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On Matrix Properties of Generalised Srivastava's Triple Hypergeometric Matrix Function $H_{C,P,Q}(\cdot)$

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Abstract The main objects of the present paper is to investigate and study a matrix version $(P, Q)-$ generalization of Srivastava's triple hypergeometric function $H_C(\cdot)$, together with its integral representations, by using the extended Beta matrix function $B_{P,Q}(x, y)$. For a matrix that satisfies an appropriate spectral property. An exact integral expression of $H_{C,P,Q}(\cdot)$ function in terms of the Laguerre matrix polynomials is obtained. Moreover, we give some of its main matrix version properties, namely a derivative matrix identity, recursive matrix relations and a bounded matrix inequality.

1 Introduction

The theory of special matrix functions has influenced in the last two decades. The attention of special matrix function has been initiated by Jódar, Cortés,[4, 6] and this work has been carried for several variable special functions in [3, 12, 14]. This matrix functions appear in the literature allied to the Gauss hypergeometric matrix function, Appell matrix functions, Lauricella matrix functions, Beta and gamma matrix functions, and Srivastava's triple hypergeometric matrix functions (see; for example [1, 2, 9]). A number of authors have intended integral representations, recursion relations, differential and transformations formulae of the extended special hypergeometric matrix functions; see [1, 3, 8, 9]. Inspired by this study, the authors establishes matrix version properties, namely a derivative identity, recursion relations and a bounded inequality for a generalised Srivastava's triple hypergeometric matrix function $H_{C,P,Q}(\cdot)$.

2 Preliminaries

Let $\mathbb{C}^{N\times N}$ be a N-square matrices contains complex entries. Then for any matrix $M \in \mathbb{C}^{N\times N}$, its spectrum $\sigma(M)$ is the set of eigenvalues of M. If $h(z)$ and $f(z)$ are two holomorphic functions of a complex variable z, which are defined in an open set Ω of the complex plane and $M \in \mathbb{C}^{N\times N}$ with $\sigma(M) \subset \Omega$, then from the properties of the matrix functional calculus; see [15], we consider $h(M)f(M) = f(M)h(M)$. If $L \in \mathbb{C}^{N\times N}$, is a matrix for which $\sigma(L) \subset \Omega$, and if $ML = LM$, then $h(M)f(L) = f(L)h(M)$. A square matrix M in $\mathbb{C}^{N\times N}$ is said to be positive stable if $\Re(\eta) > 0$ for all $\eta \in \sigma(M)$.

The image of $\Gamma^{-1}(z)$ acting on a matrix M, denoted by $\Gamma^{-1}(M)$, is a well defined matrix in $\mathbb{C}^{N\times N}$. If $M + nI$ is an invertible for all integers $n \geq 0$, then the reciprocal gamma matrix function is defined in [4] as $\Gamma^{-1}(M) = (M)_n\Gamma^{-1}(M + nI)$, where $(M)_n$ is a Pochhammer's (or the shifted factorial) matrix function for $M \in \mathbb{C}^{N\times N}$, defined by [6]

$$(M)_n := \begin{cases} I, & (n = 0) \\ M(M + I)...(M + n - I), & (n \geq 1). \end{cases} \tag{2.1}$$

for I being identity N-square matrix in $\mathbb{C}^{N\times N}$.
If U, V and $U + V$ be positive stable matrices in $\mathbb{C}^{N\times N}$ such that $UV = VU$, then the classical

simple beta matrix function is defined by [4]

$$\mathfrak{B}(U,V)=\begin{cases}\displaystyle\int_0^1 t^{U-1}(1-t)^{V-1}dt\\ \\ \Gamma(U)\Gamma(V)\Gamma^{-1}(U+V).\end{cases}\tag{2.2}$$

Let M_1, M_2, M_3 be positive stable matrices in $\mathbb{C}^{N\times N}$ such that $M_3+\ell I$ is an invertible for all integers $\ell>0$. Then the Gauss hypergeometric matrix function is defined by [4]

$${}_2F_1\left(\begin{matrix}M_1,M_2\\ M_3\end{matrix};z\right)=\sum_{n=0}^{\infty}\frac{(M_1)_n(M_2)_n}{(M_3)_n}\frac{z^n}{n!}.\tag{2.3}$$

Here (2.3) converges absolutely for $|z|<1$ and for $z=1$. A generalisation of this matrix function to add P numerator matrices M_j $(1\le j\le P)$ and Q denominator matrices M_j $(1\le j\le Q$; see[4]. Srivastava and Karlsson [19, Chapter 3] introduced and explored a table of distinct 205 triple hypergeometric functions. Some triple hypergeometric functions denoted as H_A, H_B and H_C of the second order are introduced by Srivastava, see [17, 18].

