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CC-DOMINATION IN GRAPHS

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Abstract In this paper, we introduce the closely-connected vertices and the concept of cc-domination for simple finite undirected graphs. A cut path in a graph G is a path whose edges when deleted increases the number of components of G. A subset D of $V(G)$ is called a cc-dominating set if for every vertex $u \notin D$, there exists a vertex $v \in D$ such that $\Gamma_{cc}(u,v) \geq 1$, where $\Gamma_{cc}(u,v)$ is the number of geodesics connecting u and v except the cut paths. The minimum cardinality of a cc-dominating set is called the cc-domination number of G, denoted by $\gamma_{cc}(G)$. Analogously, we introduce the cc-covering number $\alpha_{cc}(G)$, cc-independence number $\beta_{cc}(G)$ and cc-domatic number $d_{cc}(G)$ of G. Moreover, bounds and some interesting results are established.

1 Introduction

A characteristic feature that distinguishes distributed systems from single machine systems is the notion of partial failure. Hence while designing a distributed system paramount importance should be given to fault tolerance mechanisms so that the system can automatically recover from partial failures without seriously affecting its overall performance. In the modern era of cloud computing, fault-tolerance mechanisms are indispensable to ensure high availability and authenticity to the users. The faults in the cloud environment may occur due to physical faults, network faults, processor faults, service expiry faults, etc , and so on [11]. Among these, network faults arise mainly due to link failures. To minimize link failures, the nodes of the network must be associated “more closely" in such a way that unpredictable disruptions may not fail the whole network.

Mathematically, the motivation to introduce cycle graphs C_n in graph theory was to overcome the difficulties that may arise due to the lack of multiple paths between graph vertices. Being free of cut edges, the adjacent vertices of C_n are “so close" in the sense that they share no cut paths. But this is not true for non-adjacent vertices, since every path in C_n except the edges are cut paths. Thus the property enjoyed by adjacent vertices in terms of “closeness" does not hold for non-adjacent vertex pairs in C_n. This structural deficiency of C_n along with the necessity to tolerate link failures in distributed systems motivated the authors to study *closely-connected* vertices of a graph.

By graph G, we mean a finite simple undirected graph. For $u, v \in V(G)$, $N(u)$ and $N[u]$ respectively denote the open and closed neighborhoods of u, whereas the distance $d(u,v)$ between u and v is the number of edges in a geodesic connecting them. The eccentricity of a vertex u is the greatest distance from u to any other vertex and is denoted by $e(u)$. A dominating set in G is a set $D \subseteq V(G)$ such that every vertex not in D is adjacent to at least one vertex in D. The domination number $\gamma(G)$ is the minimum cardinality of a dominating set in G. The *line graph* $L(G)$ is the graph whose vertices correspond to the edges of G and two vertices in $L(G)$ are adjacent iff the corresponding edges in G are adjacent. For terminologies and notations not specifically defined here, we refer the reader to [2]. For more details about domination number and their related parameters, we refer to [3], [4],[5], [8] and for variations in ordinary domination [1],[12] and [13] can be referred.

In this paper, we introduce the concept of *cc-domination* in graphs. In ordinary domination, a vertex needs to be either dominated by itself or by its neighbor. But in practice, this model is not always economic. For instance, we have a complicated network system in which control units have to be placed at various places to monitor the faults and disturbances in the system. Thus control units have to be introduced at those vertices that can monitor the vertices "closer" to it. That is, we would like to find out the minimum number of control units required so that the whole system can be controlled with minimum cost. This idea has motivated the authors to introduce the concept of *cc-domination* in graphs. Throughout this paper, G denotes a finite simple undirected graph with vertex set $V(G)$ and edge set $E(G)$ unless specified otherwise.

2 CC-Dominating sets of graphs

Definition 2.1. Let $G = (V, E)$ be a graph. A path P in G is a cut path if there exists a graph $G^{'} = (V, E \setminus E^{'})$ for some $E^{'} \subseteq E(P)$ such that $\omega(G^{'}) > \omega(G)$.

Definition 2.2. Let $G = (V, E)$ be a graph. For $i \neq j$, the vertices $v_i, v_j \in V$ are closely-connected if at least one of the geodesics connecting them is not a cut path in G.
The cc-number of v_i and v_j is defined as

$$\Gamma_{cc}(v_i, v_j) = \{p \in P(v_i, v_j) |\, p \text{ is not a cut path in } G\},$$

where $P(v_i, v_j)$ is the set of all geodesics connecting v_i and v_j in G.

