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THE G-VERTEX COLORED PARTITION ALGEBRA AS A CENTRALIZER ALGEBRA OF $A_n \times G$

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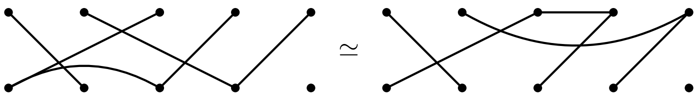
Abstract. The generalized Jones result in A J Kennedy *et al.*(2004) which says $P_k(x, G)$, a G-vertex colored partition is a centralizer algebra of the action of the direct product of S_n, symmetric group and G (i.e., $S_n \times G$) on the tensor products of its permutation representation. In this paper, specifically we restricted the action of the direct product of S_n, symmetric group and G (i.e., $S_n \times G$) to the action of the direct product of an alternating group and G (i.e., $A_n \times G$). Herein, we determine the basis for the centralizer algebra and exhibit that at that moment, the centralizer is isomorphic to $P_k(x, G)$. Also, we do the same for $\widehat{P}_k(x, G)$, the Extended G-vertex colored partition algebras.

1 Introduction

At the beginning of the 1990s, V.F.R. Jones and P. Martin studied partition algebra individually as generalizations of the Temperley-Lieb algebras and the Potts model in statistical mechanics. In [5, 6], and exclusively in [7], the partition algebras were presented completely. They started putting effort into understanding partition algebra by taking inspiration from [6] and related problems. After that, Martin and others minutely studied partition algebra's representation theory and structure. In [3], Jones observed that the partition algebra is the centralizer algebra of the symmetric group S_n on $V^{\otimes k}$. Therefore, generalized Jones result in A J Kennedy *et al.* (2004) which says "$P_k(x, G)$ G-vertex colored partition is centralizer algebra of action of the direct product of S_n symmetric group and G (i.e., $S_n \times G$) on the tensor products of its permutation representation" is restricted to the action of the direct product of A_n alternating group and G (i.e., $A_n \times G$). Hence, we determine the basis for the centralizer algebra and exhibit at that moment, the centralizer is isomorphic to $P_k(x, G)$. Also, we do the same for $\widehat{P}_k(x, G)$, Extended G-vertex colored partition algebras.
Let permutation representation of S_n symmetric group is V. Let A_n be an alternating group that acts diagonally on $V^{\otimes k}$ by restriction. In [2], Bloss proved that for $n \geq 2k+2$, $End_{A_n}(V^{\otimes k}) \cong P_k(x)$. Let $S_n \times G$ be the direct product of any finite group G and the symmetric group S_n. In this paper, Let $W = \mathbb{C}^{n|G|}$ in [8] shown that for $n \geq 2k$, $End_{S_n \times G}(W^{\otimes k}) \cong P_k(x, G)$, and here we show that for $n \geq 2k+2$, $End_{A_n \times G}(W^{\otimes k}) \cong P_k(x, G)$, where $P_k(x, G)$ denotes the G-vertex color partition algebra. We have given a clear calculation of the basis for $End_{A_n \times G}(W^{\otimes k})$. We compute the dimension of $End_{A_n \times G}(W^{\otimes} k)$ and describe when $End_{A_n \times G}(W^{\otimes} k)$ is simply the G-vertex color partition algebra $P_k(x, G)$ and also we do the same for $\hat{P}_k(x, G)$, Extended G-vertex colored partition algebras which were studied in [10].

2 Partition Algebras $P_k(x)$

Here two rows each have k-vertices; one above another is a simple graph called *k-partition diagram.* The partition diagram p, with $2k$ vertices into l distinct subsets where $1 \leq l \leq 2k$ is connected components of p. It is clear that if any k-partition diagrams form a similar partition with the same number of vertices, it is equivalent. The following two 5-diagrams are equivalent to the aforementioned.

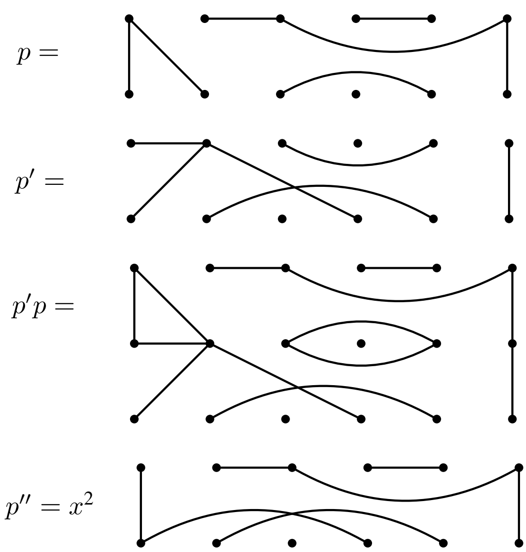


One thing essential here is when talking about diagrams, it precisely means the equivalence class. Let us name the vertices of the top row in k-diagram from left to right $1, 2, \ldots, k$ and vertices of the bottom row from left to right $k+1, k+2, \ldots, 2k$.

In this article, a field of any arbitrary characteristic is denoted by F, and an element of F will be denoted as x. Here the multiplication is defined by diagram concatenation as follows where p and p' be k-partition diagrams :

- Put p above p'.
- Combine the $(k+j)^{\text{th}}$ vertex in p with j^{th} vertex in p' where $j = 1, 2, \ldots, k$. Now the partition diagram has three rows of vertices, top, bottom, and middle.
- Let p'' is a diagram obtained by taking the upper and lower row particularly, exchange every "component" by a variable x where the components are entirely present in the middle row (i.e., $p'p = x^{\lambda}p''$ such that λ represent the number of components we exchange).

See the following diagram,



The above multiplication is well-defined and associative.

Let $x \in F$ and $k \in \mathbb{Z}_{\geq 1}$, then the *partition algebra* is indicated by $P_k(x)$, which is F-span of k-partition diagrams, and it is an associative algebra with identity e. Here e is an *identity* of this algebra, defined as the k partition diagram in which every j^{th} vertex in the first row is connected to j' vertex in the second row corresponding to each j where $j = 1, 2, \ldots, k$ and $j' = k+1, k+2, \ldots, 2k$. The dimension of partition algebra $P_k(x)$ is *Bell number* $B(2k)$, that is

$$B(2k) = \sum_{l=1}^{2k} S(2k, l)$$

$S(2k, l)$ is a *Stirling number* [11]. By convenience, $P_0(x) = F$.