The matrix analogs of the three Srivastava's triple hypergeometric functions H_A, H_B and H_C; see [3]. We study Srivastava's hypergeometric matrix function H_C of three variables given by $H_C(\cdot)$; see [1]. Let L_1, L_2, L_3 and L_4 be positive stable and commutative matrices in $\mathbb{C}^{N\times N}$ such that $L_4+\ell I$ is an invertible matrix for all positive integer $\ell>0$, and I is a square matrix in $\mathbb{C}^{N\times N}$. Then

$$\begin{aligned}H_C(L_1,L_2,L_3;L_4;z_1,z_2,z_3)&:=\sum_{h,m,k=0}^{\infty}(L_1)_{h+k}(L_2)_{h+m}(L_3)_{m+k}[(L_4)_{h+m+k}]^{-1}\frac{z_1^h}{h!}\frac{z_2^m}{m!}\frac{z_3^k}{k!},\\ &=\sum_{h,m,k=0}^{\infty}(L_2)_{h+m}(L_3)_{m+k}[(L_4)_m]^{-1}\mathfrak{B}(L_1+(h+k)I,L_4+mI-L_1)\\ &\qquad\times[\mathfrak{B}(L_1,L_4+mI-L_1)]^{-1}\frac{z_1^h}{h!}\frac{z_2^m}{m!}\frac{z_3^k}{k!}.\end{aligned}\tag{2.4}$$

We introduce an additional parameters r and s into matrix function $H_C(\cdot)$ in the form

$$\begin{aligned}&H_C^{(r,s)}(L_1,L_2,L_3;L_4;z_1,z_2,z_3)\\ &=\sum_{h,m,k=0}^{\infty}(L_2)_{h+m}(L_3)_{m+k}[(L_4)_m]^{-1}\mathfrak{B}(L_1+r+(h+k)I,L_4+s+mI-L_1)\\ &\qquad\times[\mathfrak{B}(L_1,L_4+mI-L_1)]^{-1}\frac{z_1^h}{h!}\frac{z_2^m}{m!}\frac{z_3^k}{k!}.\end{aligned}\tag{2.5}$$

The region of convergence for the Srivastava's triple hypergeometric matrix function $H_C(\cdot)$ is defined by [3] as $\alpha(L_1)<1, \alpha(L_2)<1, \alpha(L_3)<1$ and $\beta(L_4)>1$, where $|z_1|<1, |z_2|<1, |z_3|<1$. For L_1, L_2, L_3, L_4 and L_4-L_1 are positive stable matrices in $\mathbb{C}^{N\times N}$ such that $L_4+\ell I$ is an invertible matrix for all positive integers $\ell>0$ and I is a square matrix in $\mathbb{C}^{N\times N}$. Then the Appell hypergeometric matrix function $F_1(\cdot)$ is defined by [12]

$$\begin{aligned}&F_1(L_1,L_2,L_3;L_4;z_1,z_2)\\ &:=\sum_{n,m=0}^{\infty}(L_2)_n(L_3)_m\,\mathfrak{B}(L_1+(m+n)I,L_4-L_1)[\mathfrak{B}(L_1,L_4-L_1)]^{-1}\frac{z_1^m}{m!}\frac{z_2^n}{n!},\end{aligned}\tag{2.6}$$

where $|z_1|<1, |z_2|<1$ and this function has been extended in matrix form by Ashish Verma; see [12]. An integral representation of Gauss hypergeometric matrix function ${}_2F_1(\cdot)$ is given by [4]

$${}_2F_1\left(\begin{matrix}M_1,M_2\\ M_3\end{matrix};z\right)=\int_0^1 t^{M_2-1}(1-t)^{M_3-M_2-1}(1-zt)^{-M_1}dt[\mathfrak{B}(L_2,L_4-L_2)]^{-1},\tag{2.7}$$

for positive stable matrices M_1, M_2, M_3 and $M_3 - M_2$ in $\mathbb{C}^{N\times N}$. The generalised beta matrix function $\mathfrak{B}(W_1, W_2; P)$ in terms of integral is defined by [1]

$$\mathfrak{B}(W_1, W_2; P) = \int_0^1 t^{W_1 - 1}(1-t)^{W_2 - 1} \exp\left[\frac{-P}{t(1-t)}\right] dt. \tag{2.8}$$

where W_1, W_2 and P be positive stable and commuting matrices in $\mathbb{C}^{N\times N}$ and are invertible for all integers $\ell > 0$. A further extension of the Beta matrix function is defined by

$$\mathfrak{B}(W_1, W_2; P, Q) \equiv \mathfrak{B}_{P,Q}(W_1, W_2) = \int_0^1 t^{W_1 - 1}(1-t)^{W_2 - 1} \exp\left\{-\frac{P}{t} - \frac{Q}{1-t}\right\} dt, \tag{2.9}$$

where W_1, W_2, P and Q are positive stable and commuting matrices in $\mathbb{C}^{N\times N}$. When $P = Q$ this matrix function reduces to $\mathfrak{B}(W_1, W_2; P)$. Also, Ashish Verma et al.[1] discussed the further extension of the beta and gamma matrix function, Gauss hypergeometric and confluent hypergeometric matrix functions.

3 On matrix version generalized Srivastava's triple hypergeometric function $H_{C,P,Q}(\cdot)$

The matrix analog of the Srivastava's hypergeometric triple functions associated with its recursive relations have been discussed in [7]. Here we investigate (P, Q)-generalized of the Srivastava's triple hypergeometric matrix function $H_C(\cdot)$, which we identified by $H_{C,P,Q}(\cdot)$, depend on the extended Beta matrix function $\mathfrak{B}_{P,Q}(M_1, M_2)$. We have

