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# A FURTHER EXTENSIONS OF BETA AND RELATED FUNCTIONS

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**Abstract** In this article, we propose a new extension of beta function by utilizing the Bessel-Struve kernel function. Here, first we derive some basic properties of this new beta function and thereafter present a new extension of the well-known beta dispersion as an application of our proposed beta function. Additional to that we elaborate and explore one more extension of Gauss and confluent hypergeometric functions by utilizing the definition of the same. Some significant properties of the above mentioned hypergeometric functions like integral representations, differential formulae, transformation formulae, summation formulae and a generating relation are are also pointed out in a systematic manner.

## 1 Introduction and preliminaries

Special functions have a great significance in various fields of Mathematics, Physics, engineering and other related research areas including function analysis, differential equations, quantum mechanics, mathematical physics, and so on. Recently, a function has captivated various researcher thought due on very basic level to different applications, which are more wide than the beta type function $\mathbb{B}(\zeta_1, \zeta_2)$, prominently known as generalized beta type functions. In addition, generalized beta functions have played significant part in the headway of additional investigation and have end up being excellent in nature.

All through in this exploration article, let $\mathbb{N}$, $\mathbb{R}$ and $\mathbb{C}$ be the arrangements of regular numbers, real numbers and complex numbers, respectively, and let

$$\mathbb{N} := \{1, 2, 3, ...\},\ \mathbb{N}_0 := \{0, 1, 2, 3, ...\} = \mathbb{N} \cup \{0\}.$$

For our purpose, we begin by recalling here the definitions of some known functions and their generalizations. The classical beta function is a function of two complex variable $\zeta_1$ and $\zeta_2$,

$$\mathbb{B}(\zeta_1, \zeta_2) = \int_0^1 w^{\zeta_1 - 1}\,(1-w)^{\zeta_2 - 1} dw \tag{1.1}$$

$$(\Re(\zeta_1) > 0,\ \Re(\zeta_2) > 0).$$

was defined and studied by Leonhard Euler (see [15], see additionally [16]).

On account of arranged usages of this function in a wide scope of engineering furthermore sciences, various researchers have presented and explored several extensions of this important function (1.1) (see, for example, [1]–[5], [8], [10], [12], [13], [14] and [17]).

In 1997, Chaudhry *et al.* [5] presented an exceptionally valuable generalization of the classical beta function (1.1) by

$$\mathbb{B}_\kappa(\zeta_1, \zeta_2) = \int_0^1 w^{\zeta_1 - 1}\,(1-w)^{\zeta_2 - 1}\ \exp\left[-\frac{\kappa}{w(1-w)}\right] dw \tag{1.2}$$

$$(\Re(\zeta_1) > 0,\ \Re(\zeta_2) > 0,\ \Re(\kappa) > 0).$$

It is effortlessly observed that for $\kappa = 0$, (1.2) reduces to (1.1). By using (1.2), Chaudhry *et al.* [6] generalized the Gauss hypergeometric function (GHF) and the confluent hypergeometric function (CHF), respectively, as follows:

$$F_\kappa(\eta_1,\eta_2;\eta_3;z) = \sum_{\ell=0}^{\infty} \frac{(\eta_1)_\ell\, \mathbb{B}_\kappa(\eta_2+\ell,\eta_3-\eta_2)}{\mathbb{B}(\eta_2,\eta_3-\eta_2)} \frac{z^\ell}{\ell!} \tag{1.3}$$
$$(\kappa \geq 0,\ |z|<1,\ \Re(\eta_3) > \Re(\eta_2) > 0)$$

and

$$\Phi_\kappa(\eta_2;\eta_3;z) = \sum_{\ell=0}^{\infty} \frac{\mathbb{B}_\kappa(\eta_2+\ell,\eta_3-\eta_2)}{\mathbb{B}(\eta_2,\eta_3-\eta_2)} \frac{z^\ell}{\ell!} \tag{1.4}$$
$$(\kappa \geq 0,\ \Re(\eta_3) > \Re(\eta_2) > 0).$$