2.1 G-vertex Colored Partition Algebras $P_k(x, G)$

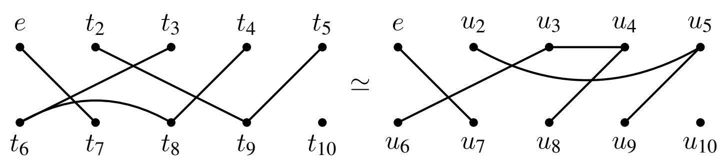
Here we take any finite group G, and in k-partition diagram, every vertex is named by an element of G, then it is (G, k)-*partition diagram*. Hereafter, let us denote (G, k)-partition diagrams by G-diagrams whenever k is known. In G-diagram, whenever we say *bottom* (resp. *top*) *label sequence* are the k-sequence of the bottom (resp. top) row labels, read from left to right. Combine *bottom* and *top* label sequence of G-diagram which are $2k-$sequence and is defined as *label sequence* of G-diagram.

Now G-diagram $\dot{p}$ with underlying partition diagram p. Every $h \in G$, we formulate G-diagram $h(\dot{p})$, by left multiple of h to the label sequence of $\dot{p}$. In this, we name the first vertex of the top row by e, an identity element of G.

Two G-diagrams $\dot{p}_1$ and $\dot{p}_2$ are said to be equivalent iff $\dot{p}_1 = h(\dot{p}_2)$ for some $h \in G$. For any two G-vertex colored diagrams are equivalent if

- Given partition diagrams are equivalent.
- Vertex labels on (G, k) are equal correspondingly.

Below, we show the equivalence of diagrams as follows: let $t_b, u_c \in G$ $(2 \leq b, c \leq 10)$:



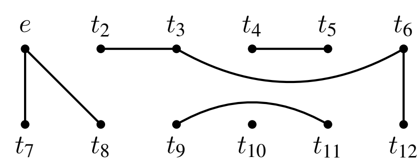
If and only if $t_2 = u_2, t_3 = u_3, \ldots, t_{10} = u_{10}$. Here, the G-diagrams are nothing but their equivalence class. Whenever G is an infinite group, then there exists an infinite number of (G, k)-diagrams; otherwise, the number of (G, k)-diagrams are $|G|^{2k-1}B(2k)$.

Now the product of the two (G, k)-diagrams $\dot{p}_1$ and $\dot{p}_2$ are:

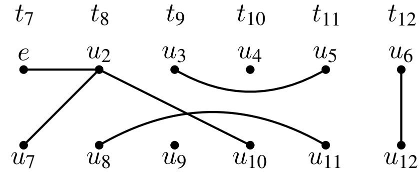
- The underlying partition diagram for the G-diagrams $\dot{p}_2\dot{p}_1$ was obtained by multiplying the underlying partition diagrams $\dot{p}_1$ and $\dot{p}_2$.
- The bottom and top label sequence of $\dot{p}_2\dot{p}_1$ are bottom and top label sequence of $h(\dot{p}_2)$ and $\dot{p}_1$, provided the top label sequence of $h(\dot{p}_2)$ is same as bottom label sequence of $\dot{p}_1$ for certain $h \in G$. If $\dot{p}_1\dot{p}_2 = 0$, provided top label sequence of $h(\dot{p}_2)$ differs from that of the bottom label sequence of $\dot{p}_1$.
- A power of x is obtained from every connected component, which is completely in the middle row when multiplying.

It can be illustrated for $t_b, u_c \in G$ $(2 \leq b, c \leq 12)$, and $e \in G$ is an identity element.

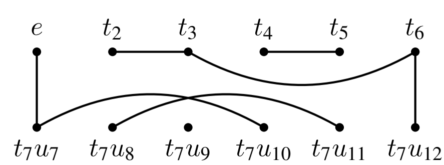
$\dot{p}_1 =$



$\dot{p}_2 =$

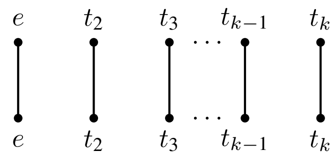


$\dot{p}_2\dot{p}_1 = x^2\delta^{(t_7,t_8,\ldots,t_{12})}_{(t_7,t_7u_2,\ldots,t_7u_6)}$



The multiplication is generally associative in nature and well-defined until the equivalence of G-diagrams. Hence, $P_k(x, G)$ is F-span of (G, k)-diagrams under the overhead product and it is an associative algebra with an identity element, then it is called *G-vertex colored partition* *algebra*. Identity element of $P_k(x, G)$

$$\sum_{t_2, t_3, \ldots, t_k \in G}$$



The dimension of $P_k(x, G)$ is $|G|^{2k-1}B(2k)$ which denotes the number of equivalence relations on $2k$ vertices and dimension signifies the number of (G, k)-diagrams.

In case, H being a subgroup of G, the subalgebra of $P_k(x, G)$, which is denoted as $P_k(x, G_H)$, where it is spanned by the diagrams in $P_k(x, G)$, which are named here by using elements of H, is isomorphic to $P_k(x, H)$. For $H = \{e\}$ we have $P_k(x, H) \simeq P_k(x)$. For G being an infinite group, $P_k(x, G)$ is defined as an infinite dimensional associative algebra.

2.2 Structure of $\widehat{P}_k(x, G)$

We postulate without specifying anything on equivalence relation to the definition of a different multiplication $(*)$ on the (G, k)-diagrams:

Two diagrams possessing the nature of (G, k) - diagrams are denoted as (p, t) and (p', t') and $t = (t_1, t_2),\ t' = (t'_1, t'_2) \in G^{2k}$.