Theorem 3.1. *Let L_1, L_2, L_3, L_4 and $L_4 - L_1$ be positive stable and commutative matrices in $\mathbb{C}^{N\times N}$ such that $L_4 + \ell I$ is an invertible matrix for all positive integer $\ell > 0$ and I is a square matrix in $\mathbb{C}^{N\times N}$. Then*

$$\begin{aligned} &H_{C,P,Q}(L_1, L_2, L_3; L_4; z_1, z_2, z_3) \\ &\quad = \sum_{h,m,k=0}^{\infty} (L_2)_{h+m}(L_3)_{m+k}[(L_4)_m]^{-1}\, \mathfrak{B}_{P,Q}(L_1 + (h+k)I, L_4 + mI - L_1) \\ &\qquad \times [\mathfrak{B}(L_1, L_4 + mI - L_1)]^{-1} \frac{z_1^h}{h!}\frac{z_2^m}{m!}\frac{z_3^k}{k!}, \end{aligned} \tag{3.1}$$

The region of convergence for this generalised matrix function $H_{C,P,Q}(\cdot)$ is defined as $\alpha(L_1) < 1, \alpha(L_2) < 1, \alpha(L_3) < 1$ and $\beta(L_4) > 1$, where $|z_1| < 1, |z_2| < 1, |z_3| < 1$. This definition implies to the original classical function (2.4) when $P = 0 = Q$.

Theorem 3.2. *Let L_1, L_2, L_3, L_4 and $L_4 - L_1$ be positive stable and commutative matrices in $\mathbb{C}^{N\times N}$ such that $L_4 + \ell I$ is an invertible matrix for all positive integer $\ell > 0$. Then the hypergeometric matrix function $H_{C,P,Q}(\cdot)$ holds the following integral representations*

$$\begin{aligned} &H_{C,P,Q}(L_1, L_2, L_3; L_4; z_1, z_2, z_3) \\ &\quad = \int_0^1 t^{L_1 - I}(1-t)^{L_4 - L_1 - I}(1 - z_1 t)^{-L_2}(1 - z_3 t)^{-L_3} \times \\ &\qquad \times \exp\left\{-\frac{P}{t} - \frac{Q}{1-t}\right\} {}_2F_1\left(\begin{array}{c} L_2, L_3; \\ L_4 - L_1; \end{array} X\right) dt\, \Gamma_{(L_4, L_1)}, \end{aligned} \tag{3.2}$$

where

$$X := \frac{z_2(1-t)}{(1 - z_1 t)(1 - z_3 t)}, \quad \Gamma_{(L_4, L_1)} = \Gamma(L_4)[\Gamma(L_1)\Gamma(L_4 - L_1)]^{-1}$$

$$H_{C,P,Q}(L_1,L_2,L_3;L_4;z_1,z_2,z_3)=\int_0^\infty \mu^{L_1-I}(1+\mu)^{L_2+L_3-L_4}\{\Omega_1\}^{-L_2}\{\Omega_2\}^{-L_3}\times$$
$$\times\exp\left\{-\frac{P(1+\mu)I}{\mu}-Q(1+\mu)I\right\}{}_2F_1\left(\begin{array}{ll} L_2,L_3; & z_2\Omega \\ L_4-L_1; & \end{array}\right)d\mu\ \Gamma_{(L_4,L_1)}, \quad (3.3)$$

where $\Omega_1=1+\mu-z_1\mu$, $\Omega_2=1+\mu-z_3\mu$, $\Omega=\frac{(1+\mu)}{\Omega_1\Omega_2}$, $\Gamma_{(L_4,L_1)}=\Gamma(L_4)[\Gamma(L_1)\Gamma(L_4-L_1)]^{-1}$,

$$H_{C,P,Q}(L_1,L_2,L_3;L_4;z_1,z_2,z_3)=\int_0^{\frac{\pi}{2}}(sin^2\mu)^{L_1-\frac{I}{2}}(cos^2\mu)^{L_4-L_1-\frac{I}{2}}(\vartheta_1)^{-L_2}(\vartheta_2)^{-L_3}\times$$
$$\times\exp\left\{-\frac{P}{sin^2\mu}-\frac{Q}{cos^2\mu}\right\}{}_2F_1\left(\begin{array}{ll} L_2,L_3; & \frac{z_2cos^2\mu}{\vartheta_1\vartheta_2} \\ L_4-L_1; & \end{array}\right)d\mu\ \Gamma_{(L_4,L_1)} \quad (3.4)$$

where $\vartheta_1=1-z_1sin^2\mu$, *and* $\vartheta_2=1-z_3sin^2\mu$, $\Gamma_{(L_4,L_1)}=2\Gamma(L_4)[\Gamma(L_1)\Gamma(L_4-L_1)]^{-1}$

$$H_{C,P,Q}(L_1,L_2,L_3;L_4;z_1,z_2,z_3)=\frac{(\beta-\gamma)^{L_1}(\alpha-\gamma)^{L_4-L_1}}{(\beta-\alpha)^{L_4-L_2-L_3-I}}\int_\alpha^\beta\frac{(\mu-\alpha)^{L_1-I}(\beta-\mu)^{L_4-L_1-I}}{(\mu-\gamma)^{L_4-L_2-L_3}}\times$$
$$\times\{\sigma_1\}^{-L_2}\{\sigma_2\}^{-L_3}\exp\{-P\sigma_3-Q\sigma_4\}{}_2F_1\left(\begin{array}{ll} L_2,L_3; & \sigma z_2 \\ L_4-L_1; & \end{array}\right)d\mu\ \Gamma_{(L_4,L_1)} \quad (3.5)$$