Definition 2.3. Let $G = (V, E)$ be a graph. The open cc-neighborhood of a vertex $u \in V$ is defined as

$$N_{cc}(u) = \{v \in V : \Gamma_{cc}(u, v) \geq 1\}.$$

and the closed cc-neighborhood of u is defined as $N_{cc}[u] = N_{cc}(u) \cup \{u\}$. The cardinality of $N_{cc}(u)$ is called the cc-degree of the vertex u, denoted by $deg_{cc}(u)$, in G. The maximum and minimum cc-degree of a vertex in G are given by

$$\Delta_{cc}(G) = \max_{u \in V} |N_{cc}(u)|,$$
$$\delta_{cc}(G) = \min_{u \in V} |N_{cc}(u)|.$$

The vertex u is said to be cc-isolated if $N_{cc}(u) = \phi$. An edge $e = uv$ is a cc-edge if $v \in N_{cc}(u)$. The cc-complement of G is the graph $\bar{G_{cc}} = (V, E^{'})$, where $u^{'}v^{'} \in E^{'}$ iff $v^{'} \notin N_{cc}(u^{'})$ in G for $u^{'}, v^{'} \in V$.

Remark 2.4. For a graph G,

(i) a vertex $u \in V(G)$ is cc-isolated iff either u is isolated or u has $deg(u)$ cut edges incident to it.

(ii) an edge in G is a cc-edge iff it is not a cut edge.

Definition 2.5. Let $G = (V, E)$ be a graph. A subset S of V is called a cc-vertex covering of G if for every cc-edge $e = uv$ either $u \in S$ or $v \in S$. The minimum cardinality of a cc-vertex covering of G is called the cc-covering number, denoted by $\alpha_{cc}(G)$. A subset S' of V is called cc-independent if $\Gamma_{cc}(u, v) = 0\; \forall u, v \in S'$. The maximum cardinality of a cc-independent set in G is called the cc-independence number and is denoted by $\beta_{cc}(G)$.

Definition 2.6. Let G be a graph. The cc-neighborhood graph of G is the graph $ccn(G)$ satisfying the following properties:

(i) $V(G) = V(ccn(G))$.

(ii) For $u, v \in V(G)$ with $u \neq v$, u and v are adjacent in $ccn(G)$ iff $\{u, v\}$ is not cc-independent in G.

The following are immediate consequences of definition 2.6 :

(i) $ccn(G) = K_n$ iff $\beta_{cc}(G) = 1$.

(ii) $ccn(G) = \bar{K_n}$ iff G is acyclic.

(iii) If G is connected, then $ccn(G) = \bar{K_n}$ iff G is a tree.

(iv) $ccn(C_n) = C_n \ \forall n \geq 3$.

(v) For $k \geq 2$ and $2 \leq n_1 \leq \cdots \leq n_k$,

$$ccn(K_{n_1,\cdots,n_k}) = \begin{cases} K_{n_1,n_2}, & \text{if } k, n_1 = 2 \\ K_{n_1+\cdots+n_k}, & \text{if } k \geq 2 \textit{ and } n_1 > 2. \end{cases}$$

Proposition 2.7. *There does not exist a graph G of order $n > 2$ satisfying $ccn(G) = \bar{G}$.*

Proof. If possible there exists a graph G such that $ccn(G) = \bar{G}$.

Case(i) $E(G) = \phi$.
In this case, $G = \bar{K_n}$ so that $ccn(G) = \bar{K_n} = G$, a contradiction since G is not self complementary.

Case(ii) $E(G) \neq \phi$.
Sub case(i): G has only cut edges.
Since $n > 2$, the edge set of $\bar{G}$ is non empty, but $ccn(G) = \bar{K_n}$.
Sub case(ii): G has at least one cc-edge.
Let uv be a cc-edge in G. Then, $\Gamma_{cc}(u, v) \geq 1$ so that u and v are adjacent in $ccn(G) = \bar{G}$, which is not possible.

This completes the proof. □

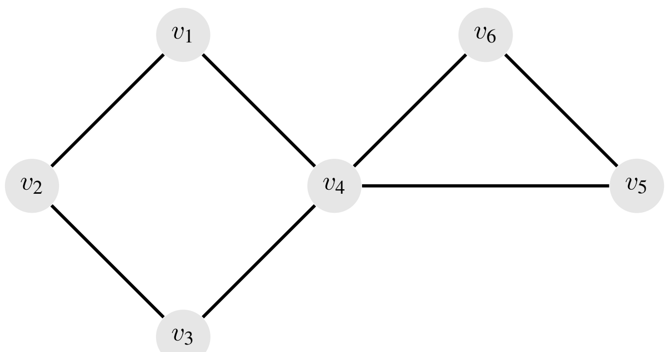
Corollary 2.8. *For a graph G, $ccn(G) = \bar{G}$ iff $G = P_2$.*