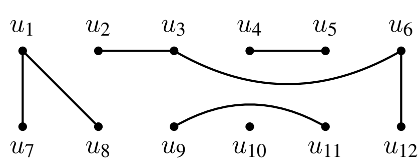
$$(p', t') * (p, t) = \begin{cases} x^{\lambda}(p'', (t_1, t'_2)) & \text{if } t_2 = t'_1 \\ 0 & \text{otherwise,} \end{cases}$$

in which $p'p = x^{\lambda}p''$. From the above definition, we equivalently state that the $*$ of two G-diagrams (p, t) and (p', t') are:

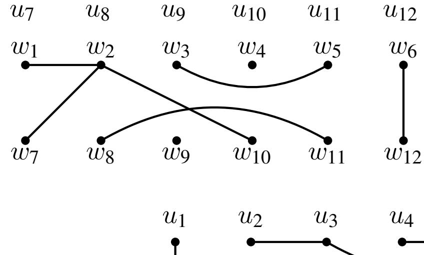
- Let $(p', t') * (p, t)$ is obtained by multiplying the underlying partition diagrams (p, t) and (p', t').
- The top and bottom label sequence of $(p', t') * (p, t)$ are the top and bottom label sequence of (p, t) and (p', t'), provided the bottom label sequence of (p, t) is equal to the top label sequence of (p', t'). If $(p', t') * (p, t) = 0$, provided the top label sequence of (p', t') differs from the bottom label sequence of (p, t).
- A factor of x is obtained in the multiplication from every entirely connected component in the middle row.

It can be illustrated for $u_b, w_c \in G$ $(1 \leq b, c \leq 12)$.

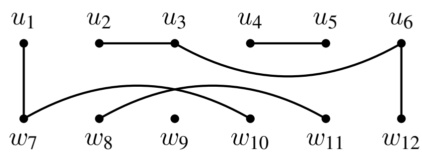
$(p, t) =$



$(p', t') =$



$(p', t') * (p, t) = x^2\delta^{(u_7, u_8, \ldots, u_{12})}_{(w_1, w_2, \ldots, w_6)}$

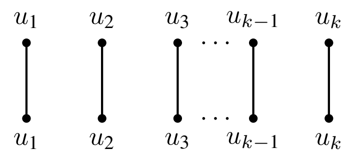


Notation $\delta^{(u_7,u_8,\ldots,u_{12})}_{(w_1,w_2,\ldots,w_6)}$ is the Kronecker delta, which is defined as

$$\delta^{(u_7,u_8,\ldots,u_{12})}_{(w_1,w_2,\ldots,w_6)} = \begin{cases} 1 & \text{if } (u_7,u_8,...,u_{12}) = (w_1,w_2,...,w_6) \\ 0 & \text{if } (u_{7,8},...,u_{12}) \neq (w_1,w_2,...,w_6). \end{cases}$$

The multiplication $*$ is generally associative, the equivalence of (G,k)-diagrams are well defined. Let $\widehat{P}_k(x,G)$ is *Extended G-Vertex Colored Partition Algebra* is an associative algebra with an identity element e, as is the F-span of the (G,k)-diagrams under $*$. Take p to be the identity partition diagram, then identity element of $\widehat{P}_k(x,G)$ is $\sum_{\substack{f\in G^{2k}\\ t_1=t_2}}(p,t)$ (i.e).,

$$\sum_{u_1,u_2,u_3,\ldots,u_k\in G}$$



Let N is a subgroup of group G, then $\widehat{P}_k(x,N)$ is a subalgebra of $\widehat{P}_k(x,G)$. For $N=\{e\}$, we have $\widehat{P}_k(x,H)\simeq P_k(x)$. For G being finite, then the dimension of $\widehat{P}_k(x,G)$ is $|G|^{2k}B(2k)$ which is denoted by the number of equivalence relations of $2k$ vertices and dimension signifies the number of (G,k)-diagrams. Otherwise, $\widehat{P}_k(x,G)$ is defined as an infinite dimensional associative algebra.

3 Two bases

3.1 Two bases for $\mathbf{End}_{S_n\times G}(W^{\otimes k})$

Here for finite group which is denoted by G and $W=\mathbb{C}^{n|G|}$, we define two bases for End$_{S_n\times G}(W^{\otimes k})$. For a symmetric group S_n and a finite group G, which is arbitrary, the direct product of $S_n\ and\ G$ is denoted as $S_n\times G$. For all such permutations in S_n, which fixes the n^{th} symbol and it possesses the group structure and it is a subgroup of S_n isomorphic to S_{n-1}; therefore, S_{n-1} is a subgroup of $S_n\times G$ and which in turn is isomorphic to $S_{n-1}\times e$.

Let direct product $S_n\times G$ consist of elements that can be specified as π_g where $\pi\in S_n$ and $g\in G$. For any two different elements $\pi_g,\ \sigma_g\in S_n\times G$, the multiplication is defined as $\pi_g\sigma_h=(\pi\sigma)_{gh}$. Moreover, the order of such direct product is $|G|n$.

Define $W=$ Span$_{\mathbb{C}}\{w_{(i,h)}\mid h\in G;\ 1\le i\le n\}$ is $S_n\times G$-permutation module by the action $\pi_g w_{(i,h)}=w_{\pi_g(i,h)}=w_{(\pi(i),gh)}$. To be precise, G is a group with a unique element, $S_n\times\{e\}\cong S_n$ and then W is similar to V, which is the permutation representation of S_n.

Define $I=((i_1,g_1),(i_2,g_2),\ldots,(i_k,g_k)),\ \ J=((i_{k+1},g_{k+1}),(i_{k+2},g_{k+2}),\ldots,(i_{2k},g_{2k}))\ in\,\mathbb{S}^k$. Definition of action of $S_n\times G$ on $\mathbb{S}^{2k}$ is defined by $\pi_g(I,J)=(\pi_g(I),\pi_g(J))$, where it is an extension of componentwise action of $S_n\times G$ on $\mathbb{S}$ which is $\pi_g(i,h)=(\pi(i),gh)$.

Let the action of $S_n\times G$ on W diagonally extend to an action of $S_n\times G$ on $W^{\otimes k}$ is:

$$\pi_g(w_{(i_1,g_1)}\otimes\cdots\otimes w_{(i_k,g_k)})=w_{(\pi(i_1),gg_1)}\otimes\cdots\otimes w_{(\pi(i_k),gg_k)},$$

such that $\pi_g\in S_n\times G$. The action overhead is written as $\pi_g(w_I)=w_{\pi_g(I)}$.

Suppose $A\in$ End$(W^{\otimes k})$. Define $A(w_J)=\sum_I A^J_I(w_I)$, such that $A^J_I\in\mathbb{C}$ is the $(I,J)^{th}$ entry of A $(I,J\in\mathbb{S}^k)$ and w_I is a basis element of $W^{\otimes k}$.

One of the analogs of Jones's result is stated by AJ Kennedy [9].

Lemma 3.1. $A\in$ End$_{S_n\times G}(W^{\otimes k})\ \Leftrightarrow\ A^J_I=A^{\pi_g(J)}_{\pi_g(I)}$ for all $\pi_g\in S_n\times G$.