where

$$\sigma_1=[(\beta-\alpha)(\mu-\gamma)-z_1(\beta-\gamma)(\mu-\alpha)]$$
$$\sigma_2=[(\beta-\alpha)(\mu-\gamma)-z_3(\beta-\gamma)(\mu-\alpha)],$$
$$\sigma_3=\frac{(\beta-\alpha)(\mu-\gamma)}{(\beta-\gamma)(\mu-\alpha)}\quad and\quad \sigma_4=\frac{(\beta-\alpha)(\mu-\gamma)}{(\alpha-\gamma)(\beta-\mu)},$$
$$\sigma=\frac{(\alpha-\gamma)(\beta-\mu)}{\sigma_1\sigma_2},$$
$$\Gamma_{(L_4,L_1)}=\Gamma(L_4)[\Gamma(L_1)\Gamma(L_4-L_1)]^{-1}$$

and

$$H_{C,P,Q}(L_1,L_2,L_3;L_4;z_1,z_2,z_3)=\int_0^1\frac{(\mu)^{L_1-I}(1-\mu)^{L_4-L_1-I}}{(1+\lambda\mu)^{L_4-L_2-L_3}}\{\nabla_1\}^{-L_2}\{\nabla_2\}^{-L_3}(1+\lambda)^{L_1}\times$$
$$\times\exp\left\{\frac{-P(1+\lambda\mu)I}{\mu(1+\lambda)}-\frac{Q(1+\lambda\mu)I}{1-\mu}\right\}{}_2F_1\left(\begin{array}{ll} L_2,L_3; & \nabla z_2 \\ L_4-L_1; & \end{array}\right)d\mu\,\Gamma_{(L_4,L_1)}, \quad (3.6)$$

where

$$\nabla_1=[1+\lambda\mu-z_1(1+\lambda)\mu],$$
$$\nabla_2=[1+\lambda\mu-z_3(1+\lambda)\mu],$$
$$\nabla=\frac{(1-\mu)(1+\lambda\mu)}{\nabla_1\nabla_2};\quad \lambda>-1.$$
$$\Gamma_{(L_4,L_1)}=\Gamma(L_4)[\Gamma(L_1)\Gamma(L_4-L_1)]^{-1}.$$

Proof: We can prove first integral (3.2) by using extended beta matrix function (2.9) in (3.1), then change the order of integration and summation (since integral is uniform convergent) and finally using Gauss hypergeometric matrix function (2.3). We get after simplification the right-hand side of (3.2). Furthermore, we can prove the integrals expressed by (3.3)-(3.6), by using the below transformations

$$t=\frac{\mu}{1+\mu},\quad \frac{dt}{d\mu}=\frac{1}{(1+\mu)^2}, \quad (3.7)$$

$$t = sin^2\mu, \quad \frac{dt}{d\mu} = 2\sin\mu\cos\mu, \tag{3.8}$$

$$t = \frac{(\beta-\gamma)(\mu-\alpha)}{(\beta-\alpha)(\mu-\gamma)}, \quad \frac{dt}{d\mu} = \frac{(\beta-\alpha)(\beta-\gamma)(\alpha-\gamma)}{(\beta-\alpha)^2(\mu-\gamma)^2}, \tag{3.9}$$

$$t = \frac{(1+\lambda)\mu}{1+\lambda\mu}, \quad \frac{dt}{d\mu} = \frac{(1+\lambda)}{(1+\lambda\mu)^2}, \tag{3.10}$$

in (3.2), we get the right hand side of respective results.

Theorem 3.3. *Let L_1, L_2, L_3, C_1, $L_4 - L_1, P, Q$ and $P - Q$ be positive stable matrices in $\mathbb{C}^{N\times N}$ such that $L_4 + \ell I$ is an invertible matrix for all positive integers $\ell > 0$. Then integral representation of matrix function $H_{C,P,Q}(\cdot)$ involving infinite sum of Laguerre matrix polynomials*

$$H_{C,P,Q}(L_1,\ L_2,\ L_3; L_4;\ z_1,\ z_2,\ z_3) = \sum_{n,m=0}^{\infty} L_n(P)L_m(Q)\exp(-P-Q)\Gamma(L_4)[\Gamma(L_1)\Gamma(L_4-L_1)]^{-1}$$
$$\times \int_0^1 \kappa^{L_1+mI}(1-\kappa)^{L_4-L_1+nI}(1-z_1\kappa)^{-L_2}(1-z_3\kappa)^{-L_3}{}_2F_1\left(\begin{array}{c} L_2, L_3; \\ L_4-L_1; \end{array} X\right) dt, \tag{3.11}$$

where $X := z_2(1-\kappa)/(1-z_1\kappa)(1-z_3\kappa)$ and I is an identity square matrix in $\mathbb{C}^{N\times N}$.

Proof. We can get exponential factor expressed in (2.9) involving Laguerre polynomials $L_m(z_1)(m \in N_0)$ associated with generating function [16, P. 202]

$$e^{\left(-\frac{z_1\kappa}{1-\kappa}\right)} = (1-\kappa)\sum_{n=0}^{\infty} \kappa^n L_n(z_1), \quad -1 < \kappa < 1, \tag{3.12}$$

for $z_1 > 0$. From this we see that P and Q are positive stable matrices in $\mathbb{C}^{N\times N}$, then we have

$$e^{\left(-\frac{Q}{1-\kappa}\right)} = e^{-Q}(1-\kappa)\sum_{m=0}^{\infty} \kappa^m L_m(Q), \quad -1 < \kappa < 1, \tag{3.13}$$

Replacement of κ by $1-\kappa$, which gives

$$e^{\left(-\frac{P}{\kappa}\right)} = e^{-P}\ \kappa\sum_{n=0}^{\infty}(1-\kappa)^n L_n(P), \quad 0 < \kappa < 2. \tag{3.14}$$

From (3.13) and (3.13), we get

$$e^{\left(-\frac{P}{\kappa}-\frac{Q}{1-\kappa}\right)} = e^{-P-Q}\sum_{n,m=0}^{\infty}(1-\kappa)^{n+I}\kappa^{m+I}L_n(P)L_m(Q), \quad 0 < \kappa < 1. \tag{3.15}$$

Using (3.15) in (3.2), we get the required result stated in (3.11).