Proposition 2.9. *Let $G = (V, E)$ be a cut edge free graph and $u, v \in V$ be such that $d(u, v) = 2$. Then, u and v are closely-connected iff there exists a path $P(u, v)$ such that both the edges of P cannot be characterized by a unique cycle in G.*

Theorem 2.10. *Let $G = (V, E)$ be a graph and $u, v \in V$ are closely-connected. Then, there exists a path free of cut edges with every intermediate vertex of degree atleast 3 in G.*

Proof. Assume that u and v are closely-connected. Then, there exists a path $P(u, v)$ in G such that none of its edges are cut edges. Let w be an intermediate vertex of P so that $deg(w) \geq 2$. If $deg(w) = 2$, then the deletion of the edges incident to w in G increases the number of components of G by isolating w. Therefore, $deg(w) \geq 3$. □

Remark 2.11. The converse of the above theorem is not true. For example, consider the graph shown below.



Here the path $P = v_1v_4v_3$ has the intermediate vertex v_4 with $deg(v_4) = deg_{cc}(v_4) = 4$, but v_1 and v_3 are not closely-connected.

Definition 2.12. Let $G = (V, E)$ be a graph. A subset D of V is called cc-dominating set if for every vertex $v \in V - D$ there exists a vertex $u \in D$ such that $\Gamma_{cc}(u, v) \geq 1$.
The minimum cardinality of a cc-dominating set in G is called the cc-domination number, denoted by $\gamma_{cc}(G)$.

For a graph G of order n, the following are some basic properties of $\gamma_{cc}(G)$:

(i) $1 \leq \gamma_{cc}(G) \leq n$.

(ii) $\gamma_{cc}(G) = \gamma(ccn(G))$.

(iii) $\gamma_{cc}(G) = n$ iff G is acyclic.

(iv) If G has no cut edge, then $\gamma_{cc}(G) \leq \gamma(G)$.

(v) If G is connected, then $\gamma_{cc}(G) = n$ iff G is a tree.

Proposition 2.13. *We have the following:*

(i) For a graph G if $ccn(G) = K_n$, then $\gamma_{cc}(G) = 1$.

(ii) For any 2-regular graph G, $\gamma_{cc}(G) = \gamma(G)$.

(iii) For the complete graph K_n,

$$\gamma_{cc}(K_n) = \begin{cases} 2, & \text{if } n = 2 \\ 1, & \text{if } n \neq 2. \end{cases}$$

(iv) For the complete bipartite graph $K_{m,n}$,

$$\gamma_{cc}(K_{m,n}) = \begin{cases} 2, & \text{if } m, n = 2 \\ 1, & \text{if } m \geq 2, n > 2. \end{cases}$$

(v) For $n > 2$, $\gamma_{cc}(K_{m_1,\ldots,m_n}) = 1$, where $K_{m_1,\ldots,m_n}$ is the complete n-partite graph.

(vi) For $n \geq 3$, $\gamma_{cc}(L_n) = 1$, where L_n is the ladder graph obtained by joining the corresponding vertices of two copies of the path graph P_n.

(vii) For the path graph $P_n, \gamma_{cc}(L(P_n)) = n - 1$, where $n > 1$.

(viii) For the cycle graph C_n, $\gamma_{cc}(L(C_n)) = \gamma(C_n)$, where $n \geq 3$.

(ix) For the star graph $K_{1,n}$, $\gamma_{cc}(L(K_{1,n})) = 1$, where $n > 1$.

(x) For the bistar graph $B_{m,n}$,

$$\gamma_{cc}(L(B_{m,n})) = \begin{cases} 1, & \text{both } m > 1 \text{ and } n > 1 \\ 2, & \text{either } m = 1 \text{ or } n = 1, \text{ not both} \\ 3, & \text{both } m = 1 \text{ and } n = 1. \end{cases}$$

Proposition 2.14. *Let G be a graph and $u \in V(G)$ be cc-isolated. Then, u belongs to every cc-dominating set of G.*

Proposition 2.15. *Let G be a graph. Then, we have the following.*

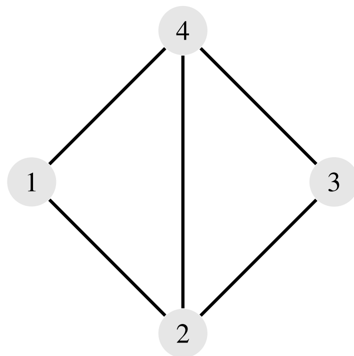
(i) If G is acyclic, then $\gamma(G) \leq \gamma_{cc}(G)$.

(ii) If G has k cc-isolated vertices, then $\gamma_{cc}(G) \geq k$.