Lemma 3.2.

$$\text{dim End}_{S_n\times G}(W^{\otimes k})=|G|^{2k-1}\sum_{l=1}^{n}S(2k,l).$$

If $n\ge 2k$,

$$\text{dim End}_{S_n\times G}(W^{\otimes k})\ =\ |G|^{2k-1}\,B(2k).$$

A matrix $T^I_J \in \text{End}(W^{\otimes k})$ is defined for each orbit $[(I,J)]$, in $S_n \times G$ as

$$T^I_J = \sum_{(I',J')\in[(I,J)]} E^{I'}_{J'}, \tag{3.1}$$

in which $E^{I'}_{J'}$ is the matrix unit, where for every nonzero entry at $(J',I')^{th}$ position, the value is 1. Interestingly, $T^I_J \in \text{End}_{S_n\times G}(W^{\otimes k})$, as it satisfies the requirements of the lemma 3.1, according to which the matrix entries are equivalent on $S_n \times G$-orbits. Now with the application of lemma 3.2, we obtain

$$T^{(i_1,g_1),(i_2,g_2),\ldots,(i_k,g_k)}_{(i_{k+1},g_{k+1}),(i_{k+2},g_{k+2}),\ldots,(i_{2k},g_{2k})} = \sum E^{(j_1,gg_1),(j_2,gg_2),\ldots,(j_k,gg_k)}_{(j_{k+1},gg_{k+1}),(j_{k+2},gg_{k+2}),\ldots,(j_{2k},gg_{2k})}, \tag{3.2}$$

For the summation being taken over $g \in G$ and $i_p = i_q \Leftrightarrow j_p = j_q$, $(1 \le p,q \le 2k)$. As every T^I_J is nothing but the sum of different matrix units and the set $\{T^I_J \mid [(I,J)] \text{ is an } S_n \times G-\text{orbit}\}$ is a linearly independent set. Given any $A \in \text{End}_{S_n\times G}(W^{\otimes k})$, the lemma 3.1 is used to obtain: $A = \sum_{[(I,J)]} A^I_J T^I_J$. Thus, the matrices T^I_J span $\text{End}_{S_n\times G}(W^{\otimes k})$, hence they are a basis for $\text{End}_{S_n\times G}(W^{\otimes k})$.

Lemma 3.3. *[[4] (2.2.4)]*

$$\left(L^{(i_1,g_1),(i_2,g_2),\ldots,(i_k,g_k)}_{(i_{k+1},g_{k+1}),(i_{k+2},g_{k+2}),\ldots,(i_{2k},g_{2k})}\right)\left(L^{(j_1,h_1),(j_2,h_2),\ldots,(j_k,h_k)}_{(j_{k+1},h_{k+1}),(j_{k+2},h_{k+2}),\ldots,(j_{2k},h_{2k})}\right) = 0$$

$\Leftrightarrow\ g'(g_1,g_2,\ldots,g_k) \neq (h_{k+1},h_{k+2},\ldots,h_{2k})$ for all $g' \in G$.

Lemma 3.4. *[[4](2.2.5)] Given any $g' \in G$,*

$$\begin{aligned}&\left(L^{(i_1,g_1),(i_2,g_2),\ldots,(i_k,g_k)}_{(i_{k+1},g_{k+1}),(i_{k+2},g_{k+2}),\ldots,(i_{2k},g_{2k})}\right)\left(L^{(j_1,h_1),(j_2,h_2),\ldots,(j_k,h_k)}_{(j_{k+1},g'g_1),(j_{k+2},g'g_2),\ldots,(j_{2k},g'g_k)}\right)\\ &= x^\lambda \sum_{\substack{h\in G\\ 1\le s_1,s_2,\ldots,s_{2k}\le n\\ p\sim q \text{ in } d\ \Rightarrow\ s_p=s_q}} E^{(s_1,hh_1),(s_2,hh_2),\ldots,(s_k,hh_k)}_{(s_{k+1},hg'g_{k+1}),(s_{k+2},hg'g_{k+2}),\ldots,(s_{2k},hg'g_{2k})},\end{aligned} \tag{3.3}$$

in which λ denotes the number of middle components in the product
$[d(i_1,i_2,\ldots,i_{2k})][d(j_1,j_2,\ldots,j_{2k})] := x^\lambda\ d$*. Furthermore, when $n \ge 2k$,* (3.3) *is equal to*

$$x^\lambda\ L^{(s_1,h_1),(s_2,h_2),\ldots,(s_k,h_k)}_{(s_{k+1},g'g_{k+1}),(s_{k+2},g'g_{k+2}),\ldots,(s_{2k},g'g_{2k})},$$

where $[d(i_1,i_2,\ldots,i_{2k})][d(j_1,j_2,\ldots,j_{2k})] = x^\lambda\ d(s_1,s_2,\ldots,s_{2k})$.

3.2 Two bases for $\text{End}_{S_n}(W^{\otimes k})$

We restricted the action of S_n on W and it is defined as $\pi(v_{(i,g)}) = v_{(\pi(i),g)}$.

For $W = \mathbb{C}^{n|G|}$, here we form two bases for $\text{End}_{S_n}(W^{\otimes k})$, and also we define the action of S_n on $W^{\otimes k}$ which is a diagonal action as:

Now $\mathbb{S} = [n] \times G$ and $I = ((i_1,g_1),(i_2,g_2),\ldots,(i_k,g_k))$,
$J = ((i_{k+1},g_{k+1}),(i_{k+2},g_{k+2}),\ldots,(i_{2k},g_{2k}))$ be in $\mathbb{S}^k$. The action of S_n on $\mathbb{S}$ is given by

$$\pi(i,g) = (\pi(i),g) \tag{3.4}$$

possibly we extended component-wise to an action on $\mathbb{S}^{2k}$ which is defined by $\pi(I,J) = (\pi(I),\pi(J))$, and Diagonally the action of S_n on W extends to an action of S_n on $W^{\otimes k}$: for $\pi \in S_n$

$$\pi(v_{(i_1,g_1)} \otimes \cdots \otimes v_{(i_k,g_k)}) = v_{(\pi(i_1),g_1)} \otimes \cdots \otimes v_{(\pi(i_k),g_k)} \tag{3.5}$$

The action above is written as $\pi(v_I) = v_{\pi(I)}$.