4 A derivative matrix identity for $H_{C,P,Q}(\cdot)$

Theorem 4.1. *Let L_1, L_2, L_3, L_4, $L_4 - L_1, P$ and Q be positive stable and commutative matrices in $\mathbb{C}^{N\times N}$ such that $L_4 + \ell I$ is an invertible matrix for all positive integers $\ell > 0$. Then the following derivative identity for hypergeometric matrix function $H_{C,P,Q}(\cdot)$ holds:*

$$\frac{\partial^{J+M+K}}{\partial z_1^J \partial z_2^M \partial z_3^K} H_{C,P,Q}(L_1, L_2, L_3; L_4; z_1, z_2, z_3) = \frac{(L_1)_{J+K}(L_2)_{J+M}(L_3)_{M+K}}{(L_4)_{J+M+K}} \times$$
$$\times H_{C,P,Q}(L_1+(J+K)I, L_2+(J+M)I, L_3+(M+K)I; L_4+(J+M+K)I; z_1, z_2, z_3), \tag{4.1}$$

where $J, M, K \in \mathbb{N}_0$ and I is a square identity matrix in $\mathbb{C}^{N\times N}$.

Proof: Differentiate the matrix series for $\mathcal{H} \equiv H_{C,P,Q}(L_1, L_2, L_3; L_4; z_1, z_2, z_3)$ in (3.1) with respect to z_1 we obtain

$$\frac{\partial \mathcal{H}}{\partial z_1} = \sum_{j=1}^{\infty} \sum_{m,k=0}^{\infty} (L_2)_{j+m}(L_3)_{m+k}(L_4)_m \mathfrak{B}_{P,Q}(L_1 + (j+k)I, L_4 + mI - L_1) \times [\mathfrak{B}(L_1, L_4 + mI - L_1)]^{-1} \frac{z_1^{j-1}}{(j-1)!} \frac{z_2^m}{m!} \frac{z_3^k}{k!}, \quad (4.2)$$

Using matrix identity

$$\mathfrak{B}(L_1, L_4 + mI - L_1) = [L_1]^{-1}(L_4 + mI)\, \mathfrak{B}(L_1 + I, L_4 + mI - L_1), \quad (4.3)$$

and algebraic matrix property $(A)_{j+m} = (A)_j (A + jI)_m$, we consider $j \to j+1$

$$\frac{\partial \mathcal{H}}{\partial z_1} = L_1\, L_2 [L_4]^{-1} \sum_{j,m,k=0}^{\infty} (L_2+I)_{j+m}(L_3)_{m+k}[(L_4+I)_m]^{-1}\mathfrak{B}_{P,Q}(L_1+I+(j+k)I, L_4+mI-L_1) \times [\mathfrak{B}(L_1 + I, L_4 + mI - L_1)]^{-1} \frac{z_1^j}{j!} \frac{z_2^m}{m!} \frac{z_3^k}{k!}, \quad (4.4)$$

$$= L_1\, L_2 [L_4]^{-1}\, H_{C,P,Q}\left(L_1 + I,\ L_2 + I,\ L_3;\ L_4 + I; z_1, z_2, z_3\right). \quad (4.5)$$

Repeated differentiation of (4.5) then gives for J = 1,2,...

$$\frac{\partial^J \mathcal{H}}{\partial z_1^J} = (L_1)_J\ (L_2)_J [(L_4)_J]^{-1}\ H_{C,P,Q}\left(L_1 + J,\ L_2 + J,\ L_3;\ L_4 + J; z_1, z_2, z_3\right). \quad (4.6)$$

A uniform processing shows that

$$\frac{\partial^{J+1} \mathcal{H}}{\partial z_1^J \partial z_2} = (L_1)_J\ (L_2)_J)[(L_4)_J]^{-1} \sum_{j,m,k=0}^{\infty} (L_2 + J)_{j+m}(L_3)_{m+k}[(L_4 + J)_m]^{-1} \times \mathfrak{B}_{P,Q}(L_1 + J + (j+k)I, L_4 + J + mI - L_1) \times [\mathfrak{B}(L_1 + J, L_4 + J + mI - L_1)]^{-1} \frac{z_1^j}{j!} \frac{z_2^m}{m!} \frac{z_3^k}{k!} \quad (4.7)$$

$$= (L_1)_J\ (L_2)_{J+I}\ (L_3)[(L_4)_{J+I}]^{-1} H_{C,P,Q}\left(L_1 + J,\ L_2 + J + I,\ L_3 + I;\ L_4 + J + I; z_1, z_2, z_3\right), \quad (4.8)$$

Replacing $m \to m+1$ and employing the Beta matrix function in (4.8). Continued differentiation of (4.8) M times with respect to z_2 then produces

$$\frac{\partial^{J+M} \mathcal{H}}{\partial z_1^J \partial z_2^M} = (L_1)_J\ (L_2)_{J+M}\ (L_3)_M [(L_4)_{J+M}]^{-1} \times H_{C,P,Q}\left(L_1 + J,\ L_2 + J + M,\ L_3 + M;\ L_4 + J + M; z_1, z_2, z_3\right). \quad (4.9)$$

Following the same methods and differentiation with respect to z_3 then we will get the result stated in (4.1).

