2.7. *The variety of semigroups defined by the identity* $axy = axaya$ *is closed.*

Proof. Take any $U, S \in \mathcal{V}$ with U a subsemigroup of S and let $d \in Dom(U, S) \backslash U$. Suppose that d has zigzag of type (1.1) in S over U with value d of shortest possible length m.

Lemma 2.8.

$$(\prod_{i=0}^{m-1} a_{2i}) t_m = y_1 a_1 y_2 a_3 \cdots y_{m-1} a_{2m-3} y_m a_{2m-1} (\prod_{i=1}^{m-1} y_i) a_{2m-1} t_m a_{2m-1}.$$

Proof.

$$(\prod_{i=0}^{m-1} a_{2i}) t_m$$

$$= a_0 a_2 a_4 \cdots a_{2m-4} a_{2m-2} t_m$$

$$= (y_1 a_1 (a_2 a_4 \cdots a_{2m-4} a_{2m-2})) t_m \text{ (by zigzag equations)}$$

$$= (y_1 a_1 y_1 (a_2 a_4 \cdots a_{2m-4} a_{2m-2}) y_1) t_m \text{ (since } S \text{ satisfies the identity } axy = axaya)$$

$$= y_1 a_1 y_2 a_3 a_4 \cdots a_{2m-4} a_{2m-2} y_1 t_m \text{ (by zigzag equations)}$$

$$= y_1 a_1 (y_2 a_3 (a_4 \cdots a_{2m-4} a_{2m-2} y_1)) t_m$$

$$= y_1 a_1 (y_2 a_3 y_2 (a_4 \cdots a_{2m-4} a_{2m-2} y_1) y_2) t_m \text{ (since } S \text{ satisfies the identity } axy = axaya)$$

$$\vdots$$

$$= y_1 a_1 y_2 a_3 \cdots y_{m-1} a_{2m-3} y_{m-1} a_{2m-2} y_1 y_2 \cdots y_{m-1} t_m$$

$$= y_1 a_1 y_2 a_3 \cdots y_{m-1} a_{2m-3} y_m a_{2m-1} y_1 y_2 \cdots y_{m-1} t_m \text{ (by zigzag equations)}$$

$$= y_1 a_1 y_2 a_3 \cdots y_{m-1} a_{2m-3} y_m (a_{2m-1} (y_1 y_2 \cdots y_{m-1}) t_m)$$

$$= y_1 a_1 y_2 a_3 \cdots y_{m-1} a_{2m-3} y_m (a_{2m-1} (y_1 y_2 \cdots y_{m-1}) a_{2m-1} t_m a_{2m-1})$$

(since S satisfies the identity $axy = axaya$)

$$= y_1 a_1 y_2 a_3 \cdots y_{m-1} a_{2m-3} y_m a_{2m-1} (\prod_{i=1}^{m-1} y_i) a_{2m-1} t_m a_{2m-1},$$