For $A \in \text{End}(W^{\otimes k})$, we state $A(v_J) = \sum_I A^J_I(v_I)$, where $A^J_I \in \mathbb{C}$ is the $(I,J)^{th}$ entry of A, $(I,J \in \mathbb{S}^k)$ and v_I is a basis element of $W^{\otimes k}$. We have,

$$\text{End}_{S_{n|G|}}(W^{\otimes k}) \subseteq \text{End}_{G\wr S_n}(W^{\otimes k}) \subseteq \text{End}_{S_n\times G}(W^{\otimes k}) \subseteq \text{End}_{S_n}(W^{\otimes k}).$$

The below-mentioned lemma is the corresponding analogue of the Jones result.

Lemma 3.5. $A \in \mathrm{End}_{S_n}(W^{\otimes k}) \Leftrightarrow A_I^J = A_{\pi(I)}^{\pi(J)}$ for all $\pi \in S_n$.

Lemma 3.6.

$$\dim\ \mathrm{End}_{S_n}(W^{\otimes k}) = |G|^{2k} \sum_{l=1}^{n} S(2k, l).$$

When $n \geq 2k$,

$$\dim\ \mathrm{End}_{\ S_n}(W^{\otimes k}) \ = \ |G|^{2k}\ B(2k).$$

4 Schur - Weyl Duality

4.1 Schur - Weyl Duality of $P_k(x, G)$

Let W is the permutation representation of $S_n \times G$ (with respect to S_{n-1}), then we have the diagonal action of $S_n \times G$ on $W^{\otimes k}$. We obtain an action of $P_k(x, G)$ on $W^{\otimes k}$, stated here by numbering the vertices of a (G, k)-diagram as $1, 2, \ldots k$ from left to right in the top row, and in the similar format $k+1, k+2, \ldots, 2k$ for the bottom row. A map $\phi : P_k(x, G) \longrightarrow \mathrm{End}(W^{\otimes k})$ is defined on a G-diagram $\dot{d}$ with any label sequence $(e, g_2, \ldots, g_k, g_{k+1}, g_{k+2}, \ldots, g_{2k})$, then

$$\begin{aligned} \phi(\dot{d}) &= (\phi(\dot{d})^{(i_1,h_1),(i_2,h_2),\ldots,(i_k,h_k)}_{(i_{k+1},h_{k+1}),(i_{k+2},h_{k+2}),\ldots,(i_{2k},h_{2k})}) \\ &= (\psi(d)^{i_1,i_2,\ldots,i_k}_{i_{k+1},i_{k+2},\ldots,i_{2k}} \quad \delta^{h_1(e,g_2,g_3,\ldots,g_{2k})}_{(h_1,h_2,\ldots,h_{2k})}) \ , \end{aligned}$$

in which $\delta^{h_1(e,g_2,g_3,\ldots,g_{2k})}_{(h_1,h_2,\ldots,h_{2k})}$ is the Kronocker delta and $\psi(d)^{i_1,i_2,\ldots,i_k}_{i_{k+1},i_{k+2},\ldots,i_{2k}}$ is defined as in equation [4](1.7). In other words, with respect to the matrix unit, we have

$$\phi(\dot{d}) = \sum_{\substack{g\in G \\ p\sim q \ in \ d \Rightarrow i_p = i_q \\ 1\leq i_1,i_2,\ldots,i_{2k}\leq n}} E^{(i_1,g),(i_2,gg_2),\ldots,(i_k,gg_k)}_{(i_{k+1},gg_{k+1}),(i_{k+2},gg_{k+2}),\ldots,(i_{2k},gg_{2k})}. \tag{4.1}$$

With respect to which an action of $P_k(x, G)$ on $W^{\otimes k}$ is defined by

$$\dot{d}(v_J) = \phi(\dot{d})(v_J) \ \text{ for all } \ J \in \mathbb{S}^k.$$

Let G be a group with only one element then the above-defined action restricts the action of the partition algebra as defined in [3] on tensors.

Therefore, the action of a G-partition diagram $\dot{d} \in P_k(x, G)$ on $W^{\otimes k}$ is stated with respect to standard basis as

$$\begin{aligned} &\dot{d}.(v_{(i_1,h_1)} \otimes v_{(i_2,h_2)} \otimes \cdots \otimes v_{(i_k,h_k)}) \\ = \delta^{h_1(e,g_2,g_3,\ldots,g_{2k})}_{(h_1,h_2,\ldots h_{2k})} &\sum_{1\leq i_{k+1},i_{k+2},\ldots,i_{2k}\leq n} \psi(d)^{i_1,i_2,\ldots,i_k}_{i_{k+1},i_{k+2},\ldots,i_{2k}} v_{(i_{k+1},h_{k+1})} \otimes \cdots \otimes v_{(i_{2k},h_{2k})} \end{aligned}$$

Lemma 4.1. *The map* $\phi : P_k(x, G) \longrightarrow \mathrm{End}(W^{\otimes k})$ *is an algebra homomorphism onto* $\mathrm{End}_{S_n\times G}(W^{\otimes k})$.

Proof. We have the (G, k)-diagram $\dot{d}$ with underlying partition diagram d, which are the label sequence $(g_1, g_2, \ldots, g_{2k})$. From (4.1), we have,

$$\phi(\dot{d}) = \sum_{\substack{g\in G \\ d(i_1,i_2,\ldots,i_{2k})\leq d}} E^{(i_1,gg_1),(i_2,gg_2),\ldots,(i_k,gg_k)}_{(i_{k+1},gg_{k+1}),(i_{k+2},gg_{k+2}),\ldots,(i_{2k},gg_{2k})} \tag{4.2}$$

where $1 \leq i_1, i_2, \ldots, i_{2k} \leq n$.

$$\text{(i.e.,)} \quad \phi(\dot{d}) = \sum_{d(i_1,i_2,\ldots,i_{2k})\leq d} T^{(i_1,g_1),(i_2,g_2),\ldots,(i_k,g_k)}_{(i_{k+1},g_{k+1}),(i_{k+2},g_{k+2}),\ldots,(i_{2k},g_{2k})}, \tag{4.3}$$

where the sum is over one representative $(i_1, g_1), (i_2, g_2), \ldots, (i_{2k}, g_{2k})$ for one $S_n \times G$-orbit. Thus, $\phi(\dot{d}) \in \mathrm{End}_{S_n \times G}(W^{\otimes k})$, for all (G, k)-diagrams $\dot{d}$.