Proposition 2.16. *For a cut edge free graph G, $\gamma_{cc}(G) \leq \gamma(G)$. The equality holds if $N(v) = N_{cc}(v)$ $\forall v \in V(G)$.*

Proof. Being free of cut edges, the adjacent vertices are closely-connected in G so that every dominating set is a cc-dominating set. Now, if $N(v) = N_{cc}(v)$ holds $\forall v \in V(G)$, we can conclude that $\Gamma_{cc}(u, v) \geq 1$ iff u and v are adjacent $\forall$ $u, v \in V(G)$ with $u \neq v$. Thus every cc-dominating set of G will also be its dominating set. Hence $\gamma_{cc}(G) = \gamma(G)$. □

Remark 2.17. The converse for the equality in the above proposition is not true, in general. That is, $\gamma_{cc}(G) = \gamma(G)$ need not always imply $N(v) = N_{cc}(v)\ \forall v \in V(G)$. For example, consider the graph shown below. Observe that, every vertex pair is closely-connected so that $\gamma_{cc}(G) = \gamma(G) = 1$. But, $N(1) = \{2, 4\}$ whereas $N_{cc}(1) = \{2, 3, 4\}$ so that $N(1) \neq N_{cc}(1)$.



Proposition 2.18. *Let G be a connected graph in which vertex pairs having common neighbors are closely-connected. Then $\gamma_{cc}(G) = 1$ if there exists a vertex $v \in V(G)$ such that $N(v) = N_{cc}(v)$ and $e(v) \leq 2$.*

Proof. Let $v \in V(G)$ be such that $N(v) = N_{cc}(v)$ and $e(v) \leq 2$. If $G = K_1$, there is nothing to prove. Otherwise, for any $u \in V(G)$,

Case(i) If u is adjacent to v, then $u \in N(v) = N_{cc}(v)$ so that $\Gamma_{cc}(u, v) \geq 1$.

Case(ii) If u is not adjacent to v, then $d(u, v) = 2$ since $e(v) \leq 2$. Since there exists a path of length 2 in G from u to v, the vertex pair (u, v) shares a common neighbor. Hence by our assumption it follows that u and v are closely-connected.

Thus in both the cases, $\Gamma_{cc}(u, v) \geq 1$ and hence $\gamma_{cc}(G) = 1$. □

Corollary 2.19. *Let G be a connected graph of diameter at most 2 such that the vertex pairs sharing a common neighbor are closely-connected. Then $\gamma_{cc}(G) = 1$ iff there exists a vertex $v \in V(G)$ such that $N(v) \subseteq N_{cc}(v)$.*

Corollary 2.20. *Let G be either disconnected or having diameter at least 3 such that the vertex pairs sharing a common neighbor in $\bar{G}$ are closely-connected. Then, $\gamma_{cc}(\bar{G}) = 1$ iff there exists a vertex $v \in V(\bar{G})$ such that $N(v) \subseteq N_{cc}(v)$ in $\bar{G}$.*

Proof. This follows from the fact that the diameter of the complement of G is at most 2. □

Theorem 2.21. *Let $G = (V, E)$ be a graph having no cc-isolated vertices. Then, the complement of every minimal cc-dominating set in G is again a cc-dominating set.*

Proof. Let D be a minimal cc-dominating set in G. If possible, assume that $V - D$ is not a cc-dominating set of G. Then, there exists $u \in D$ such that $\Gamma_{cc}(u, v) = 0\ \forall v \in V - D$. But since G has no cc-isolated vertices, there exists some vertex $w \in D - \{u\}$ such that u and w are closely-connected. Thus $D - \{u\}$ is a cc-dominating set in G, a contradiction to the minimality property of D. □

Corollary 2.22. *Let $G = (V, E)$ be a graph of order n such that $\delta_{cc}(G) \geq 1$. Then, $\gamma_{cc}(G) \leq \frac{n}{2}$.*

Proof. Let D be a minimal cc-dominating set of G. Since $\delta_{cc}(G) \geq 1$, G has no cc-isolated vertices so that it follows from theorem 2.21 that $V - D$ is a cc-dominating set of G. Therefore, $\gamma_{cc}(G) \leq \min\{|D|, |V - D|\} \leq \frac{n}{2}$. □

Theorem 2.23. *Let $G = (V, E)$ be a graph. A cc-dominating set D of G is minimal iff for every vertex $u \in D$ one of the following conditions hold:*

(i) $N_{cc}(u) \subseteq V - D$.

(ii) $\exists\, v \in V - D$ *such that* $N_{cc}(v) \cap D = \{u\}$.