as required. □

Now

$$
\begin{aligned}
d &= a_0t_1 \text{ (by zigzag equations)}\\
&= y_1a_1t_1 \text{ (by zigzag equations)}\\
&= y_1a_1y_1t_1y_1 \text{ (since } S \text{ satisfies the identity } axy = axaya)\\
&= y_1(a_1y_1(t_1y_1))\\
&= y_1(a_1y_1a_1(t_1y_1)a_1) \text{ (since } S \text{ satisfies the identity } axy = axaya)\\
&= y_1a_1y_1a_2t_2y_1a_1 \text{ (by zigzag equations)}\\
&= y_1a_1y_2(a_3t_2(y_1a_1)) \text{ (by zigzag equations)}\\
&= y_1a_1y_2(a_3t_2a_3(y_1a_1)a_3) \text{ (since } S \text{ satisfies the identity } axy = axaya)\\
&= (y_1(a_1y_2a_3)t_2)a_3y_1a_1a_3\\
&= (y_1(a_1y_2a_3)y_1t_2y_1)a_3y_1a_1a_3 \text{ (since } S \text{ satisfies the identity } axy = axaya)\\
&= y_1a_1y_2(a_3y_1(t_2y_1))a_3y_1a_1a_3\\
&= y_1a_1y_2(a_3y_1a_3(t_2y_1)a_3)a_3y_1a_1a_3 \text{ (since } S \text{ satisfies the identity } axy = axaya)\\
&= y_1a_1y_1a_2y_1a_3t_2y_1a_3a_3y_1a_1a_3 \text{ (by zigzag equations)}\\
&= y_1a_1y_1a_2(y_1(a_3t_2y_1)a_3)a_3y_1a_1a_3\\
&= y_1a_1y_1a_2(y_1(a_3t_2y_1)y_1a_3y_1)a_3y_1a_1a_3 \text{ (since } S \text{ satisfies the identity } axy = axaya)\\
&= (y_1a_1y_1a_2y_1)a_3t_2y_1y_1a_3y_1a_3y_1a_1a_3\\
&= (y_1a_1a_2)a_3t_2y_1y_1a_3y_1a_3y_1a_1a_3 \text{ (since } S \text{ satisfies the identity } axy = axaya)\\
&= y_1a_1a_2a_3t_2y_1y_1(a_3y_1a_3(y_1a_1)a_3)\\
&= y_1a_1a_2a_3t_2y_1y_1(a_3y_1(y_1a_1)) \text{ (since } S \text{ satisfies the identity } axy = axaya)\\
&= (y_1(a_1a_2a_3t_2y_1)y_1a_3y_1)y_1a_1\\
&= (y_1(a_1a_2a_3t_2y_1)a_3)y_1a_1 \text{ (since } S \text{ satisfies the identity } axy = axaya)\\
&= (y_1(a_1a_2a_3t_2)y_1a_3y_1)a_1\\
&= (y_1(a_1a_2a_3t_2)a_3)a_1 \text{ (since } S \text{ satisfies the identity } axy = axaya)\\
&= a_0a_2a_3t_2a_3a_1 \text{ (by zigzag equations)}\\
&= (\prod_{i=0}^{1} a_{2i})(a_3t_2)a_3a_1\\
&\vdots\\
&= (\prod_{i=0}^{m-2} a_{2i})(a_{2m-3}t_{m-1})a_{2m-3}\cdots a_3a_1\\
&= a_0a_2a_4\cdots a_{2m-4}a_{2m-2}t_ma_{2m-3}\cdots a_3a_1 \text{ (by zigzag equations)}\\
&= y_1a_1y_2a_3\cdots y_{m-1}a_{2m-3}y_ma_{2m-1}(\prod_{i=1}^{m-1} y_i)a_{2m-1}t_ma_{2m-1}a_{2m-3}\cdots a_3a_1 \text{ (by Lemma 2.8)}\\
&= y_1a_1y_2a_3\cdots y_{m-2}a_{2m-5}y_{m-1}a_{2m-3}y_{m-1}a_{2m-2}y_1y_2\cdots y_{m-2}y_{m-1}a_{2m}a_{2m-1}\cdots a_3a_1\\
&\quad \text{(by zigzag equations)}\\
&= y_1a_1y_2a_3\cdots y_{m-2}a_{2m-5}(y_{m-1}a_{2m-3}y_{m-1}(a_{2m-2}y_1y_2\cdots y_{m-2})y_{m-1})a_{2m}a_{2m-1}\cdots a_3a_1\\
&= y_1a_1y_2a_3\cdots y_{m-2}a_{2m-5}(y_{m-1}a_{2m-3}(a_{2m-2}y_1y_2\cdots y_{m-2}))a_{2m}a_{2m-1}\cdots a_3a_1\\
&\quad \text{(since } S \text{ satisfies the identity } axy = axaya)
\end{aligned}
$$

$= y_1a_1y_2a_3\cdots y_{m-2}a_{2m-5}y_{m-2}a_{2m-4}a_{2m-2}y_1y_2\cdots y_{m-2}a_{2m}a_{2m-1}\cdots a_3a_1$
(by zigzag equations)

$\vdots$

$= (y_1a_1y_1(a_2a_4\cdots a_{2m-2})y_1)a_{2m}a_{2m-1}\cdots a_3a_1$
$= (y_1a_1(a_2a_4\cdots a_{2m-2}))a_{2m}a_{2m-1}\cdots a_3a_1$ (since S satisfies the identity $axy = axaya$)
$= a_0a_2a_4\cdots a_{2m-2}a_{2m}a_{2m-1}\cdots a_3a_1$ (by zigzag equations)
$\in U$

