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THE (WEAKLY) SIGN SYMMETRIC Q_1-MATRIX COMPLETION PROBLEM

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Abstract In this article the (weakly) sign symmetric $[(w)ss]$ Q_1-matrix completion problems are studied. Some necessary and sufficient conditions for a digraph to have (weakly) sign symmetric Q_1-completion are shown. Lastly the digraphs of order at most four having (w)ss Q_1-completion have been sorted out.

1 Introduction

Matrix Completion Problems was initiated by Professor Burg in 1984. He studied positive definite matrix completion problem [1] in his thesis in geophysical perspective. After that a couple of researchers carried out research on matrix completion problems in their own way and applied it to different engineering fields (e.g.,[3, 4, 5, 7, 9, 10, 11, 12, 13, 14, 15]). The main aim of the matrix completion problem is to complete a real partial $m \times m$ matrix to a desired type in a combinatorial approach.

A *partial matrix* is a matrix in which some entries are specified and others are not. A *completion* of a partial matrix is a process to obtain a desired type of matrix by choosing value for the unspecified entries. A real $m \times m$ matrix $B = [b_{rs}]$ is a *Q-matrix* if for every $l \in \{1, 2, \ldots, m\}$, $S_l(B) > 0$, where $S_l(B)$ is the sum of all $l \times l$ principal minors of B. The matrix B is Q_1-matrix if all diagonals are positive. The matrix B is *sign symmetric* (ss) if $b_{rs}b_{sr} > 0$ or $b_{rs} = 0 = b_{sr}$ for each pair of $r, s \in \{1, \ldots, m\}$. The matrix B is weakly sign symmetric (wss) if $b_{rs}b_{sr} \geq 0$. The Q-matrix completion problem and its related classes are discussed in paper [6, 16, 17, 18, 19, 20, 21] respectively. Here we use the term (w)ss in a result to mean that the result is true for the cases wss as well as ss matrix completion problems.

Graphs and digraphs are closely associated with the matrix completion problems. A preliminary concept regarding graph theory can be found in any standard book i.e. [2, 8]. However we request all the readers of the article to follow any of the reference [16, 17, 21] for preliminary terms and definitions which is used through out this article.

2 Partial (w)ss Q_1-matrix

A partial Q-matrix $B = [b_{ij}]$ with specified positive diagonal entries is said to be a partial Q_1-matrix. A *partial (w)ss Q_1-matrix* is a partial Q_1-matrix in which all fully specified principal submatrices are (w)ss. We characterize a partial (w)ss Q_1-matrix as following:

Proposition 2.1. *Suppose $B = [b_{ij}]$ is a partial (w)ss matrix. Then B is a partial (w)ss Q_1-matrix if and only if exactly one of the following occurs:*

(i) *B excludes one diagonal entry.*

(ii) *B has specified diagonals and trace$(B) \geq 0$. B excludes an off diagonal entry.*

(iii) *B is complete as well as a (w)ss Q_1-matrix.*

A completion A of a partial (w)ss Q_1-matrix B is called a *(w)ss Q_1-completion* of B if A is a (w)ss Q_1-matrix.

3 The (w)ss Q_1-matrix completion problem

A *digraph* $D = (V_D, A_D)$ of order $k > 0$ is a finite set V_D with $|V_D| = k$ of objects defined as *vertices* along a set (possibly empty) A_D of ordered pairs of vertices, defined as *arcs*. We associate a $m \times m$ partial matrix B with $D = (\{1, 2, \ldots, m\}, A_D)$ by drawing an arc $(i, j) \in A_D,\ 1 \leq i, j \leq m$ for a specified (i, j)-th entry of B. A digraph D has (w)ss Q_1-*completion* if every partial (w)ss Q_1-matrix specifying D can be completed to a (w)ss Q_1-matrix. The (w)ss Q_1-*matrix completion problems* are studied for classifying all digraphs based on (w)ss Q_1-matrix completion.