In [6], the authors additionally characterized the resulting Euler's type integral representations of $F_\kappa(\eta_1,\eta_2;\eta_3;z)$ and $\Phi_\kappa(\eta_2;\eta_3;z)$, respectively:

$$F_\kappa(\eta_1,\eta_2;\eta_3;z) = \frac{1}{\mathbb{B}(\eta_2,\eta_3-\eta_2)} \tag{1.5}$$
$$\times \int_0^1 w^{\eta_2-1}\,(1-w)^{\eta_3-\eta_2-1}\,(1-zw)^{-\eta_1} \exp\left[-\frac{\kappa}{w(1-w)}\right] dw$$
$$(\kappa \geq 0,\ |\arg(1-z)| < \pi,\ \Re(\eta_3) > \Re(\eta_2) > 0)$$

and

$$\Phi_\kappa(\eta_2;\eta_3;z) = \frac{1}{\mathbb{B}(\eta_2,\eta_3-\eta_2)} \tag{1.6}$$
$$\times \int_0^1 w^{\eta_2-1}\,(1-w)^{\eta_3-\eta_2-1} \exp\left[zw-\frac{\kappa}{w(1-w)}\right] dw$$
$$(\kappa \geq 0,\ \Re(\eta_3) > \Re(\eta_2) > 0).$$

If we set $\kappa = 0$ in (1.5) and (1.6) then we easily recover the integral representations of the classical GHF and CHF as follows (see [15] and also [16]):

$$F(\eta_1,\eta_2;\eta_3;z) = \frac{1}{\mathbb{B}(\eta_2,\eta_3-\eta_2)} \int_0^1 w^{\eta_2-1}\,(1-w)^{\eta_3-\eta_2-1}\,(1-zw)^{-\eta_1} dw \tag{1.7}$$
$$(|\arg(1-z)| < \pi,\ \Re(\eta_3) > \Re(\eta_2) > 0)$$

and

$$\Phi(\eta_2;\eta_3;z) = \frac{1}{\mathbb{B}(\eta_2,\eta_3-\eta_2)} \int_0^1 w^{\eta_2-1}\,(1-w)^{\eta_3-\eta_2-1} \exp(zw) dw \tag{1.8}$$
$$(\Re(\eta_3) > \Re(\eta_2) > 0).$$

By presenting an additional parameter $\mu$, Lee *et al.* [12] defined a further extension of (1.2) as follows:

$$\mathbb{B}_\kappa^\mu(\zeta_1,\zeta_2) = \int_0^1 w^{\zeta_1-1}\,(1-w)^{\zeta_2-1} \exp\left[-\frac{\kappa}{w^\mu(1-w)^\mu}\right] dw \tag{1.9}$$
$$(\Re(\zeta_1) > 0,\ \Re(\zeta_2) > 0,\ \Re(\kappa) > 0,\ \mu > 0).$$

The case $\mu = 1$ in (1.9), yields the extended beta function given in (1.2). Further setting $\kappa = 0$, this definition clearly reduces to the classical beta function.

The Bessel-Struve kernel function $S_\eta(\lambda u), \lambda \in \mathbb{C}$ is the unique solution of the initial value problem $L_\eta w(u) = \lambda^2 w(u)$ subject to the initial conditions $w(0) = 1$ and $w'(0) = \frac{\lambda\Gamma(\eta+1)}{\sqrt{\pi}\,\Gamma(\eta+\frac{3}{2})}$, where

$$L_\eta = \frac{d^2w(u)}{du^2} + \frac{2\eta+1}{u}\left(\frac{dw(u)}{du} - \frac{dw(0)}{du}\right)$$

is the Bessel-Struve differential operator. This function is given by (see [7] and also [11])

$$S_\eta(\lambda u) = j_\eta(i\lambda u) - ih_\eta(i\lambda u), \qquad \forall\, u \in \mathbb{C},$$

where $j_\eta$ and $h_\eta$ are the normalized Bessel and Struve functions. The series representation of the Bessel-Struve kernel function is given as follows:

$$S_\eta(u) = \frac{\Gamma(\eta+1)}{\sqrt{\pi}} \sum_{k=0}^{\infty} \frac{u^k \Gamma(\frac{k+1}{2})}{k!\,\Gamma(\frac{k}{2}+\eta+1)}. \tag{1.10}$$

Additionally, we have the following relations of Bessel-Struve kernel function with the exponential functions, modified Bessel function and Struve function (see, [7], see also [11]):

$$S_{-\frac{1}{2}}(u) = e^u, \tag{1.11}$$

$$S_{\frac{1}{2}}(u) = \frac{e^u - 1}{u}, \tag{1.12}$$

$$S_0(u) = I_0(u) + L_0(u) \tag{1.13}$$

and

$$S_1(u) = \frac{2I_1(u) + L_1(u)}{u}. \tag{1.14}$$

where $I_0, L_0$ and $I_1, L_1$ are the modified Bessel and Struve functions of order zero and one individually (see, [15], see additionally [16]).