5 On matrix version bounded inequality for $H_{C,P,Q}(\cdot)$

Theorem 5.1. *Let P, Q, L_4 and $L_j (1 \le j \le 3)$ be positive stable matrices in $\mathbb{C}^{N\times N}$ such that $L_4 + \ell I$ is an invertible matrix for all positive integers $\ell > 0$ and the variables $z_1,\ z_2,\ z_3 \in \mathbb{C}$. Then*

$$|H_{C,P,Q}(L_1, L_2, L_3; L_4; z_1, z_2, z_3)| < \varphi_E\ H_C(L_1, L_2, L_3; L_4; |z_1|,\ |z_2|,\ |z_3|), \quad (5.1)$$

where $\varphi_E := e^{[-P-Q-2\sqrt{P\,Q}]}$.

Proof. Assume $P, Q, L_4,\ L_j(1 \le j \le 3)$ be positive stable matrices in $\mathbb{C}^{N\times N}$. Then

$$|H_{C,P,Q}(L_1,L_2,L_3;L_4;z_1,z_2,z_3)| \le \sum_{h,m,k\ge 0} (L_2)_{h+m}(L_3)_{m+k}[(L_4)_m]^{-1}$$
$$\times \mathfrak{B}_{P,Q}(L_1+(h+k)I, L_4+mI-L_1)\mathfrak{B}(L_1,L_4+mI-L_1)\,\frac{|z_1|^h}{h!}\frac{|z_2|^m}{m!}\frac{|z_3|^k}{k!}. \quad (5.2)$$

Let U, V, P and Q be positive stable matrices in $\mathbb{C}^{N\times N}$. Then extended Beta matrix function $\mathfrak{B}_{P,Q}(U,V)$ defined in (2.9) can be written as

$$B_{P,Q}(U,V) \le \int_0^1 t^{U-1}(1-t)^{V-1}\,E_{P,Q}(t)dt,\ \ E_{P,Q}(t) := exp\left(-\frac{P}{t}-\frac{Q}{1-t}\right),$$
$$< \int_0^1 t^{U-1}(1-t)^{V-1}\,E_{P,Q}(t)dt,$$

Since, $E_{P,Q}(t)$ is maximum at $t^* = r/(1+r),\ r=\sqrt{P\,Q}$. We then derive that

$$B_{P,Q}(U,V) < \varphi_E\,B(U,V),\ \ \varphi_E := exp[-P-Q-2\sqrt{P\,Q}].$$

Therefore, from (5.2), we have

$$|H_{C,P,Q}(L_1,L_2,L_3;L_4;z_1,z_2,z_3)|$$
$$< \varphi_E \sum_{h,m,k\ge 0}(L_2)_{h+m}(L_3)_{m+k}[(L_4)_m]^{-1}\mathfrak{B}(L_1+(h+k)I, L_4+mI-L_1)$$
$$\times [\mathfrak{B}(L_1,L_4+mI-L_1)]^{-1}\,\frac{|z_1|^h}{h!}\frac{|z_2|^m}{m!}\frac{|z_3|^k}{k!}, \quad (5.3)$$

Comparing of this last sum by (2.4) then employ the result stated in (5.1).
Note that if $P=Q=\mu I$, then $\varphi_E = e^{-4\mu I}$.

6 On recursive matrix relations for $H_{C,P,Q}(\cdot)$

We investigate two recursive matrix relations for the generalized Srivastava's triple hypergeometric matrix function $H_{C,P,Q}(\cdot)$.

Theorem 6.1. *Let $L_1, L_2, L_3, L_4,\ L_4-L_1, P$ and Q be positive stable and commutative matrices in $\mathbb{C}^{N\times N}$ such that $L_4+\ell I$ is an invertible matrix for all positive integers $\ell > 0$. Then the recursive relations for a matrix function $H_{C,P,Q}(\cdot)$ with respect to the matrices L_2 and L_3 holds true:*

$$H_{C,P,Q}(L_1,L_2+I,L_3;L_4;z_1,z_2,z_3) = H_{C,P,Q}(L_1,L_2,L_3;L_4;z_1,z_2,z_3)$$
$$+ z_1\ L_1[L_4]^{-1}H_{C,P,Q}(L_1+I,L_2+I,L_3;\ L_4+I;z_1,z_2,z_3)$$
$$+ z_2\ L_3[L_4]^{-1}H_{C,P,Q}(L_1,L_2+I,L_3+I;L_4+I;z_1,z_2,z_3), \quad (6.1)$$

and

$$H_{C,P,Q}(L_1,L_2,L_3+I;L_4;z_1,z_2,z_3) = H_{C,P,Q}(L_1,L_2,L_3;L_4;z_1,z_2,z_3)$$
$$+ z_2\ L_2[L_4]^{-1}H_{C,P,Q}(L_1,L_2+I,L_3+I;L_4+I;z_1,z_2,z_3)$$
$$+ z_3\ L_1[L_4]^{-1}H_{C,P,Q}(L_1+I,L_2,L_3+I;L_4+I;z_1,z_2,z_3), \quad (6.2)$$

for I is an identity square matrix in $\mathbb{C}^{N\times N}$.