$\Rightarrow dom(U,S) = U.$
Thus the proof of the theorem is completed. □

Dually, we can prove the following result:

Theorem 2.9. *The variety of semigroups defined by the identity $axy = yayxy$ is closed.*

Theorem 2.10. *The variety of semigroups defined by the identity $axy = xaxyx$ is closed.*

Proof. Take any $U, S \in \mathcal{V}$ with U a subsemigroup of S and let $d \in Dom(U,S)\backslash U$. Suppose that d has zigzag of type (1.1) in S over U with value d of shortest possible length m.

Lemma 2.11.

$$(\prod_{i=0}^{m-1} a_{2i})t_m = a_{2m-1}(\prod_{i=0}^{m} a_{2i}).$$

Proof.

$(\prod_{i=0}^{m-1} a_{2i})t_m$
$= a_0a_2a_4\cdots a_{2m-4}a_{2m-2}t_m$
$= y_1(a_1a_2(a_4\cdots a_{2m-4}a_{2m-2}))t_m$ (by zigzag equations)
$= y_1(a_2a_1a_2(a_4\cdots a_{2m-4}a_{2m-2})a_2)t_m$ (since S satisfies the identity $axy = xaxyx$)
$= y_2a_3a_1a_2a_4\cdots a_{2m-4}a_{2m-2}a_2t_m$ (by zigzag equations)
$= y_2((a_3a_1a_2)a_4(a_6\cdots a_{2m-4}a_{2m-2}a_2))t_m$
$= y_2(a_4(a_3a_1a_2)a_4(a_6\cdots a_{2m-4}a_{2m-2}a_2)a_4)t_m$ (since S satisfies the identity $axy = xaxyx$)

$\vdots$

$= y_{m-1}a_{2m-2}a_{2m-3}\cdots a_3a_1a_2a_4\cdots a_{2m-2}a_2a_4\cdots a_{2m-2}t_m$
$= y_ma_{2m-1}a_{2m-3}\cdots a_3a_1a_2a_4\cdots a_{2m-2}a_2a_4\cdots a_{2m-2}t_m$ (by zigzag equations)
$= (y_ma_{2m-1}(a_{2m-3}\cdots a_3a_1a_2a_4\cdots a_{2m-2}a_2a_4\cdots a_{2m-2}))t_m$
$= (a_{2m-1}y_ma_{2m-1}(a_{2m-3}\cdots a_3a_1a_2a_4\cdots a_{2m-2}a_2a_4\cdots a_{2m-2})a_{2m-1})t_m$
(since S satisfies the identity $axy = xaxyx$)
$= a_{2m-1}y_{m-1}a_{2m-2}a_{2m-3}\cdots a_3a_1a_2a_4\cdots a_{2m-2}a_2a_4\cdots a_{2m-2}a_{2m-1}t_m$
(by zigzag equations)
$= a_{2m-1}y_{m-1}(a_{2m-2}(a_{2m-3}\cdots a_3a_1a_2a_4\cdots a_{2m-4})a_{2m-2}(a_2a_4\cdots a_{2m-4})a_{2m-2})a_{2m-1}t_m$
$= a_{2m-1}y_{m-1}((a_{2m-3}\cdots a_3a_1a_2a_4\cdots a_{2m-4})a_{2m-2}(a_2a_4\cdots a_{2m-4}))a_{2m-1}t_m$
(since S satisfies the identity $axy = xaxyx$)

$$
\begin{aligned}
&= a_{2m-1}y_{m-2}a_{2m-4}a_{2m-5}\cdots a_3a_1a_2a_4\cdots a_{2m-4}a_{2m-2}a_2a_4\cdots a_{2m-4}a_{2m} \\
&\quad \text{(by zigzag equations)} \\
&\vdots \\
&= a_{2m-1}y_1(a_2a_1a_2(a_4a_6\cdots a_{2m-4}a_{2m-2})a_2)a_{2m} \\
&= a_{2m-1}y_1(a_1a_2(a_4a_6\cdots a_{2m-4}a_{2m-2}))a_{2m} \text{ (since } S \text{ satisfies the identity } axy = xaxyx) \\
&= a_{2m-1}a_0a_2a_4\cdots a_{2m-4}a_{2m-2}a_{2m} \text{ (by zigzag equations)} \\
&= a_{2m-1}(\prod_{i=0}^{m} a_{2i}),
\end{aligned}
$$