As in the proofs of Lemma 3.3 and Lemma 3.4, we have $\phi(\dot{d}_2\dot{d}_1) = \phi(\dot{d}_2)\phi(\dot{d}_1)$, where $\dot{d}_1, \dot{d}_2$ are G-diagrams, and hence ϕ is an algebra homomorphism.

Observe that every L^I_J has a pre image $\dot{d}$:

$$\phi(\dot{d}) = L^{(i_1,g_1),(i_2,g_2),\ldots,(i_k,g_k)}_{(i_{k+1},g_{k+1}),\ldots,(i_{2k},g_{2k})},$$

in which the underlying partition diagram of $\dot{d}$ is $d(i_1, i_2, \ldots, i_{2k})$ with label sequence $g_1, g_2, \ldots, g_{2k}$. Hence, ϕ is onto $\mathrm{End}_{S_n \times G}(W^{\otimes k})$. Hence the lemma is proved. □

Theorem 4.2. $\mathbb{C}[S_n \times G]$ *and* $P_k(x, G)$ *generate full centralizers of each other in* $\mathrm{End}(W^{\otimes k})$. *That is for* $n \geq 2k$*, we have*
(i) $P_k(x, G) \cong \mathrm{End}_{S_n \times G}(W^{\otimes k})$,
(ii) $S_n \times G$ *generates* $\mathrm{End}_{P_k(x,G)}(W^{\otimes k})$.

Proof. (i). For $n \geq 2k$, $\dim P_k(x, G) = \dim \mathrm{End}_{S_n \times G}(W \otimes k)$. From Lemma 4.1, we obtained $\phi(P_k(x, G)) \subseteq \mathrm{End}_{S_n \times G}(W^{\otimes k})$. As $\dot{d}$ ranges overall G-diagrams, all $L_{\dot{d}}$ are obtained. Hence, the representation ϕ takes a basis of $P_k(x, G)$ into a basis of $\mathrm{End}_{S_n \times G}(W^{\otimes k})$, therefore $P_k(x, G) \cong \mathrm{End}_{S_n \times G}(W^{\otimes k})$.

(ii). The proof from (i) and the double centralizer Theorem. □

As the centralizer of the semisimple group algebra $\mathbb{C}[S_n \times G]$, the $\mathbb{C}$-algebra $P_k(x, G)$ is semisimple for $n \geq 2k$.

4.2 Schur - Weyl Duality of $\widehat{P_k}(x, G)$

The action of $\widehat{P}_k(x, G)$ on $W^{\otimes k}$ is stated here by numbering the vertices of a (G, k)-diagram as $1, 2, \ldots k$ in the top row from the left to the right-hand side and in a similar format $k + 1, k + 2, \ldots, 2k$ for the bottom row. A map $\widehat{\phi} : \widehat{P}_k(x, G) \longrightarrow \mathrm{End}(W^{\otimes k})$ is defined on a G-diagram (d, f) as:
for $f = (g_1, g_2, \ldots, g_k, g_{k+1}, g_{k+2}, \ldots, g_{2k})$ be any label sequence of d, then

$$\begin{aligned}\widehat{\phi}(d, f) &= \left(\widehat{\phi}(d, f)^{(i_1,h_1),(i_2,h_2),\ldots,(i_k,h_k)}_{(i_{k+1},h_{k+1}),(i_{k+2},h_{k+2}),\ldots,(i_{2k},h_{2k})}\right)\\ &= \left(\psi(d)^{i_1,i_2,\ldots,i_k}_{i_{k+1},i_{k+2},\ldots,i_{2k}} \quad \delta^{(g_1,g_2,\ldots,g_{2k})}_{(h_1,h_2,\ldots,h_{2k})}\right)\end{aligned}$$

in which $\delta^{(g_1,g_2,\ldots,g_{2k})}_{(h_1,h_2,\ldots,h_{2k})}$ is the Kronecker delta and $\psi(d)^{i_1,i_2,\ldots,i_k}_{i_{k+1},i_{k+2},\ldots,i_{2k}}$ is similar to the definition in equation [4] 1.7. In other words, with respect to the matrix unit, we have

$$\widehat{\phi}(d, f) = \sum_{\substack{p \sim q \ in\ d \Rightarrow i_p = i_q \\ 1 \leq i_1, i_2, \ldots, i_{2k} \leq n}} E^{(i_1,g_1),(i_2,g_2),\ldots,(i_k,g_k)}_{(i_{k+1},g_{k+1}),(i_{k+2},g_{k+2}),\ldots,(i_{2k},g_{2k})}. \tag{4.4}$$

With respect to which an action of $\widehat{P}_k(x, G)$ on $W^{\otimes k}$ is defined by

$$(d, f)(v_J) = \widehat{\phi}(d, f)(v_J), \quad \text{for all } J \in \mathbb{S}^k.$$

Let G be a group with only one element, then the above-defined action restricts the action of the partition algebra as defined in [3] on tensors.

Therefore, the action of a G-partition diagram $(d, f) \in \widehat{P}_k(x, G)$ on $W^{\otimes k}$ is stated with respect to standard basis as

$$(d, f).(v_{(i_1,h_1)} \otimes v_{(i_2,h_2)} \otimes \cdots \otimes v_{(i_k,h_k)})$$

$$= \delta^{(g_1,g_2,\ldots,g_{2k})}_{(h_1,h_2,\ldots h_{2k})} \sum_{i_{k+1},i_{k+2},\ldots,i_{2k}} \psi(d)^{i_1,i_2,\ldots,i_k}_{i_{k+1},i_{k+2},\ldots,i_{2k}} v_{(i_{k+1},h_{k+1})} \otimes v_{(i_{k+2},h_{k+2})} \otimes \cdots \otimes v_{(i_{2k},h_{2k})}$$

Lemma 4.3. *The map* $\widehat{\phi} : \widehat{P}_k(x,G) \longrightarrow \mathrm{End}(W^{\otimes k})$ *is an algebra homomorphism onto* $\mathrm{End}_{S_n}(W^{\otimes k})$.

Proof. It holds from the lemma 4.1. □

The following is our analogue of Theorem 4.2.