4 Relationship between wss Q_1-completion and ss Q_1-completion

Consider X and Y are two different classes of matrices. It is impossible for us to get a conclusion that Y-completion of a digraph always implies X-completion or vice versa since a partial Y-matrix (X-matrix) always is not a X-matrix (Y-matrix). But there are some instances in which it is possible to give a decision that X-completion implies Y-completion or vice versa for two classes of matrices X and Y. Prof L. Hogben studied this types of results in [10] and called them as “Relationship theorem" . In this section, we will study the relationship theorems of ss and wss Q_1-completion problem. Here for a symmetric pair b_{ij}, x_{ij} in a partial matrix B we denote x_{ij} as the unspecified entry corresponding to the specified entry b_{ij}.

Theorem 4.1. *Suppose a digraph D has wss Q_1-completion, where for any partial wss Q_1-matrix specifying D, there is a wss Q_1-completion in which zero is allotted to any unspecified entry whose corresponding specified entry is zero. Then D has ss Q_1-completion.*

Proof. Suppose B be a partial ss Q_1-matrix specifying D. Then B is also a partial wss Q_1-matrix. Consider a wss Q_1-completion $A = [a_{ij}]$ of B obtained by putting 1 and 0 respectively to unspecified diagonal entries and to those unspecified entries whose specified entries are 0. But there may exist some nonzero a_{ij} in A where as corresponding a_{ji} is zero. Since a few principal minors of A are to be considered, which are continuous functions of the entries of A, we will slightly perturb originally unspecified zero entries keeping the sum of all principal minors of same order as positive. Then A can be converted into a ss Q_1-matrix. □

Corollary 4.2. *Any symmetric digraph D that has wss Q_1-completion has ss Q_1-completion.*

The following result is quite obvious.

Theorem 4.3. *Any asymmetric digraph D that has ss Q_1-completion if and only if it has wss Q_1-completion.*

5 Some results on the (w)ss Q_1 Completion

It is quite clear that any (w)ss partial matrix with unspecified diagonals must have (w)ss Q_1-completion. A desired completion can be obtained with the choice of extremely big values for the unspecified diagonals. Suppose a partial (w)ss Q_1-matrix B has unspecified diagonals at (i, i) positions $(i = k + 1, \ldots, n)$. If $B[1, \ldots, k]$ is fully specified, a (w)ss Q_1-completion of B may not be obtained as seen from the partial wss Q_1-matrix B_1 where

$$B_1 = \begin{bmatrix} 2 & 2 & 0.1 \\ 2 & 2 & 0.1 \\ 2 & 2 & * \end{bmatrix},$$

with unspecified entries labeled a $*$. For any value of unspecified entry $*$, we always have $\det(B_1) = 0$. Hence completion of B_1 to a (w)ss Q_1-matrix cannot be possible. Again if $B[1, \ldots, k]$ has an unspecified entry as well as a (w)ss Q_1-completion, then B has a (w)ss Q_1-completion which is obtained with sufficiently large choice of unspecified diagonals.

Theorem 5.1. *Suppose a partial wss Q_1-matrix B has unspecified diagonal at $(r + 1, r + 1)$ position. Then the wss Q_1-completion of a not fully specified principal submatrix $B[1, \ldots, r]$ of B implies wss Q_1-completion of B.*

Proof. Consider B is of the form,

$$B = \begin{bmatrix} B_{11} & B_{12} \\ B_{21} & B_{22} \end{bmatrix},$$

where, $B_{11} = B[1, \ldots, r]$ and $B_{22} = B[r+1, r+1]$. Take a wss Q_1-matrix completion A_1 of $B[1, \ldots, r]$. In that case

$$\widetilde{B} = \begin{bmatrix} A_1 & B_{12} \\ B_{21} & B_{22} \end{bmatrix},$$

is a partial wss Q_1-matrix due to the presence of an unspecified diagonal entry in B_{22}. For $t > 0$, consider a completion $A = [a_{ij}]$ of $\widetilde{B}$ obtained by choosing $a_{ii} = t, \quad i = r+1$ and $a_{ij} = 0$ against all other unspecified entries in $\widetilde{B}$. Then A is of the form,

$$A = \begin{bmatrix} A_1 & A_{12} \\ A_{21} & t \end{bmatrix}.$$

Since A_1 is a wss Q_1-matrix, $S_i(A_1) > 0$ for $1 \leq i \leq r$. For $2 \leq j \leq r+1$,

$$S_j(A) \quad = \quad S_j(A_1) + tS_{j-1}(A_1) + s_j,$$

where s_j is a constant. For sufficiently large values of t A turns in to a wss Q_1-matrix. □