Proof. Assume that D is a minimal cc-dominating set of G. Then, for every vertex $u \in D, D - \{u\}$ is not a cc-domination of G. That is, there exists $v \in (V - D) \cup \{u\}$ such that $\Gamma_{cc}(v, w) = 0$ is $\forall w \in D - \{u\}$.

Case(i) If $v = u$, then $N_{cc}(u) \subseteq V - D$.

Case(ii) Let $v \in V - D$. Since $\Gamma_{cc}(v, w) = 0$ is $\forall\, w \in D - \{u\}$ and since D is a cc-domination of G, it follows that $\Gamma_{cc}(v, u) \geq 1$. Hence it can be concluded that the only vertex in D to which v is closely-connected is u.

Conversely, assume that D is not a minimal cc-dominating set in G. Thus there exists $u \in D$ such that $D - \{u\}$ is a cc-domination of G. That is, $\Gamma_{cc}(u, w) \geq 1$ for some $w \in D - \{u\}$ so that $N_{cc}(u) \cap D \neq \phi$. Also since every vertex of $V - D$ is closely-connected to some vertex of $D - \{u\}$, there does not exists a vertex $v \in V - D$ satisfying $N_{cc}(v) \cap D = \{u\}$, a contradiction. Hence D is a minimal cc-dominating set in G. □

Theorem 2.24. *A graph $G = (V, E)$ has a unique minimal cc-dominating set iff the set of all cc-isolated vertices constitutes a cc-domination of G.*

Proof. Let D be the unique minimal cc-dominating set in G and let $S = \{v \in V :\ v \text{ is cc-isolated}\}$. From proposition 2.14, it folows that $S \subseteq D$.

Case(i) If $S = D$, then there is nothing to prove.

Case(ii) If $S \neq D$, then $\exists\, u \in D$ such that u is not cc-isolated. That is, $\Gamma_{cc}(u, v) \geq 1$ for some $v \in V$ so that $V - \{u\}$ is a cc-dominating set in G. This proves the existence of a minimal cc-dominating set $D^{'}$ different from D and contained in $V - \{u\}$, a contradiction to the uniqueness of D.

The converse of the theorem holds trivially since the set of all cc-isolated vertices is contained in every cc-dominating set of G. □

Theorem 2.25. *Let G be a cut edge free graph of diameter at most 2 and maximum degree $\Delta(G)$. Then, $\gamma_{cc}(G) \leq \Delta(G)$.*

Proof. Let $v \in V(G)$ be such that $deg(v) = \Delta(G)$.

Case(i) If diameter of G is 1, then $G = K_n$ for some n. Hence the result holds trivially.

Case(ii) If diameter of G is 2, let $V_1(G) = \{u \in V \mid d(u, v) = 1\}$. Since the edges of G are cc-edges, adjacent vertices are closely-connected in G. Thus $V_1(G)$ itself constitutes a cc-dominating set of G.

Hence, $\gamma_{cc}(G) \leq \Delta(G)$. □

Theorem 2.26. *For a graph $G = (V, E)$ of order n, $\gamma_{cc}(G) \leq n - \Delta_{cc}(G)$. Further, the equality holds if $\gamma_{cc}(G) = 1$.*

Proof. Let $v \in V$ be such that $d_{cc}(v) = \Delta_{cc}(G)$. Then, $|N_{cc}(v)| = \Delta_{cc}(G)$. Therefore, $V - N_{cc}(v)$ is a cc-dominating set of G so that

$$\gamma_{cc}(G) \leq |V - N_{cc}(v)| = n - \Delta_{cc}(G).$$

If $\gamma_{cc}(G) = 1$, then $\Delta_{cc}(G) = n - 1$, so that the equality holds. □

Theorem 2.27. *Let G be a connected graph free of cut edges. Then, G has a cc-dominating set whose complement is again a cc-dominating set.*

Proof. Since G is connected, it has a spanning tree, say T. Let $u \in V(G)$ and D be the set of all vertices in T which are at odd unit distance from u. Then, both D and $\bar{D}$ are dominating sets in G. The result follows from the fact that dominating sets in cut edge free graphs are also its cc-dominations. □

Theorem 2.28. *Let $G = (V, E)$ be a disconnected graph with components $G_1, \dots, G_m$ such that $|V(G_i)| \geq 3$ $\forall i = 1, \dots, m$. Then, $\gamma_{cc}(\bar{G}) = 1$.*

Proof. Let u be a fixed vertex and v be an arbitrary vertex of the connected graph $\bar{G}$.

Case(i) v is adjacent to u in $\bar{G}$.
Since $|V(G_i)| \geq 3$ $\forall i = 1, \dots, m$, $\bar{G}$ has no cut edges. Hence $\Gamma_{cc}(u, v) \geq 1$ in $\bar{G}$.