Theorem 4.4. $\mathbb{C}[S_n]$ *and* $\widehat{P}_k(x,G)$ *generate full centralizers of each other in* $\mathrm{End}(W^{\otimes k})$, *which is, for* $n \geq 2k$, *we obtain*
(i) $\widehat{P}_k(x,G) \cong \mathrm{End}_{S_n}(W^{\otimes k})$,
(ii) S_n *generates* $\mathrm{End}_{S_n}(W^{\otimes k})$.

Proof. (i). For $n \geq 2k$, dim $\widehat{P}_k(x,G) =$ dim $\mathrm{End}_{S_n}(W \otimes k)$. From Lemma 4.3, we obtained $\widehat{\phi}(\widehat{P}_k(x,G)) \subseteq \mathrm{End}_{S_n}(W^{\otimes k})$. As (d,f) ranges over all G-diagrams, all $L_{(d,f)}$ are obtained. Hence, the representation ϕ takes a basis of $\widehat{P}_k(x,G)$ to a basis of $\mathrm{End}_{S_n}(W^{\otimes k})$, therefore $\widehat{P}_k(x,G) \cong \mathrm{End}_{S_n}(W^{\otimes k})$.
(ii). The proof from (i) and the double centralizer Theorem. □

5 ALGEBRA $End_{A_n \times G}(W^{\otimes k})$

Let A_n signify the group of even permutations on n elements, and W represents the permutation module of the symmetric group $S_{n|G|}$. Similar to Section 3.2, $S_n \times G$ acts diagonally on $W^{\otimes k}$, and this action restricted to $A_n \times G \subset S_n \times G$. We formulate a basis for the centralizer algebra $End_{A_n \times G}(W^{\otimes k})$. Clearly $End_{S_n \times G}(W^{\otimes k}) \subseteq End_{A_n \times G}(W^{\otimes k})$. We calculate the dimension of $End_{A_n \times G}(W^{\otimes k})$ and describe when $End_{A_n \times G}(W^{\otimes k})$ is simply the G-vertex color partition algebra $P_k(x,G)$.

Lemma 5.1. $A \in \mathrm{End}_{A_n \times G}(W^{\otimes k}) \Leftrightarrow A_I^J = A_{\pi_g(I)}^{\pi_g(J)}$ for all $\pi_g \in A_n \times G$.

Proof. Given is

$$\begin{aligned} A \in \mathrm{End}_{A_n \times G}(W^{\otimes k}) &\Leftrightarrow \pi_g A = A \pi_g \quad \forall \quad \pi_g \in A_n \times G \\ &\Leftrightarrow \pi_g A(w_J) = A\pi_g(w_J) \; \forall \; w_J \\ &\Leftrightarrow \pi_g \sum_I A_I^J(w_I) = A(w_{\pi_g(J)}) \\ &\Leftrightarrow \sum_I A_I^J \pi_g(w_I) = \sum_I A_I^{\pi_g(J)}(v_I) \\ &\Leftrightarrow \sum_I A_I^J(w_{\pi_g(I)}) = \sum_I A_{\pi_g(I)}^{\pi_g(J)}(w_{\pi_g(I)}) \end{aligned}$$

As the action of $A_n \times G$ is by the permutation representation, an ensuing conclusion is deduced from linear independence and equating the scalars.

Because the action of $A_n \times G$ is by the permutation representation. Hence, it holds by linear independence and equating the scalars. □

Therefore, we can form a basis of $End_{A_n \times G}(W^{\otimes k})$ by describing $A_n \times G$-orbits on

$$\mathbb{S}^{2k} = (I,J) = \{[\begin{smallmatrix}((i_1,g_1),(i_2,g_2),\ldots,(i_k,g_k))\\((i_{k+1},g_{k+1}),(i_{k+2},g_{k+2}),\ldots,(i_{2k},g_{2k}))\end{smallmatrix}] : 1 \leq i_1, i_2, \ldots, i_{2k} \leq n\}.$$

The action of $S_n \times G$ on $\mathbb{S}^{2k}$ by $\pi_g(I,J) = (\pi_g(I), \pi_g(J))$. As we saw in Section 3, the number of $S_n \times G$-orbits on $\mathbb{S}^{2k}$ gives the dimension of $End_{S_n \times G}(W^{\otimes k})$.

Every orbit corresponds to a basis element $T_{\sim_l}$, of $End_{S_n \times G}(W^{\otimes k})$ (from Equation 3.2). From Section 3, we obtained that the $S_n \times G$-orbits are in one-to-one correspondence with the set partitions of $1, 2, \ldots, 2k$ and a $2k$-tuple $(e, g_2', g_3', \ldots, g_{2k}')$ and vice-versa. Whereas, we know that the $A_n \times G$-orbits are not necessarily in $1-1$ correspondence with the equivalence relations

on $1, 2, \ldots, 2k$ and a $2k-$tuple $(e, g'_2, g'_3, \ldots, g'_{2k})$. Because what was an entire $S_n \times G$-orbits can be regarded as the disjoint union of more than one $A_n \times G$-orbits.
Let $\sim_l$ denote an equivalence relation with l classes on $1, 2, \ldots, 2k$ and a $2k-$tuple $(e, g'_2, g'_3, \ldots, g'_{2k})$. There are $|G|^{2k-1}S(2k, l)$ such equivalence relations; then corresponding to $\sim_l$ are $|G|^{2k-1}S(2k, l)$ basis elements T of $End_{S_n \times G}(W^{\otimes k})$. Now $T_{\sim_l}$ represent a basis element of $End_{S_n \times G}(W^{\otimes k})$ from the nature of the partition $\sim_l$, we conclude that $T_{\sim_l}$ is of type $|G|^{2k-1}S(2k, l)$. Once again $T_{\sim_l}$ be a basis element of type $|G|^{2k-1}S(2k, l)$ in $End_{S_n \times G}(W^{\otimes k})$, such that the entries of $T_{\sim_l}$ have indices partitioned according to $\sim_l$. Define

$$\mathbb{S}^{2k}(T_{\sim_l})=\{[{}^{((i_1,g_1),(i_2,g_2),\ldots,(i_k,g_k))}_{((i_{k+1},g_{k+1}),(i_{k+2},g_{k+2}),\ldots,(i_{2k},g_{2k}))}] : 1 \le i_1, i_2, \ldots, i_{2k} \le n,$$
$$where\ i_1, i_2, \ldots, i_{2k}\ are\ partitioned\ according\ to\ \sim_l\},$$

then $\mathbb{S}^{2k}(T_{\sim_l})$ represents the positions of the nonzero entries in $T_{\sim_l}$. When $\alpha \,\epsilon\, \mathbb{S}^{2k}(T_{\sim_l})$ and O_α represent the $A_n \times G$-orbit of α. Now, $|\mathbb{S}^{2k}(T_{\sim_l})|$=$n!/(n-l)!$ along with $|O_\alpha| = |G|^{2k-1}n!/2|(A_n \times G)_\alpha|$, where $(A_n \times G)_\alpha \subseteq A_n \times G$ is the stabilizer of α. If $\mathbb{S}^{2k}(T_{\sim_l})$ is the disjoint union of two $A_n \times G$-orbits, then it is concluded that $T_{\sim_l} \in End_{S_n \times G}(W^{\otimes k})$ splits when lifted to $End_{A_n \times G}(W^{\otimes k})$.