The key object of this paper is to presented a new extension of the beta function by utilizing the Bessel-Struve kernel function (1.10). This is applied to extend the notable beta dispersion emerging in statistical distribution theory. We additionally characterize another class of the usual GHF and CHF regarding our presented beta function.

## 2 An Extended beta function

This section deals with the new development of beta function by utilizing the Bessel-Struve kernel function $S_\eta(\lambda u)$ given in (1.10):

**Definition 2.1.** The new extended beta function $\mathbb{B}_\kappa^{\mu,\eta}(\zeta_1,\zeta_2)$ for $\Re(\eta) > -1$ is defined by

$$\mathbb{B}_\kappa^{\mu,\eta}(\zeta_1,\zeta_2) = \int_0^1 w^{\zeta_1-1}\,(1-w)^{\zeta_2-1}\, S_\eta\left[-\frac{\kappa}{w^\mu(1-w)^\mu}\right] dw \tag{2.1}$$

$$(\Re(\zeta_1) > 0,\ \Re(\zeta_2) > 0,\ \kappa \geq 0,\ \mu > 0)$$

where $S_\eta(u)$ denotes the Bessel-Struve kernel function given by (1.10).

**Remark 2.2.** We note that the case $\eta = -\frac{1}{2}$ in (2.1) leads to the corresponding results given in Lee *et al.* [12], which further for $\mu = 1$ gives the familiar extension of the beta function given by Chaudhry *et al.* [5]. Obviously, when $\kappa = 0$, (2.1) reduces to the traditional beta function (1.1).

**Integral representation of $\mathbb{B}_\kappa^{\mu,\eta}(\zeta_1,\zeta_2)$**

**Theorem 2.3.** *For $\Re(\eta) > -1$, $\kappa \geq 0$ and $\mu > 0$, we have the resulting integral representations of $\mathbb{B}_\kappa^{\mu,\eta}(\zeta_1,\zeta_2)$:*

$$\mathbb{B}_\kappa^{\mu,\eta}(\zeta_1,\zeta_2) = 2\int_0^{\frac{\pi}{2}} \cos^{2\zeta_1-1} u\, \sin^{2\zeta_2-1} u\; S_\eta(-\kappa \sec^2 u\, \csc^2 u) du; \tag{2.2}$$

$$\mathbb{B}_{\kappa}^{\mu,\eta}(\zeta_1,\zeta_2)=\int_0^\infty \frac{u^{\zeta_1-1}}{(1+u)^{\zeta_1+\zeta_2}}\, S_\eta\left[-\kappa\left(2+u+\frac{1}{u}\right)\right] du; \tag{2.3}$$

$$\mathbb{B}_{\kappa}^{\mu,\eta}(\zeta_1,\zeta_2)=2^{1-\zeta_1-\zeta_2}\int_{-1}^1 (1+u)^{\zeta_1-1}\,(1-u)^{\zeta_2-1}\, S_\eta\left[-\frac{4\kappa}{(1-u^2)}\right] du. \tag{2.4}$$

*Proof.* On putting $w=\cos^2 u$, $w=\frac{u}{1+u}$ and $w=\frac{1+u}{2}$ in (2.1) gives, respectively, the integral representations (2.2)–(2.4). □

## 3 Properties of extended beta function

This section makes do with some crucial properties of our presented beta function $\mathbb{B}_{\kappa}^{\mu,\eta}(\zeta_1,\zeta_2)$.

**Theorem 3.1.** *The following functional relation holds true for extended beta function $\mathbb{B}_{\kappa}^{\mu,\eta}(\zeta_1,\zeta_2)$:*

$$\mathbb{B}_{\kappa}^{\mu,\eta}(\zeta_1,\zeta_2)=\sum_{s=0}^{l}\begin{pmatrix} l \\ s \end{pmatrix}\mathbb{B}_{\kappa}^{\mu,\eta}(\zeta_1+s,\zeta_2+l-s),\ \text{where } l\in\mathbb{N}_0. \tag{3.1}$$