Case(ii) v is not adjacent to u in $\bar{G}$.
Since u and v are adjacent in G, they belong to the same component of G, say G_k. Let $w \in G_j$ for some $j \neq k$. Since $|V(G_i)| \geq 3$ $\forall i = 1, \dots, m$, $\omega(\bar{G}') = \omega(\bar{G})$, where $\bar{G}' = (V, \bar{E} - \{uw, vw\})$. Thus, $\Gamma_{cc}(u, v) \geq 1$.

Hence it follows that $\{u\}$ is a cc-dominating set in $\bar{G}$ so that $\gamma_{cc}(\bar{G}) = 1$. □

Theorem 2.29. *Let G be a graph of order n such that $\frac{n}{1+\Delta_{cc}(G)} \leq \gamma_{cc}(G)$. Further, the equality holds if and only if for every minimum cc-dominating set D in G the following conditions are satisfied:*
(i) for any vertex $v \in D$, $deg_{cc}(v) = \Delta_{cc}(G)$.
(ii) D is cc-independent in G.
(iii) every vertex in $V - D$ is closely-connected to exactly one vertex in D.

Proof. Let D be a minimum cc-dominating set in G. Clearly, every vertex in G cc-dominates at most $\Delta_{cc}(G) + 1$ vertices so that

$$n = |N_{cc}[D]| \leq \gamma_{cc}(G)(\Delta_{cc}(G) + 1).$$

Hence, $\frac{n}{1+\Delta_{cc}(G)} \leq \gamma_{cc}(G)$.
Now if the given conditions are satisfied, $n = \gamma_{cc}(G)\Delta_{cc}(G) + \gamma_{cc}(G)$ so that $\frac{n}{1+\Delta_{cc}(G)} = \gamma_{cc}(G)$. Conversely, if one of the above conditions is not satisfied, then $n < \gamma_{cc}(G)\Delta_{cc}(G) + \gamma_{cc}(G)$ so that the equality fails. □

Example 2.30. Consider the cycle graph C_{3k} $\forall$ $k \geq 1$. Then,

$$deg_{cc}(v) = 2 = \Delta_{cc}(G), \forall v \in V(C_{3k}).$$

Also, $\gamma_{cc}(C_{3k}) = \gamma(C_{3k}) = k$ by 2.13. Therefore,

$$\frac{n}{1 + \Delta_{cc}(G)} = \frac{3k}{3} = \gamma_{cc}(C_{3k}).$$

Lemma 2.31. *Every maximal cc-independent set in a graph G is its minimal cc-dominating set.*

Proof. Let D be a maximal cc-independent set and $v \in V - D$. If $v \notin N_{cc}(u)$ for every $u \in D$, then $D \cup \{v\}$ is cc-independent, a contradiction to the maximality of D. Therefore $v \in N_{cc}(u)$ for some $u \in D$ so that D is a cc-dominating set in G. Now if possible assume that D is not a minimal cc-domination of G. Then, there exists $w \in D$ such that $D - \{w\}$ is a cc-dominating set. That is, there exists a vertex closely-connected to w in D, a contradiction since D is cc-independent . Hence D is a minimal cc-dominating set in G. □

Theorem 2.32. *For any graph G, $\gamma_{cc}(G) \leq \beta_{cc}(G)$.*

Proof. The proof of the theorem is a direct consequence of the above lemma. □

Theorem 2.33. *Let $G = (V, E)$ be a graph of order n. Then,*

$$\alpha_{cc}(G) + \beta_{cc}(G) = n.$$

Proof. Assume that $S \subseteq V$ be a cc-vertex covering of G of cardinality $\alpha_{cc}(G)$ and $e = uv$ be a cc-edge of G. Since either u or v belongs to S, it follows that $V - S$ is cc-independent in G. Thus,

$$\beta_{cc}(G) \geq |V - S| = n - \alpha_{cc}(G).$$

□

3 CC-Domatic number of graphs

A domatic partition of a graph G is a partition of $V(G)$, all of whose classes are dominating sets in G. The maximum number of classes of a domatic partition of G is called the domatic number of G, denoted by $d(G)$. For applications of domatic partition [6], [9] and [14] can be inferred. Analogously, we introduce the cc-domatic number $d_{cc}(G)$ of a graph G and establish some of its properties. Moreover, we obtain some bounds for $d_{cc}(G)$ as well.

Definition 3.1. Let G be a graph. The cc-domatic partition of G is a partition $\{V_1, \dots, V_k\}$ of $V(G)$ in which each V_i is a cc-dominating set of G. The maximum order of a cc-domatic partition of G is called the cc-domatic number of G and is denoted by $d_{cc}(G)$.
For every graph G there exists at least one cc-domatic partition of $V(G)$, namely $\{V(G)\}$ so that $d_{cc}(G)$ is well-defined.