The Theorem 5.1 also holds for ss Q_1-completion and in that case the proof is quite similar. A may not be a ss Q_1-matrix if some a_{rs} is nonzero where as corresponding a_{sr} is zero. Here also we will change zero entries in a very small manner to turn A into a ss Q_1-matrix.

Corollary 5.2. *Suppose a partial (w)ss Q_1-matrix B has unspecified diagonals at at (i, i) positions where $(i = r+1, \ldots, n)$. Then the (w)ss Q_1-completion of a not fully specified principal submatrix $B[1, \ldots, r]$ of B implies (w)ss Q_1-completion of B.*

However the counter part of the above Corollary 5.2 is not valid.

Example 5.3. Consider a partial (w)ss Q_1-matrix

$$B = \begin{bmatrix} d_1 & b_{12} & b_{13} & ? & b_{15} \\ b_{21} & d_2 & ? & ? & ? \\ b_{31} & ? & d_3 & ? & ? \\ ? & ? & ? & d_4 & b_{45} \\ b_{51} & b_{52} & b_{53} & ? & ? \end{bmatrix},$$

where $d_i > 0, i = 1, 2, 3, 4$ and $b_{ij}, i \neq j, \ i, j = 1, 2, 3, 4, 5$ are specified diagonal and off-diagonal entries respectively and ? denotes the unspecified entries. Now for $t, \epsilon > 0$ we take a matrix A where:

$$A = \begin{bmatrix} d_1 & b_{12} & b_{13} & \epsilon & b_{15} \\ b_{21} & d_2 & \epsilon & t & \epsilon \\ b_{31} & t & d_3 & \epsilon & \epsilon \\ \epsilon & \epsilon & t & d_4 & b_{45} \\ b_{51} & b_{52} & b_{53} & \epsilon & t \end{bmatrix},$$

in which we define,

$$\epsilon = \begin{cases} \gamma, & \text{if } b_{ij} > 0 \text{ and the } \{j, i\} - \text{th entry of A is unspecified} \\ -\gamma, & \text{if } b_{ij} < 0 \text{ and the } \{j, i\} - \text{th entry of A is unspecified} \\ 0 & \text{otherwise} \end{cases}$$

$$\begin{aligned} S_k(A(t,\epsilon)) &= t^k + g(t,\epsilon), \ \forall\, k = 1,2,3,4, \\ \det A &= \beta t^4 + g(t,\epsilon). \end{aligned}$$

where $\beta > 0$ is a constant and $g(t,\epsilon)$ is a polynomial of degree $\leq k-1$. Thus by choosing $t > 0$ sufficiently large and $\epsilon > 0$ as a sufficiently small we can conclude that A is (w)ss Q_1-matrix completion of B. But the principal partial submatrix $B[1,2,3]$ does not have (w)ss Q_1-completion. To verify this consider a partial submatrix $B[1,2,3]$ of B as follows;

$$B[1,2,3] = \begin{bmatrix} 1 & 10 & 10 \\ 10 & 1 & ? \\ 10 & ? & 1 \end{bmatrix}.$$

It is quite clear that $B[1,2,3]$ does not have (w)ss Q_1-completion.