*Proof.* From (2.1), we have

$$\mathbb{B}_{\kappa}^{\mu,\eta}(\zeta_1,\zeta_2)=\int_0^1 w^{\zeta_1-1}\,(1-w)^{\zeta_2-1}\{w+(1-w)\}\, S_\eta\left[-\frac{\kappa}{w^\mu(1-w)^\mu}\right] dw$$

$$\mathbb{B}_{\kappa}^{\mu,\eta}(\zeta_1,\zeta_2)=\mathbb{B}_{\kappa}^{\mu,\eta}(\zeta_1+1,\zeta_2)+\mathbb{B}_{\kappa}^{\mu,\eta}(\zeta_1,\zeta_2+1). \tag{3.2}$$

Once more, applying the similar argument on the right hand side of (3.2), we acquire

$$\mathbb{B}_{\kappa}^{\mu,\eta}(\zeta_1,\zeta_2)=\mathbb{B}_{\kappa}^{\mu,\eta}(\zeta_1+2,\zeta_2)+2\mathbb{B}_{\kappa}^{\mu,\eta}(\zeta_1+1,\zeta_2+1)+\mathbb{B}_{\kappa}^{\mu,\eta}(\zeta_1,\zeta_2+2),$$

continuing this process, by induction on $s$, we obtain the stated result. □

**Theorem 3.2.** *The extended beta function satisfies the following summation formula:*

$$\mathbb{B}_{\kappa}^{\mu,\eta}(\zeta_1,1-\zeta_2)=\sum_{s=0}^{\infty}\frac{(\zeta_2)_s}{s!}\mathbb{B}_{\kappa}^{\mu,\eta}(\zeta_1+s,1). \tag{3.3}$$

*Proof.* We have

$$(1-w)^{-\zeta_2}=\sum_{s=0}^{\infty}\frac{(\zeta_2)_s}{s!}\, w^s \qquad (|w|<1),$$

where $(a)_\ell=\Gamma(a+\ell)/\Gamma(a)$ is the Pochhammer symbol, therefore (2.1) can be written as

$$\mathbb{B}_{\kappa}^{\mu,\eta}(\zeta_1,1-\zeta_2)=\int_0^1 w^{\zeta_1-1}\,(1-w)^{-\zeta_2}\, S_\eta\left[-\frac{\kappa}{w^\mu(1-w)^\mu}\right] dw$$

$$=\int_0^1 w^{\zeta_1-1}\sum_{s=0}^{\infty}\frac{(\zeta_2)_s\, w^s}{s!}\, S_\eta\left[-\frac{\kappa}{w^\mu(1-w)^\mu}\right] dw.$$

Interchanging the order of integration and summation (which is verified by uniform convergence of the involved progression) in the last expression and further by using (2.1), we easily obtain the stated result (3.3). □

**Theorem 3.3.** *The following identity holds true for extended beta function $\mathbb{B}_{\kappa}^{\mu,\eta}(\zeta_1,\zeta_2)$:*

$$\mathbb{B}_{\kappa}^{\mu,\eta}(\zeta_1,\zeta_2)=\sum_{s=0}^{\infty}\mathbb{B}_{\kappa}^{\mu,\eta}(\zeta_1+s,\zeta_2+1). \tag{3.4}$$

*Proof.* Replacing $(1-w)^{\zeta_2-1}$ in (2.1) by its series representation

$$(1-w)^{\zeta_2-1}=(1-w)^{\zeta_2}\sum_{s=0}^{\infty}w^s$$

We easily obtain the stated result (3.4). We omit the details. □

## 4 The beta distribution of $\mathbb{B}_{\kappa}^{\mu,\eta}(\zeta_1, \zeta_2)$

In this section, we consider the possible utilization of our extended beta function by characterizing the following new extended beta distribution:

$$f(w) = \begin{cases} \frac{1}{\mathbb{B}_{\kappa}^{\mu,\eta}(\zeta_1,\zeta_2)}\, w^{\zeta_1-1}\,(1-w)^{\zeta_2-1}\, S_\eta\left[-\frac{\kappa}{w^\mu(1-w)^\mu}\right] & (0<w<1) \\ 0 & \text{otherwise} \end{cases} \tag{4.1}$$

$$(\Re(\eta) > -1,\ \zeta_1,\ \zeta_2 \in \mathbb{R},\ \kappa \geq 0,\ \mu > 0).$$

Next, we currently introduced here few crucial properties of presented beta dispersion (4.1).