Proof. Using (3.1) and the matrix identity $(L_2+I)_{h+m} = (L_2)_{h+m}(I+hI[L_2]^{-1}+mI[L_2])]^{-1}$, which implies

$$H_{C,P,Q}(L_1,L_2+I,L_3;L_4;z_1,z_2,z_3) = \sum_{h,m,k=0}^{\infty}(L_2+I)_{h+m}(L_3)_{m+k}[(L_4)_m]^{-1}$$
$$\times\mathfrak{B}_{P,Q}(L_1+(h+k)I,L_4+mI-L_1)[\mathfrak{B}(L_1,L_4+mI-L_1)]^{-1}\frac{z_1^h}{h!}\frac{z_2^m}{m!}\frac{z_3^k}{k!}, \tag{6.3}$$

$$= H_{C,P,Q}(L_1,L_2,L_3;L_4;z_1,z_2,z_3) + z_1[L_2]^{-1}\sum_{h=1}^{\infty}\sum_{m=0}^{\infty}\sum_{k=0}^{\infty}(L_2)_{h+m}(L_3)_{m+k}[(L_4)_m]^{-1}$$
$$\times\mathfrak{B}_{P,Q}(L_1+(h+k)I,L_4+mI-L_1)[\mathfrak{B}(L_1,L_4+mI-L_1)]^{-1}\frac{z_1^{h-1}}{(h-1)!}\frac{z_2^m}{m!}\frac{z_3^k}{k!}$$
$$+z_2[L_2]^{-1}\sum_{h=0}^{\infty}\sum_{m=1}^{\infty}\sum_{k=0}^{\infty}(L_2)_{h+m}(L_3)_{m+k}(L_4)_m\mathfrak{B}_{P,Q}(L_1+(h+k)I,L_4+mI-L_1)$$
$$\times[\mathfrak{B}(L_1,L_4+mI-L_1)]^{-1}\frac{z_1^h}{h!}\frac{z_2^{m-1}}{(m-1)!}\frac{z_3^k}{k!}. \tag{6.4}$$

We esteem the preceding sum in (6.4) by S_1. Set $m\to m+1$ and employ the matrix identity $(A)_{m+1}=A(A+I)_m$, we find

$$S_1 = z_1[L_2]^{-1}\sum_{h,m,k=0}^{\infty}(L_2)_{h+1+m}(L_3)_{m+k}(L_4)_m\mathfrak{B}_{P,Q}(L_1+I+(h+k)I,L_4+mI-L_1)$$
$$\times[\mathfrak{B}(L_1,L_4+mI-L_1)]^{-1}\frac{z_1^h}{h!}\frac{z_2^m}{m!}\frac{z_3^k}{k!}$$
$$= z_1[L_2]^{-1}\sum_{h,m,k=0}^{\infty}(L_2+I)_{h+m}(L_3)_{m+k}[(L_4)_m]^{-1}\mathfrak{B}_{P,Q}(L_1+I+(h+k)I,L_4+mI-L_1)$$
$$\times[\mathfrak{B}(L_1,L_4+mI-L_1)]^{-1}\frac{z_1^h}{h!}\frac{z_2^m}{m!}\frac{z_3^k}{k!}. \tag{6.5}$$

Applying (4.3), we then obtain

$$S_1 = z_1L_1[L_4]^{-1}\sum_{h,m,k=0}^{\infty}(L_2+I)_{h+m}(L_3)_{m+k}[(L_4+I)_m]^{-1}$$
$$\times\mathfrak{B}_{P,Q}(L_1+I+(h+k)I,L_4+mI-L_1)[\mathfrak{B}(L_1+I,L_4+mI-L_1)]^{-1}\frac{z_1^h}{h!}\frac{z_2^m}{m!}\frac{z_3^k}{k!},$$
$$= z_1\ L_1[L_4]^{-1}\ H_{C,P,Q}(L_1+I,L_2+I,L_3;L_4+I;z_1,z_2,z_3). \tag{6.6}$$

Applying the same procedure for second sum in (6.4) by replacing $m\to m+1$, we find

$$S_2 = z_2\ L_3[L_4]^{-1}\ H_{C,P,Q}(L_1,L_2+I,L_3+I;L_4+I;z_1,z_2,z_3). \tag{6.7}$$

Combining of the above equations (6.6) and (6.7) with (6.4) then gives the result showed in (6.1). Similarly, the solution of (6.2) is investigated in the same way by interchanging the matrix L_3.

Corollary 1. The relation (6.1) gives the following recursive relation

$$H_{C,P,Q}(L_1,L_2+sI,L_3;L_4;z_1,z_2,z_3) = H_{C,P,Q}(L_1,L_2,L_3;L_4;z_1,z_2,z_3)$$
$$+z_1\ L_1[L_4]^{-1}\sum_{n=1}^{s}H_{C,P,Q}(L_1+I,L_2+nI,L_3;L_4+I;z_1,z_2,z_3)$$
$$+z_2L_3[L_4]^{-1}\sum_{n=1}^{s}H_{C,P,Q}(L_1,L_2+nI,L_3+I;L_4+I;z_1,z_2,z_3), \tag{6.8}$$

for L_4+sI is an invertible matrix for all positive integers $s>0$ and I is an identity square matrix in $\mathbb{C}^{N\times N}$.