as required. □

Now

$$
\begin{aligned}
d &= a_0t_1 \text{ (by zigzag equations)} \\
&= y_1a_1t_1 \text{ (by zigzag equations)} \\
&= a_1y_1a_1t_1a_1 \text{ (since } S \text{ satisfies the identity } axy = xaxyx) \\
&= a_1y_1a_2t_2a_1 \text{ (by zigzag equations)} \\
&= a_1y_2a_3t_2a_1 \text{ (by zigzag equations)} \\
&= a_1(y_2a_3(t_2a_1)) \\
&= a_1(a_3y_2a_3(t_2a_1)a_3) \text{ (since } S \text{ satisfies the identity } axy = xaxyx) \\
&= a_1a_3y_2((a_3t_2)a_1a_3) \\
&= a_1a_3y_2(a_1(a_3t_2)a_1a_3a_1) \text{ (since } S \text{ satisfies the identity } axy = xaxyx) \\
&= a_1a_3y_2((a_1a_3t_2a_1)a_3a_1) \\
&= a_1a_3y_2(a_3(a_1a_3t_2a_1)a_3a_1a_3) \text{ (since } S \text{ satisfies the identity } axy = xaxyx) \\
&= a_1a_3y_1a_2a_1a_3t_2a_1a_3a_1a_3 \text{ (by zigzag equations)} \\
&= a_1(a_3(y_1a_2a_1a_3t_2a_1)a_3a_1a_3) \\
&= a_1((y_1a_2a_1a_3t_2a_1)a_3a_1) \text{ (since } S \text{ satisfies the identity } axy = xaxyx) \\
&= (a_1(y_1a_2a_1a_3t_2)a_1a_3a_1) \\
&= ((y_1a_2a_1a_3t_2)a_1a_3) \text{ (since } S \text{ satisfies the identity } axy = xaxyx) \\
&= y_1(a_2a_1(a_3t_2a_1a_3)) \\
&= y_1(a_1a_2a_1(a_3t_2a_1a_3)a_1) \text{ (since } S \text{ satisfies the identity } axy = xaxyx) \\
&= a_0a_2a_1a_3t_2a_1a_3a_1 \text{ (by zigzag equations)} \\
&= a_0a_2(a_1(a_3t_2)a_1a_3a_1) \\
&= a_0a_2((a_3t_2)a_1a_3) \text{ (since } S \text{ satisfies the identity } axy = xaxyx) \\
&= (\prod_{i=0}^{1} a_{2i})(a_3t_2)(\prod_{i=0}^{1} a_{2i+1}) \\
&\vdots \\
&= (\prod_{i=0}^{m-2} a_{2i})(a_{2m-3}t_{m-1})(\prod_{i=0}^{m-2} a_{2i+1}) \\
&= a_0a_2a_4\cdots a_{2m-4}a_{2m-2}t_ma_1a_3\cdots a_{2m-3} \text{ (by zigzag equations)}
\end{aligned}
$$