If $p$ is any real number, then the $p^{th}$ moment of the above said probability density function about the origin is given by

$$E(W^p) = \frac{\mathbb{B}_{\kappa}^{\mu,\eta}(\zeta_1+p, \zeta_2)}{\mathbb{B}_{\kappa}^{\mu,\eta}(\zeta_1, \zeta_2)} \tag{4.2}$$

$$(\zeta_1,\ \zeta_2 \in \mathbb{R},\ \mu \geq 0,\ \Re(k) > -1).$$

The particular case of (4.2) when $p = 1$

$$\mu = E(W) = \frac{\mathbb{B}_{\kappa}^{\mu,\eta}(\zeta_1+1, \zeta_2)}{\mathbb{B}_{\kappa}^{\mu,\eta}(\zeta_1, \zeta_2)} \tag{4.3}$$

represents the mean of the above said presented beta dispersion.

The variance of our presented dispersion is defined by

$$Var(W) = E[(W - E(W))^2]$$

$$Var(W) = \frac{\mathbb{B}_{\kappa}^{\mu,\eta}(\zeta_1+2, \zeta_2)\, \mathbb{B}_{\kappa}^{\mu,\eta}(\zeta_1, \zeta_2) - [\mathbb{B}_{\kappa}^{\mu,\eta}(\zeta_1+1, \zeta_2)]^2}{[\mathbb{B}_{\kappa}^{\mu,\eta}(\zeta_1, \zeta_2)]^2}. \tag{4.4}$$

The C.V. of this dispersion can be determined as follows:

$$C.V = \sqrt{\frac{\mathbb{B}_{\kappa}^{\mu,\eta}(\zeta_1+2, \zeta_2)\, \mathbb{B}_{\kappa}^{\mu,\eta}(\zeta_1, \zeta_2)}{[\mathbb{B}_{\kappa}^{\mu,\eta}(\zeta_1+1, \zeta_2)]^2} - 1}. \tag{4.5}$$

The moment generating function about origin of this presented dispersion can be determined as follows:

$$M_W(u) = \sum_{s=0}^{\infty} \frac{u^s}{s!} E(W^s)$$

$$M_W(u) = \frac{1}{\mathbb{B}_{\kappa}^{\mu,\eta}(\zeta_1, \zeta_2)} \sum_{s=0}^{\infty} \mathbb{B}_{\kappa}^{\mu,\eta}(\zeta_1+s, \zeta_2) \frac{u^s}{s!}. \tag{4.6}$$

The characteristic function of the presented dispersion can be calculated as

$$E(e^{iuw}) = \sum_{s=0}^{\infty} \frac{i^s u^s}{s!} E(W^s)$$

$$E(e^{iuw}) = \frac{1}{\mathbb{B}_{\kappa}^{\mu,\eta}(\zeta_1, \zeta_2)} \sum_{s=0}^{\infty} \mathbb{B}_{\kappa}^{\mu,\eta}(\zeta_1+s, \zeta_2) \frac{i^s u^s}{s!}. \tag{4.7}$$

In probability theory and statistics, the CDF of a random variable $W$ of above mentioned extended beta dispersion (4.1) can be written as

$$F(w) = P[W < w] = \int_0^w f(w)dw$$

$$F(w) = \frac{\mathbb{B}_{\kappa}^{\mu,\eta,w}(\zeta_1,\zeta_2)}{\mathbb{B}_{\kappa}^{\mu,\eta}(\zeta_1,\zeta_2)}, \tag{4.8}$$

where $\mathbb{B}_{\kappa}^{\mu,\eta,w}(\zeta_1,\zeta_2)$ signifies the (lower) incomplete extended beta function defined by

$$\mathbb{B}_{\kappa}^{\mu,\eta,w}(\zeta_1,\zeta_2) = \int_0^w w^{\zeta_1-1}\,(1-w)^{\zeta_2-1}\,S_\eta\left[-\frac{\kappa}{w^\mu(1-w)^\mu}\right]dw.$$

The reliability function of above mentioned dispersion can be written as

$$R(w) = P[W \geq w] = 1 - F(w) = \int_w^\infty f(w)dw$$

$$R(w) = \frac{\mathbb{B}_{\kappa}^{\mu,\eta,w}(\zeta_1,\zeta_2)}{\mathbb{B}_{\kappa}^{\mu,\eta}(\zeta_1,\zeta_2)}, \tag{4.9}$$

where $\hat{\mathbb{B}}_{\kappa}^{\mu,\eta,w}(\zeta_1,\zeta_2)$ signifies the (upper) incomplete extended beta function defined by

$$\hat{\mathbb{B}}_{\kappa}^{\mu,\eta,w}(\zeta_1,\zeta_2) = \int_w^\infty w^{\zeta_1-1}\,(1-w)^{\zeta_2-1}\,S_\eta\left[-\frac{\kappa}{w^\mu(1-w)^\mu}\right]dw.$$