Corollary 2. The relation (6.2) gives the following recursive relation

$$\begin{aligned}H_{C,P,Q}(L_1,L_2,L_3+sI;L_4;z_1,z_2,z_3) &= H_{C,P,Q}(L_1,L_2,L_3;L_4;z_1,z_2,z_3)\\ &+z_2\ L_2[L_4]^{-1}\sum_{n=1}^{s} H_{C,P,Q}(L_1,L_2+I,L_3+nI;L_4+I;z_1,z_2,z_3)\\ &+z_3\ L_1[L_4]^{-1}\sum_{n=1}^{s} H_{C,P,Q}(L_1+I,L_2,L_3+nI;L_4+I;z_1,z_2,z_3),\end{aligned} \tag{6.9}$$

for L_4+sI is an invertible matrix for all positive integers $s>0$ and I is an identity square matrix in $\mathbb{C}^{N\times N}$.

Theorem 6.2. *Let L_1,L_2,L_3,C_1, L_4-L_1, P and Q be positive stable and commutative matrices in $\mathbb{C}^{N\times N}$ such that $L_1+\ell I$ is an invertible matrix for all positive integers $\ell>0$. Then the following recursive matrix relation for $H_{C,P,Q}(\cdot)$ function with respect to the matrix L_4 holds:*

$$\begin{aligned}H_{C,P,Q}(L_1,L_2,L_3;L_4;z_1,z_2,z_3) &= H_{C,P,Q}(L_1,L_2,L_3;L_4+I;z_1,z_2,z_3)\\ &+z_2\ L_2L_3[L_4(L_4+I)]^{-1}H_{C,P,Q}(L_1,L_2+I,L_3+I;L_4+2I;z_1,z_2,z_3).\end{aligned} \tag{6.10}$$

for I is an identity square matrix in $\mathbb{C}^{N\times N}$.

Proof. The matrix L_4 is reduced by I, namely

$$\mathbb{H} :\equiv H_{C,P,Q}(L_1,L_2,L_3;L_4-I;z_1,z_2,z_3), \tag{6.11}$$

and upon the use of matrix identity $(L_4-I)_m=(L_4)_m\left\{I+m[L_4-I]^{-1}\right\}^{-1}$. Then

$$\begin{aligned}\mathbb{H} &= \sum_{h,m,k=0}^{\infty}(L_2)_{h+m}(L_3)_{m+k}[(L_4-I)_m]^{-1}\mathfrak{B}_{P,Q}(L_1+(h+k)I,L_4+mI-L_1)\\ &\qquad\times\mathfrak{B}(L_1,L_4+mI-L_1)\frac{z_1^h}{h!}\frac{z_2^m}{m!}\frac{z_3^k}{k!}\\ &= \sum_{h,m,k=0}^{\infty}(L_2)_{h+m}(L_3)_{m+k}[(L_4)_m]^{-1}\mathfrak{B}_{P,Q}(L_1+(h+k)I,L_4+mI-L_1)\\ &\qquad\times[\mathfrak{B}(L_1,L_4+mI-L_1)]^{-1}\left(I+m[L_4-I]^{-1}\right)\frac{z_1^h}{h!}\frac{z_2^m}{m!}\frac{z_3^k}{k!},\\ &= H_{C,P,Q}(L_1,L_2,L_3;L_4;z_1,z_2,z_3)+z_2[L_4-I]^{-1}\sum_{h,k=0}^{\infty}\sum_{m=1}^{\infty}(L_2)_{h+m}(L_3)_{m+k}[(L_4)_m]^{-1}\\ &\times\mathfrak{B}_{P,Q}(L_1+(h+k)I,L_4+mI-L_1)[\mathfrak{B}(L_1,L_4+mI-L_1)]^{-1}\frac{z_1^h}{h!}\frac{z_2^{m-1}}{(m-1)!}\frac{z_3^k}{k!}.\end{aligned} \tag{6.12}$$

Setting $m\to m+1$, we obtain

$$\begin{aligned}\mathbb{H} &= H_{C,P,Q}(L_1,L_2,L_3;L_4;z_1,z_2,z_3)+z_2B_2L_3[L_4(L_4-I)]^{-1}\\ &\times\sum_{h,m,k=0}^{\infty}(L_2+I)_{h+m}(L_3+I)_{m+k}[(L_4+I)_m]^{-1}\mathfrak{B}_{P,Q}(L_1+(h+k)I,L_4+I+mI-L_1)\\ &\qquad\times[\mathfrak{B}(L_1,L_4+I+mI-L_1)]^{-1}\frac{z_1^h}{h!}\frac{z_2^m}{m!}\frac{z_3^k}{k!},\\ &= H_{C,P,Q}(L_1,L_2,L_3;L_4;z_1,z_2,z_3)+z_2\ L_2L_3[L_4(L_4-I)]^{-1}\\ &\qquad\times H_{C,P,Q}(L_1,L_2+I,L_3+I;L_4+I;z_1,z_2,z_3).\end{aligned}$$

Change of matrix L_4 by L_4+I then gives the result showed in (6.10).

Conclusion

We investigate the matrix version of a generalized Srivastava's triple hypergeometric function denoted by $H_{C,P,Q}(.)$ in (3.1), together with the integral representations. Also, we have given an integral expression of the generalised Srivastava's matrix function $H_{C,P,Q}(\cdot)$ associated with Laguerre matrix polynomials. In addition, we have also investigated some properties of this matrix function, namely, a differential matrix identity, a matrix version upper bound inequality and matrix recursive relations.

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