Theorem 5.4. *Suppose B is a partial (w)ss Q_1-matrix specifying a digraph D. If the partial submatrices of B induced by every strongly connected induced subdigraph of D has (w)ss Q_1-completion then B has (w)ss Q_1-completion.*

Proof. We restrict our proof by considering two strongly connected induced subdigraph $H_{(1)}$ and $H_{(2)}$ of D. The generalization of the proof can be done by the method of induction. To prove this result we at first rename the vertices (if needed) and we obtain

$$B = \begin{bmatrix} B_{(11)} & B_{(12)} \\ X_B & B_{(22)} \end{bmatrix},$$

where the digraph $H_{(i)},\ i = 1,2$ is specified by a partial (w)ss Q_1-matrix $B_{(ii)}$ and X_B has all unspecified entries. Since by our assumption $B_{(ii)}$ has a (w)ss Q_1-completion say $A_{(ii)}$ hence we can obtain our desired completion

$$A = \begin{bmatrix} A_{(11)} & A_{(12)} \\ A_{(21)} & A_{(22)} \end{bmatrix},$$

by substituting all unspecified entries in X_B and $B_{(12)}$ as ϵ. We can choose ϵ to be a sufficiently small positive negative or zero number according to the sign of corresponding specified entry. Then for $2 \leq j \leq |D|$ and by choosing ϵ sufficiently small we have

$$S_j(A) = S_j(A_{(11)}) + S_j(A_{(22)}) + \sum_{r=1}^{j-1} S_r(A_{(11)}) S_{j-r}(A_{(22)}) + \epsilon\, h_j(\epsilon) > 0,$$

where each h_k is a polynomial in ϵ. Here we take $S_j(A_{(ii)}) = 0$ whenever j exceeds the size of $A_{(ii)}$. Now choosing ϵ sufficiently small A can be completed to a (w)ss Q_1-matrix. □

Theorem 5.5. *Let a digraph D contains strongly connected components $H_{(1)}, H_{(2)}, \ldots, H_{(j)}$ such that $|H_{(j)}| \geq 2\ \forall\ j$. If for each j, $H_{(j)}$ has (w)ss Q_1-completion then D has (w)ss Q_1-completion.*

We have omitted proof of the above Theorem 5.5 since it follows easily from the Theorem 5.4. The next theorem is obvious.

Theorem 5.6. *Let a digraph D consists components $H_{(1)}, H_{(2)}, \ldots, H_{(j)}$ such that $|H_{(j)}| \geq 2\ \forall\ j$. If for each j, $H_{(j)}$ is not complete and has (w)ss Q_1-completion, then D has (w)ss Q_1-completion.*

Remark 5.7. The counter part of the Theorem 5.4 is not valid. Here the digraph D_0 has (w)ss Q_1-completion but its strong component $D[1,2,3]$ does not have (w)ss Q_1-completion(See Example 5.3).

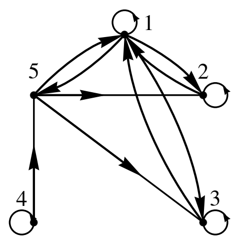

Figure 1. A digraph D_0 having (w)ss Q_1-completion

6 Sufficient conditions for (w)ss Q_1-matrix completion

Theorem 6.1. *Any spanning subdigraph of a digraph $D \neq K_n, |D| = n$ with (w)ss Q_1-completion also has (w)ss Q_1-completion.*

Proof. Consider a spanning subdigraph $\widehat{D}$ of D and $\widehat{B}$ be a partial (w)ss Q_1-matrix specifying $\widehat{D}$. Now we obtain a partial (w)ss Q_1-matrix B specifying the digraph D from partial (w)ss Q_1-matrix $\widehat{B}$ as follows:

(i) For ss Q_1-matrix we choose the entries which are associated to $(i, j) \in A_D \setminus A_{\widehat{D}}$ as μ, where μ is chosen as per with the ss condition of the corresponding twin.

(ii) For wss Q_1-matrix we choose the entries which are associated to $(i, j) \in A_D \setminus A_{\widehat{D}}$ as 0.

Considering both the cases and by Proposition 2.1, we can say that B is a partial (w)ss Q_1-matrix specifying D. Since $D \neq K_n$ has (w)ss Q_1-completion, hence we consider a (w)ss Q_1-completion A of B. In that case A is also a (w)ss Q_1-completion of $\widehat{B}$. □

Theorem 6.2. *A digraph $D \neq K_n$ without a cycle of even length has (w)ss Q_1-completion.*