## 5 Extensions of Gauss and confluent hypergeometric functions and related properties

In this section, we present the following extensions of GHF and CHF by utilizing our extended beta function $\mathbb{B}_{\kappa}^{\mu,\eta}(\zeta_1,\zeta_2)$:

$$F_{\kappa}^{\mu,\eta}(\zeta_1,\zeta_2;\zeta_3;t) = \sum_{\ell=0}^{\infty} \frac{(\zeta_1)_\ell\,\mathbb{B}_{\kappa}^{\mu,\eta}(\zeta_2+\ell,\zeta_3-\zeta_2)}{\mathbb{B}(\zeta_2,\zeta_3-\zeta_2)}\frac{t^\ell}{\ell!} \tag{5.1}$$
$$(\Re(\zeta_3) > \Re(\zeta_2) > 0,\ \Re(\eta) > -1,\ \kappa \geq 0,\ \mu > 0,\ |t| < 1)$$

and

$$\Phi_{\kappa}^{\mu,\eta}(\zeta_2;\zeta_3;t) = \sum_{\ell=0}^{\infty} \frac{\mathbb{B}_{\kappa}^{\mu,\eta}(\zeta_2+\ell,\zeta_3-\zeta_2)}{\mathbb{B}(\zeta_2,\zeta_3-\zeta_2)}\frac{t^\ell}{\ell!} \tag{5.2}$$
$$(\Re(\zeta_3) > \Re(\zeta_2) > 0,\ \Re(\eta) > -1,\ \kappa \geq 0,\ \mu > 0).$$

**Remark 5.1.** We note that the case $\eta = -\frac{1}{2}$ in (5.1) and (5.2), yields the known extensions of GHF and CHF defined by Lee *et al.* [12], which further for $\mu = 1$ gives the familiar extensions of GHF and CHF defined by Chaudhry *et al.* [5]. Unmistakably, for $\kappa = 0$, (5.1) and (5.2) reduces to the traditional Gauss and confluent hypergeometric functions (see [15] and also [16]).

**Theorem 5.2.** *For the new extended Gauss also, confluent hypergeometric functions, we have the following integral representations:*

$$\begin{aligned} F_{\kappa}^{\mu,\eta}(\zeta_1,\zeta_2;\zeta_3;t) &= \frac{1}{\mathbb{B}(\zeta_2,\zeta_3-\zeta_2)} \\ &\times \int_0^1 w^{\zeta_2-1}\,(1-w)^{\zeta_3-\zeta_2-1}\,(1-tw)^{-\zeta_1}\,S_\eta\left[-\frac{\kappa}{w^\mu(1-w)^\mu}\right]dw \end{aligned} \tag{5.3}$$
$$(\kappa \geq 0,\ \mu > 0,\ |\arg(1-t)| < \pi,\ \Re(\zeta_3) > \Re(\zeta_2) > 0,\ \Re(\eta) > -1)$$

*and*

$$\Phi_{\kappa}^{\mu,\eta}(\zeta_2;\zeta_3;t) = \frac{1}{\mathbb{B}(\zeta_2,\zeta_3-\zeta_2)}\int_0^1 w^{\zeta_2-1}\,(1-w)^{\zeta_3-\zeta_2-1}\,e^{tw}\,S_\eta\left[-\frac{\kappa}{w^\mu(1-w)^\mu}\right]dw \tag{5.4}$$
$$(\kappa \geq 0,\ \mu > 0,\ \Re(\zeta_3) > \Re(\zeta_2) > 0,\ \Re(\eta) > -1).$$

*Proof.* All of the above representations can be effortlessly settled by utilizing the integral representation of the extended beta function (2.1) on the right hand sides of (5.1) and (5.2), separately. □