Proposition 3.2. *For the complete graph K_n,*

$$d_{cc}(K_n) = \begin{cases} 1; & \text{if } n = 2, \\ n; & \text{otherwise.} \end{cases}$$

Proof. Here we have two cases.

Case(i) If $n = 2$, then $K_2 = P_2$, a tree so that $d_{cc}(K_2) = 1$.

Case(ii) If $n \neq 2$, then every vertex $v \in V(K_n)$ cc-dominates all other vertices so that the maximum order of a partition of $V(K_n)$ into cc-dominating sets is n. Hence, $d_{cc}(K_n) = n$.

□

Proposition 3.3. *For a graph G, $d_{cc}(G) = 1$ iff G has at least one cc-isolated vertex.*

Proof. It can be noted from proposition 2.14 that if G has a cc-isolated vertex v, then every cc-dominating set of G contains v. Thus $d_{cc}(G) = 1$. Conversely, assume that $d_{cc}(G) = 1$ and G has no cc-isolated vertex. Since $\delta_{cc}(G) \geq 1$, it follows from corollary 2.22 that $\gamma_{cc}(G) \leq \frac{n}{2}$. Now for a minimal cc-dominating set D in G, $V - D$ is also a cc-dominating set. Thus $d_{cc}(G) \geq 2$, a contradiction. Hence G has at least one cc-isolated vertex. □

Proposition 3.4. *Let G be a graph. If $d_{cc}(G) = |V(G)|$, then $\gamma_{cc}(G) = 1$.*

Proof. Since $d_{cc}(G) = |V(G)|$, every cc-dominating set in the maximum order cc-domatic partition of G is singleton so that $\gamma_{cc}(G) = 1$. □

Remark 3.5. The converse of the above proposition is not true. For example, consider the complete bipartite graph $K_{2,n}$, where $n > 2$. Then, $\gamma_{cc}(K_{2,n}) = 1$ by proposition 2.13 . But the vertices of degree n in $K_{2,n}$ are not closely-connected so that $d_{cc}(G) = n + 1 < |V(K_{2,n})|$.

Proposition 3.6. *For any graph G, if $N_{cc}(v) = N(v)$ for any vertex $v \in V(G)$, then $d_{cc}(G) = d(G)$.*

Proposition 3.7. *Let G be a graph without cut edges. Then, $d(G) \leq d_{cc}(G)$.*

Proof. This follows from the fact that every dominating set of a cut edge free graph is also a cc-dominating set. □

Theorem 3.8. *For any graph G of order n, $d_{cc}(G) \leq \frac{n}{\gamma_{cc}(G)}$. Further, the equality holds if the number of cc-dominating sets of minimum cardinality is $\binom{n}{\gamma_{cc}(G)}$.*

Proof. Let G be a graph of order n with cc-domatic partition $\{D_1, \dots, D_k\}$. Since $d_{cc}(G) = k$ and $|D_i| \geq \gamma_{cc}(G)$ $\forall i = 1, \dots, k$,

$$n = \sum_{i=1}^{k} |D_i| \geq k\gamma_{cc}(G).$$

Hence, $d_{cc}(G) \leq \frac{n}{\gamma_{cc}(G)}$.
Now, if the number of cc-dominating sets of cardinality $\gamma_{cc}(G)$ is $\binom{n}{\gamma_{cc}(G)}$, then $|D_i| = \gamma_{cc}(G)$ $\forall i = 1, \dots, k$ so that $d_{cc}(G) = \frac{n}{\gamma_{cc}(G)}$.

□

Theorem 3.9. *Let G be a graph of order n. Then, $d_{cc}(G) \leq \delta_{cc}(G) + 1$. Further, the equality holds if G is the complete graph K_n for $n \neq 2$.*

Proof. Assume that $d_{cc}(G) > \delta_{cc}(G) + 1$ and $D = \{D_1, \ldots, D_k\}$ be a cc-domatic partition of G. Clearly, $k \geq \delta_{cc}(G) + 2$. Let $v \in V(G)$ be such that $deg_{cc}(v) = \delta_{cc}(G)$.

Claim v is not cc-dominated by D_i for some $i = 1, \ldots, k$.
If possible, $N_{cc}[v] \cap D_i \neq \phi \; \forall i = 1, \ldots, k$. Since $D_i^{'}s$ are mutually disjoint and $|N_{cc}[v]| = \delta_{cc}(G) + 1$, the cardinality of the cc-domatic partition D of G must not exceed $\delta_{cc}(G) + 1$, a contradiction. Hence there exists a member of D, say D_j, such that $N_{cc}[v] \cap D_j = \phi$.