Proof. Consider a partial (w)ss Q_1-matrix B which specifies D. Take a completion A of B obtained by putting all unspecified diagonal entries as $t > 0$ and all unspecified off diagonal entries as ϵ, where ϵ is either zero or very small positive or negative entry, chosen according to their twin keeping the (w)ss condition of the corresponding specified entries. Since D has no cycle of even length so for $j \in \{2, 3, \dots, n\}$

$$S_j(A) = \alpha t^j + p_{j-1}(t, \epsilon)$$

where α is a real positive constant and $p_{j-1}(t, \epsilon)$ is a polynomial of degree $< j - 1$. Now we choose $\epsilon = \frac{1}{t^5}$. Then for sufficiently large t A becomes a (w)ss Q_1-matrix. □

Theorem 6.3. *Suppose $D \neq K_n$ be a asymmetric digraph s.t. D contains a cycle of even length > 2. $\overline{D}$ contains a symmetric 3-cycle $\widehat{D}$ satisfying the following:*

(i) *The digraph $\widehat{D}$ contains an absolutely asymmetric spanning 3-cycle $C = \langle u_1, u_2, u_3 \rangle$.*

(ii) *C does not form a negative permutation digraph with any permutation digraph in D.*

Then D has (w)ss Q_1-completion.

Proof. Suppose $B = [b_{ij}]$ be a partial (w)ss Q_1-matrix which specifies D. Now for $t, \epsilon > 0$, consider a completion $A = [a_{ij}]$ of B as follows:

$$a_{ij} = \begin{cases} b_{ij}, & \text{if } (i,j) \in A_D \\ t, & \text{if } (i,j) \in A_C \\ t, & \text{if } (i,i) \in A_{\overline{D}} \\ \epsilon, & \text{otherwise} \end{cases}$$

where ϵ is either zero or very small positive or negative entry, chosen according to the ss condition of the corresponding twin. Also D does not have a 2-cycle, hence $S_2(A) > 0$. Now choosing $\epsilon = \frac{1}{t^4}$ we have,

$$S_k(A) = \beta t^3 + p_{k-1}(t), \forall k = 3, 4, \ldots, n, \tag{6.1}$$

where $p_{k-1}(t)$ is a polynomial in t of total degree at most $k-1$ and β is a positive number. Since C does not form a negative permutation digraph in D, hence by choosing sufficiently large values of t, we have $S_i(A) > 0 \ \ i \in \{1, 2, \ldots, n\}$. Therefore A is a (w)ss Q_1-matrix completion of B. □

However the conditions of the Theorem 6.3 is not necessary. To see this consider the digraph D_1 in Figure 2. Although D_1 is asymmetric, contains a cycle of length 3 and $\overline{D_1}$ does not satisfy the statement of the Theorem 6.3, but the digraph D_1 has (w)ss Q_1-matrix completion. Consider

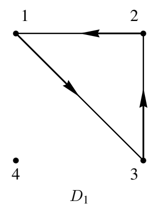

Figure 2. A digraph D_1 having (w)ss Q_1-completion

a partial (w)ss Q_1-matrix

$$B = \begin{bmatrix} d_1 & ? & b_{13} & ? \\ b_{21} & d_2 & ? & ? \\ ? & b_{32} & d_3 & ? \\ ? & ? & ? & d_4 \end{bmatrix},$$

specifying D_1, where ? denotes the unspecified entries. If all specified entries are zero or positive then we are done. Suppose not. Consider $b_{21} \neq 0$. Then for $t, \epsilon > 0$ consider a completion

$$A = \begin{bmatrix} d_1 & \epsilon & b_{13} & sgn(b_{21})t \\ b_{21} & d_2 & \epsilon & \epsilon \\ \epsilon & b_{32} & d_3 & \epsilon \\ \epsilon & t & \epsilon & d_4 \end{bmatrix},$$

of B where ϵ is chosen according to ss condition of corresponding twin. Now choosing $t > 0$ sufficiently large A becomes a (w)ss Q_1-matrix.