**Theorem 5.3.** *The extended Gauss and Confluent hypergeometric functions satisfies the following differential formulas:*

$$\frac{d^m}{dt^m}\{F_\kappa^{\mu,\eta}(\zeta_1,\zeta_2;\zeta_3;t)\} = \frac{(\zeta_1)_m(\zeta_2)_m}{(\zeta_3)_m}\, F_\kappa^{\mu,\eta}(\zeta_1+m,\zeta_2+m;\zeta_3+m;t) \tag{5.5}$$

$$(\kappa \geq 0,\ \mu > 0,\ \Re(\eta) > -1,\ m \in \mathbb{N}_0)$$

*and*

$$\frac{d^m}{dt^m}\{\Phi_\kappa^{\mu,\eta}(\zeta_2;\zeta_3;t)\} = \frac{(a_2)_m}{(a_3)_m}\,\Phi_\kappa^{\mu,\eta}(\zeta_2+m;\zeta_3+m;t) \tag{5.6}$$

$$(\kappa \geq 0,\ \mu > 0,\ \Re(\eta) > -1,\ m \in \mathbb{N}_0).$$

*Proof.* On differentiating (5.1) with regard to $t$, we get

$$\frac{d}{dt}\{F_\kappa^{\mu,\eta}(\zeta_1,\zeta_2;\zeta_3;t)\} = \sum_{l=1}^{\infty}\frac{(\zeta_1)_\ell\, \mathbb{B}_\kappa^{\mu,\eta}(\zeta_2+\ell,\zeta_3-\zeta_2)}{\mathbb{B}(\zeta_2,\zeta_3-\zeta_2)}\frac{t^{\ell-1}}{(\ell-1)!}.$$

On replacing $l$ by $l+1$, we have

$$\frac{d}{dt}\{F_\kappa^{\mu,\eta}(\zeta_1,\zeta_2;\zeta_3;t)\} = \sum_{l=0}^{\infty}\frac{(\zeta_1)_{\ell+1}\, \mathbb{B}_\kappa^{\mu,\eta}(\zeta_2+\ell+1,\zeta_3-\zeta_2)}{\mathbb{B}(\zeta_2,\zeta_3-\zeta_2)}\frac{t^{\ell}}{\ell!}.$$

Now by using $\mathbb{B}(\zeta_2,\zeta_3-\zeta_2) = \frac{\zeta_3}{\zeta_2}\,\mathbb{B}(\zeta_2+1,\zeta_3-\zeta_2)$ and $(\zeta_1)_{\ell+1} = \zeta_1(\zeta_1+1)_\ell$, we get

$$\frac{d}{dt}\{F_\kappa^{\mu,\eta}(\zeta_1,\zeta_2;\zeta_3;t)\} = \frac{\zeta_1\zeta_2}{\zeta_3}\sum_{l=0}^{\infty}\frac{(\zeta_1+1)_\ell\, \mathbb{B}_\kappa^{\mu,\eta}(\zeta_2+\ell+1,\zeta_3-\zeta_2)}{\mathbb{B}(\zeta_2+1,\zeta_3-\zeta_2)}\frac{t^{\ell}}{\ell!}. \tag{5.7}$$

$$= \frac{\zeta_1\zeta_2}{\zeta_3}F_\kappa^{\mu,\eta}(\zeta_1+1,\zeta_2+1;\zeta_3+1;t).$$

Again differentiating (5.7) with respect to $t$, we get

$$\frac{d^2}{dt^2}\{F_\kappa^{\mu,\eta}(\zeta_1,\zeta_2;\zeta_3;t)\} = \frac{\zeta_1(\zeta_1+1)\zeta_2(\zeta_2+1)}{\zeta_3(\zeta_3+1)}F_\kappa^{\mu,\eta}(\zeta_1+2,\zeta_2+2;\zeta_3+2;t).$$

Continuing this process, we acquire the desired result (5.5).
Also we can set up the result (5.6). □

**Theorem 5.4.** *The extended Gauss and Confluent hypergeometric functions satisfies the following transformation formulas:*

$$F_\kappa^{\mu,\eta}(\zeta_1,\zeta_2;\zeta_3;t) = (1-t)^{-\zeta_1}F_\kappa^{\mu,\eta}\left(\zeta_1,\zeta_3-\zeta_2;\zeta_2;-\frac{t}{(1-t)}\right) \tag{5.8}$$

$$(\kappa \geq 0,\ \mu > 0,\ \Re(\eta) > -1)$$

*and*

$$\Phi_\kappa^{\mu,\eta}(\zeta_2;\zeta_3;t) = e^t\Phi_\kappa^{\mu,\eta}(\zeta_3-\zeta_2;\zeta_3;-t) \tag{5.9}$$