Since D_j cannot cc-dominate v, it cannot be a cc-dominating set in G, a contradiction to our assumption that D is a cc-domatic partition of G. Therefore, $d_{cc}(G) \leq \delta_{cc}(G) + 1$. Now, if $G = K_n$, then $\delta_{cc}(G) = n - 1$ and hence it follows from proposition 3.2 that $d_{cc}(G) = \delta_{cc}(G) + 1$. □

Theorem 3.10. *Let G be a graph of order n without cut edges. Then,*

$$d_{cc}(G) = n - 1 \textit{ iff } G = K_{2,n-2}, \textit{ where } n \geq 5.$$

Proof. Let $G = K_{2,n-2}$ with partite sets M and N of cardinalities 2 and $n-2$ respectively. Then, the vertex pairs of G except the one belonging to M are closely-connected. Thus, $d_{cc}(G) = n-1$. Conversely, assume that $G \neq K_{2,n-2}$ and $d_{cc}(G) = n - 1$. Since $n \geq 5$, there exists adjacent vertices $u, v \in V(G)$ such that $w \in V(G)$ is a common neighbor of u and v. Since G has no cut edges, the vertex w cc-dominates u and v so that $\delta_{cc}(G) \leq 2$. From theorem 3.9, we get $d_{cc}(G) \leq \delta_{cc}(G) + 1 \leq 2 + 1 = 3$. That is, $d_{cc}(G) \leq 3 \leq n - 2$, a contradiction. Therefore, $G = K_{2,n-2}$. □

Theorem 3.11. *For any graph G of order n, $d_{cc}(G) \geq \left\lfloor \frac{n}{n-\delta_{cc}(G)} \right\rfloor$.*

Proof. Let D be any subset of $V(G)$ with $|D| \geq n - \delta_{cc}(G)$.

Claim D is a cc-dominating set of G.
If possible, let $v \in \bar{D}$ be such that $N_{cc}(v) \cap D = \phi$. Clearly, $|N_{cc}(v)| \geq \delta_{cc}(G)$. But, since $|D| \geq n - \delta_{cc}(G)$, we get $|N_{cc}(v)| < \delta_{cc}(G)$, a contradiction. Hence $N_{cc}(v) \cap D \neq \phi$.

Now, if we take any $\left\lfloor \frac{n}{n-\delta_{cc}(G)} \right\rfloor$ disjoint subsets each of cardinality $n - \delta_{cc}(G)$, it will be a cc-domatic partition of G. Therefore,

$$d_{cc}(G) \geq \left\lfloor \frac{n}{n - \delta_{cc}(G)} \right\rfloor.$$

□

Theorem 3.12. *Let G be a graph of order n. Then, $d_{cc}(G) + d(\bar{G_{cc}}) \leq n + 1$.*

Proof. It follows from theorem 3.9 that $d_{cc}(G) \leq \delta_{cc}(G) + 1$. Also, $d(\bar{G_{cc}}) \leq \delta(\bar{G_{cc}}) + 1$. Thus,

$$d_{cc}(G) + d(\bar{G_{cc}}) \leq \delta_{cc}(G) + \delta(\bar{G_{cc}}) + 2.$$

But, $\delta(\bar{G_{cc}}) = n - 1 - \Delta_{cc}(G)$ so that,

$$d_{cc}(G) + d(\bar{G_{cc}}) \leq \delta_{cc}(G) - \Delta_{cc}(G) + n + 1 \leq n + 1.$$

This completes the proof. □

Theorem 3.13. *Let G be a graph of order n. Then,*

$$2 \leq \gamma_{cc}(G) + d_{cc}(G) \leq n + 1.$$

Further, the equality holds if every vertex of G is cc-isolated.

Proof. Since $\gamma_{cc}(G) \geq 1$ and $d_{cc}(G) \geq 1$, the lower bound follows. Now, from theorem 2.26 and theorem 3.9, it follows that

$$\gamma_{cc}(G) \leq n - \Delta_{cc}(G) \leq n - \delta_{cc}(G) \text{ and } d_{cc}(G) \leq \delta_{cc}(G) + 1.$$

Thus,

$$\gamma_{cc}(G) + d_{cc}(G) \leq n - \delta_{cc}(G) + \delta_{cc}(G) + 1 = n + 1.$$

Now, if every vertex of G is cc-isolated, then $\gamma_{cc}(G) = n$ and $d_{cc}(G) = 1$ by propositions 2.15 and 3.3 respectively. Hence $\gamma_{cc}(G) + d_{cc}(G) = n + 1$. □

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