Theorem 6.4. *Suppose a digraph $D \neq K_n^*$ such that $\overline{D}$ is symmetric as well as stratified. If it is possible to sign the arcs of an absolutely asymmetric spanning stratified subdigraph $\widehat{D}$ of $\overline{D}$ such that the sign of every cycle is positive, then D has (w)ss Q_1-completion.*

Proof. Consider a (weakly) partial ss Q_1-matrix $B = [b_{ij}]$ which specifies D. Now $s, \Psi > 0$, a completion $A(t, \gamma) = [a_{ij}]$ of C can be obtained in a following manner:

$$a_{ij} = \begin{cases} b_{ij}, & \text{if } (i,j) \in A_D \\ t, & \text{if } (i,i) \in A_{\overline{D}} \\ sgn(i,j)t, & \text{if } (i,j) \in A_{\widehat{D}} \cap A_{\overline{D}}, i \neq j \\ \Psi, & \text{otherwise} \end{cases}$$

where Ψ can be chosen as small positive or negative or zero as per the ss condition of the corresponding twin. Then for $k = 1, 2, \ldots, n$,

$$S_k(A(t, \Psi)) = \beta t^k + p(t), \tag{6.2}$$

where $p(t)$ is a polynomial of degree at most $k-1$ and $\beta > 0$. Hence we are done □

Consider a digraph D_2 in Figure 3 which satisfies the statement of both Theorem 6.3 and Theorem 6.4. Hence the digraph D_2 has (w)ss Q_1-completion. However the counter part of

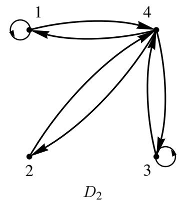

Figure 3. A digraph D_2 having (w)ss Q_1-completion

Theorem 6.4 is not valid. Take a digraph D_3 which is obtained by removing an arc $(3, 4)$ from D_2. The complement $\overline{D_3}$ of D_3 is not symmetric. However D_3 has (w)ss Q_1-completion.

7 Necessary conditions for (w)ss Q_1-completion

Theorem 7.1. *Suppose a digraph $D \neq K_m$ with $|D| = m$ with loops at each of its vertices and has (w)ss Q_1-completion. Then D omits all 2-cycle.*

Proof. Let a 2-cycle $[l, k]$ is present in D and $B = [b_{ij}]$ be a partial (w)ss Q_1-matrix specifying D s.t. $b_{ii} = 1$ $(1 \leq i \leq m)$ and $b_{lk}b_{kl} > \begin{pmatrix} m \\ 2 \end{pmatrix}$. For any (w)ss Q_1-completion $A = [a_{ij}]$ of B, we have

$$S_2(A) = \sum_{r \neq s} b_{rr}b_{ss} - \sum_{r \neq s} b_{rs}b_{sr} \ < \ - \sum_{r,s \notin \{l,k\}} a_{rs}a_{sr} \ < \ 0,$$

and therefore A is not a (w)ss Q_1-matrix. □

Our next corollary is obvious.

Corollary 7.2. *If a digraph $D \neq K_m$ has more than $\frac{1}{2}m(m+1)$ arcs then D fails to have (w)ss Q_1-completion.*

Theorem 7.3. *Suppose a digraph $D \neq K_4$ has a cycle of length > 2. If D has ss Q_1-completion then $\overline{D}$ must contain a 2-cycle.*

Proof. Suppose $\overline{D}$ is completely asymmetric. Consider $V_D = \{v_1, v_2, \ldots, v_4\}$, D contains a cycle $C_1 = \langle v_1, v_2, v_3 \rangle$ s.t. $|C_1| = 3$. Consider a partial ss Q_1-matrix $B = [b_{ij}]$ specifying D such that

$$b_{ij} = \begin{cases} 1, & \text{if } (i, i) \in A_D \\ 2, & \text{if } (v_1, v_2), (v_3, v_1) \in A_{C_1} \\ -2, & \text{if } (v_2, v_3) \in A_{C_1} \\ 0, & \text{otherwise} \end{cases}$$

Now for any ss Q_1-completion A of B, we always have $S_3(A) \leq 0$. Hence ss Q_1-completion of D is not possible. □

Example 7.4. The Theorem 7.3 is not sufficient. Consider D_4 in Figure 4. The digraph contains a 4-cycle $\langle 1, 2, 3, 4 \rangle$ and it's complement digraph $\overline{D_4}$ contains a 2-cycle $\langle 1, 3 \rangle$. Although D_4 does not have ss Q_1-completion. Take any partial matrix

$$B = \begin{bmatrix} 1 & 2 & ? & ? \\ ? & 1 & 0 & ? \\ ? & ? & 1 & 0 \\ 0 & 0 & ? & 1 \end{bmatrix},$$

specifying D_4. One can easily observe that completion of B is not possible.