$= a_{2m-1}(\prod_{i=0}^{m} a_{2i})a_1a_3\cdots a_{2m-3}$ (by Lemma 2.11)
$= (a_{2m-1}a_0(a_2a_4\cdots a_{2m-4}a_{2m-2}a_{2m}a_1a_3\cdots a_{2m-3}))$
$= (a_0a_{2m-1}a_0(a_2a_4\cdots a_{2m-4}a_{2m-2}a_{2m}a_1a_3\cdots a_{2m-3})a_0)$
(since S satisfies the identity $axy = xaxyx$)
$= (a_0a_{2m-1}(a_0a_2a_4\cdots a_{2m-4}a_{2m-2}a_{2m}a_1a_3\cdots a_{2m-3}))a_0$
$= (a_{2m-1}a_0a_{2m-1}(a_0a_2a_4\cdots a_{2m-4}a_{2m-2}a_{2m}a_1a_3\cdots a_{2m-3})a_{2m-1})a_0$
(since S satisfies the identity $axy = xaxyx$)
$= a_{2m-1}a_0a_{2m-1}a_0a_2a_4\cdots a_{2m-4}a_{2m-2}a_{2m}((a_1a_3\cdots a_{2m-3})a_{2m-1}a_0)$
$= a_{2m-1}a_0a_{2m-1}a_0a_2a_4\cdots a_{2m-4}a_{2m-2}a_{2m}(a_{2m-1}(a_1a_3\cdots a_{2m-3})a_{2m-1}a_0a_{2m-1})$
(since S satisfies the identity $axy = xaxyx$)
$= (a_{2m-1}a_0a_{2m-1}(a_0a_2a_4\cdots a_{2m-4}a_{2m-2}a_{2m})a_{2m-1})a_1a_3\cdots a_{2m-3}a_{2m-1}a_0a_{2m-1}$
$= (a_0a_{2m-1}(a_0a_2a_4\cdots a_{2m-4}a_{2m-2}a_{2m}))a_1a_3\cdots a_{2m-3}a_{2m-1}a_0a_{2m-1}$
(since S satisfies the identity $axy = xaxyx$)
$= a_0a_{2m-1}a_0a_2a_4\cdots a_{2m-4}a_{2m-2}(a_{2m}(a_1a_3\cdots a_{2m-3})a_{2m-1})a_0a_{2m-1}$
$= a_0a_{2m-1}a_0a_2a_4\cdots a_{2m-4}a_{2m-2}((a_1a_3\cdots a_{2m-3})a_{2m}(a_1a_3\cdots a_{2m-3})a_{2m-1}$
$(a_1a_3\cdots a_{2m-3}))a_0a_{2m-1}$ (since S satisfies the identity $axy = xaxyx$)
$= (a_0a_{2m-1}a_0(a_2a_4\cdots a_{2m-4}a_{2m-2}a_1a_3\cdots a_{2m-3}a_{2m}a_1a_3\cdots a_{2m-3}a_{2m-1}$
$a_1a_3\cdots a_{2m-3})a_0)a_{2m-1}$
$= (a_{2m-1}a_0(a_2a_4\cdots a_{2m-4}a_{2m-2}a_1a_3\cdots a_{2m-3}a_{2m}a_1a_3\cdots a_{2m-3}a_{2m-1}$
$a_1a_3\cdots a_{2m-3}))a_{2m-1}$ (since S satisfies the identity $axy = xaxyx$)
$= (a_{2m-1}(a_0a_2a_4\cdots a_{2m-4}a_{2m-2}a_1a_3\cdots a_{2m-3}a_{2m}a_1a_3\cdots a_{2m-3})a_{2m-1}$
$(a_1a_3\cdots a_{2m-3})a_{2m-1})$
$= ((a_0a_2a_4\cdots a_{2m-4}a_{2m-2}a_1a_3\cdots a_{2m-3}a_{2m}a_1a_3\cdots a_{2m-3})a_{2m-1}$
$(a_1a_3\cdots a_{2m-3}))$ (since S satisfies the identity $axy = xaxyx$)
$= a_0a_2a_4\cdots a_{2m-4}a_{2m-2}((a_1a_3\cdots a_{2m-3})a_{2m}(a_1a_3\cdots a_{2m-3})a_{2m-1}(a_1a_3\cdots a_{2m-3}))$
$= a_0a_2a_4\cdots a_{2m-4}a_{2m-2}(a_{2m}(a_1a_3\cdots a_{2m-3})a_{2m-1})$
(since S satisfies the identity $axy = xaxyx$)
$= a_0a_2a_4\cdots a_{2m-4}a_{2m-2}a_{2m}a_1a_3\cdots a_{2m-1}$
$= (\prod_{i=0}^{m} a_{2i})(\prod_{i=0}^{m-1} a_{2i+1})$
$\in U$

$\Rightarrow dom(U,S) = U$.
Thus the proof of the theorem is completed. □

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