$$(\kappa \geq 0,\ \mu > 0,\ \Re(\eta) > -1).$$

*Proof.* On supplanting $w$ by $1-w$ in (5.3) and afterward by utilizing $[1-t(1-w)]^{-\zeta_1} = (1-t)^{-\zeta_1}\left[1+\frac{t}{1-t}w\right]^{-\zeta_1}$, we obtain

$$F_\kappa^{\mu,\eta}(\zeta_1,\zeta_2;\zeta_3;t);t) = \frac{(1-t)^{-\zeta_1}}{\mathbb{B}(\zeta_2,\zeta_3-\zeta_2)}$$

$$\times\int_0^1 w^{\zeta_3-\zeta_2-1}(1-w)^{\zeta_2-1}\left(1+\frac{t}{1-t}w\right)^{-\zeta_1}S_\eta\left[-\frac{\kappa}{w^\mu(1-w)^\mu}\right]dw,$$

which in view of (5.3), yields the right hand side of (5.8). Likewise, we can set up (5.9). □

**Theorem 5.5.** *The extended Gauss and Confluent hypergeometric functions satisfies the following summation formulas:*

$$F_{\kappa}^{\mu,\eta}(\zeta_1,\zeta_2;\zeta_3;1)=\frac{\mathbb{B}_{\kappa}^{\mu,\eta}(\zeta_2,\zeta_3-\zeta_1-\zeta_2)}{\mathbb{B}(\zeta_2,\zeta_3-\zeta_2)} \tag{5.10}$$

$$(\kappa\geq 0,\ \mu>0,\ \Re(\eta)>-1,\ \Re(\zeta_3-\zeta_1-\zeta_2)>0).$$

*Proof.* On setting $t=1$ in (5.3) and then by using (2.1), we arrive at our stated result (5.10). □

**Theorem 5.6.** *The extended Gauss hypergeometric function satisfies the following generating function:*

$$\sum_{\ell=0}^{\infty}(\zeta_1)_\ell\, F_{\kappa}^{\mu,\eta}(\zeta_1+\ell,\zeta_2;\zeta_3;t)\frac{w^\ell}{\ell!}=(1-w)^{-\zeta_1}F_{\kappa}^{\mu,\eta}\left(\zeta_1,\zeta_2;\zeta_3;\frac{t}{1-w}\right) \tag{5.11}$$

$$(\kappa\geq 0,\ \mu>0,\ \Re(\eta)>-1,\ |w|<1).$$

*Proof.* On utilizing (5.1) in the left hand side of (5.11), we acquire

$$\sum_{\ell=0}^{\infty}(\zeta_1)_\ell\, F_{\kappa}^{\mu,\eta}(\zeta_1+\ell,\zeta_2;\zeta_3;t)\frac{w^\ell}{\ell!}$$

$$=\sum_{\ell=0}^{\infty}(\zeta_1)_\ell\left[\sum_{m=0}^{\infty}\frac{(\zeta_1+\ell)_m\,\mathbb{B}_{\kappa}^{\mu,\eta}(\zeta_2+m,\zeta_3-\zeta_2)}{\mathbb{B}(\zeta_2,\zeta_3-\zeta_2)}\frac{t^m}{m!}\right]\frac{w^\ell}{\ell!}.$$

Now by using the identity $(\zeta)_m(\zeta+m)_\ell=(\zeta)_\ell(\zeta+\ell)_m$, in the above expression, we obtain

$$\sum_{\ell=0}^{\infty}(\zeta_1)_\ell\, F_{\kappa}^{\mu,\eta}(\zeta_1+\ell,\zeta_2;\zeta_3;t)\frac{w^\ell}{\ell!}$$

$$=\sum_{m=0}^{\infty}(\zeta_1)_m\frac{\mathbb{B}_{\kappa}^{\mu,\eta}(\zeta_2+m,\zeta_3-\zeta_2)}{\mathbb{B}(\zeta_2,\zeta_3-\zeta_2)}\left[\sum_{\ell=0}^{\infty}(\zeta_1+m)_\ell\frac{w^\ell}{\ell!}\right]\frac{t^m}{m!}.$$

$$=\sum_{m=0}^{\infty}(\zeta_1)_m\frac{\mathbb{B}_{\kappa}^{\mu,\eta}(\zeta_2+m,\zeta_3-\zeta_2)}{\mathbb{B}(\zeta_2,\zeta_3-\zeta_2)}(1-w)^{-(\zeta_1)+m)}\frac{t^m}{m!},$$

which upon further use of (5.1), yields the stated result (5.11). □

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