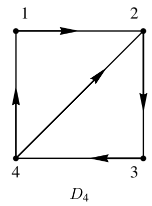

Figure 4. A digraph D_4 not having (w)ss Q_1-completion

7.1 Algorithm for (w)ss Q_1-completion of digraph D, $|D| \leq 4$

The following steps should be followed to check whether a digraph $D \neq K_4$ has (w)ss Q_1-completion or not:

(i) If D omits all loops, then D has (w)ss Q_1-completion. Otherwise, we will proceed to step (ii).

(ii) Suppose D includes all loops. If D has no cycle of even length then D has (w)ss Q_1-completion.

(iii) If D contains 2-cycle, thee (w)ss Q_1-completion of D is not possible. If not, we will proceed to step (iv).

(iv) Suppose D contains a cycle of length $m > 2$ and $\overline{D}$ is completely asymmetric. Then D does not have ss Q_1-completion. If not consider the step (v).

(v) Consider D contains a cycle of length $m > 2$ and $\overline{D}$ is not completely asymmetric. If D follows the hypothesis of Theorem 6.3, then D has (w)ss Q_1-completion. If not then we will proceed to step (vi).

(vi) Check whether $\overline{D}$ is symmetric or not. If $\overline{D}$ is symmetric and satisfies the conditions of the Theorem 6.4, then D has (w)ss Q_1-completion.

(vii) Finally D does not satisfy all of the above conditions, then we have to check whether it has (w)ss Q_1-completion or not manually.

8 A (w)ss Q_1-completion based classification

Now we will classify all digraphs (with all specified loops) up to order 4 on the basis of (w)ss Q_1-completion. For this we follow the atlas of digraphs as given in [8]. We denote a digraph $D_p(q, n)$ as the n-th digraph with p vertices and q (non-loop) arcs.

Theorem 8.1. *For $p \in \{1, 2, 3, 4\}$, the below mentioned digraphs $D_p(q, n)$ have (w)ss Q_1-completion,*

$$\begin{array}{lll} p = 2; & q = 0, 1, 2 & n = 1 \\ p = 3; & q = 0; & n = 1 \\ & q = 1; & n = 1 \\ & q = 2; & n = 2, 3, 4 \\ & q = 3; & n = 3 \\ & q = 6; & n = 1 \end{array}$$

$$\begin{array}{lll} p = 4; & q = 0, 1; & n = 1 \\ & q = 2; & n = 1\text{–}5 \\ & q = 3; & n = 1\text{–}13 \\ & q = 4; & n = 10, 11, 12, 17\text{–}23, 25\text{–}27 \\ & q = 5; & n = 4, 5, 29\text{–}31, 33\text{–}38 \\ & q = 6; & n = 1, 46\text{–}48 \\ & q = 12. & n = 1. \end{array}$$

Proof. If $q = 0$ or a complete digraph we are done. The digraphs $D_3(3,3), D_4(6,46), D_4(6,47), D_4(6,48)$ have (w)ss Q_1-completion by Theorem 6.2. The digraph $D_4(6,1)$ has (w)ss Q_1-completion by Theorem 6.4. Rest of the each above listed digraph is a spanning digraph of any one the digraph say $D_3(3,3), D_4(6,46), D_4(6,47), D_4(6,48), D_4(6,1)$ and hence it has (w)ss Q_1-completion. Hence the result follows. □

9 Conclusions

Here we have discussed the completion problem of (w)ss Q_1-matrices. Some results regarding (w)ss Q_1-completion of a digraph are obtained but our main goal of complete factorization of digraphs on the basis of (w)ss Q_1-matrix completion are still missing. In future we will try to develop to fill up the gaps of our obtained results.

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