Proposition 5.2. *i) For* $2 < n \le 2k$, *we obtain*

$$dim\ End_{A_n \times G}(W^{\otimes k}) = |G|^{2k-1}(\sum_{l=1}^{n-2} S(2k, l) + 2S(2k, n-1) + 2S(2k, n))$$

ii) For $n = 2k+1$, *we obtain*

$$dim\ End_{A_n \times G}(W^{\otimes k}) = |G|^{2k-1}(\sum_{l=1}^{2k-1} S(2k, l) + 2S(2k, 2k)) = |G|^{2k-l}(B(2k) + 1)$$

iii) For $n \ge 2k+2$, *we obtain*

$$dim\ End_{A_n \times G}(W^{\otimes k}) = |G|^{2k-1}\sum_{l=1}^{2k} S(2k, l) = |G|^{2k-1}B(2k)$$

Proof. i) As observed from lemma 3.2 dim $End_{S_n \times G}(W^{\otimes k}) = |G|^{2k-1}\sum_{l=1}^{n} S(2k, l)$. Observe that $End_{A_n \times G}(W^{\otimes k}) = |G|^{2k-1}\sum_{l=1}^{n} c_l S(2k, l)$ where c_l is the number of disjoint $A_n \times G$-orbits that $\mathbb{S}^{2k}(T_{\sim_l})$ comprises when $T_{\sim_l}$ is of type $|G|^{2k-1}S(2k, l)$ (c_l is independent of $\sim$, and only depends on l). We will see that c_l is either 1 or 2. In other words, $T_{\sim_l} \in End_{S_n \times G}(W^{\otimes k})$ of type $|G|^{2k-1}S(2k, l)$ either splits as $T_{\sim_l} = T^-_{\sim_l} + T^+_{\sim_l}$ into a sum of two basis elements of $End_{A_n \times G}(W^{\otimes k})$ when lifted to $End_{S_n \times G}(W^{\otimes k})$, or remains a basis element of type $|G|^{2k-1}$ $S(2k, l)$ in $End_{S_n \times G}(W^{\otimes k})$. Let $\alpha \in \mathbb{S}^{2k}(T_{\sim_l})$ where $T_{\sim_l} \in End_{S_n \times G}(W^{\otimes k})$ is of type $|G|^{2k-1}S(2k, n)$. Then, it can be easily deduced that the identity is the only element of $A_n \times G$ which fixes the entire n different entries of α. Hence $|O_\alpha| = n!/2$, where $\mathbb{S}^{2k}(T_{\sim_l}) = n!$, $c_n = 2$. Following the above discussion, we also obtain for $c_{n-1} = 2$. For $\alpha \in \mathbb{S}^{2k}(T_{\sim_l})$, $(T_{\sim_l}) \in End_{S_n \times G}(W^{\otimes k})$ is of the form of $|G|^{2k-1}S(2k, l)$, $1 \le l \le n-2$. We infer $(A_n \times G)_\alpha \cong A_{n-l} \times G$. Therefore $|O_\alpha| = ((n!/2)/(n-l)!/2) = n!/(n-l)! = \mathbb{S}^{2k}(T_{\sim_l})$, so $c_l = 1$, $1 \le l \le n-2$, hence the condition follows.

ii) As observed from lemma 3.2 dim $End_{S_n \times G}(W^{\otimes k}) = |G|^{2k-1}\Sigma_{l=1}^{2k} S(2k, l) = |G|^{2k-1}B(2k)$. Again, we assume dim $End_{A_n \times G}(W^{\otimes k}) = |G|^{2k-1}\Sigma_{l=1}^{2k} c_l S(2k, l)$ and from (i), the only $T_{\sim_l} \in End_{S_n \times G}(W^{\otimes k})$ that can split are of type $S(2k, n)$ or $S(2k, n-1)$. As $c_n = c_{2k+1}$ is absent in the above mentioned sum, only $T_{\sim_l} \in End_{S_n \times G}(W^{\otimes k})$ of type $S(2k, n-1) = S(2k, 2k)$ splits. Hence, $c_{2k} = 2$ and $c_l = 1$, $1 \le l \le n-2 = 2k-1$. To point out $S(2k, 2k) = 1$, so dim $End_{A_n \times G}(W^{\otimes k}) = |G|^{2k-1}(\Sigma_{l=1}^{2k-1} S(2k, l) + 2S(2k, 2k)) = |G|^{2k-1}(\Sigma_{l=1}^{2k} S(2k, l) + S(2k, 2k)) = |G|^{2k-1}(B(2k) + 1)$.

iii) Similarly, dim $End_{S_n \times G}(W^{\otimes k}) = |G|^{2k-1}\Sigma_{l=1}^{2k} S(2k, l) = |G|^{2k-1}B(2k)$. As $n \ge 2k+2$, c_n and c_{n-1} do not appear in the sum dim $End_{A_n \times G}(W^{\otimes k}) = |G|^{2k-1}\Sigma_{l=1}^{2k} S(2k, l)$ and therefore, $c_l = 1$, $1 \le l \le 2k$ $(\le n-2)$. Hence the condition holds.

□

The $End_{A_n \times G}(W^{\otimes k})$ always remains similar for higher values of n, and theorem 5.3 provides the necessary proof.

Theorem 5.3. *When $n \geq 2k+2$, $\mathbb{C}[A_n \times G]$ and $P_k(x,G)$ generate full centralizers of each other in* $\mathrm{End}(W^{\otimes k})$. *Mathematically, these are represented as $n \geq 2k+2$. We have,*
(i) $P_k(n,G) \cong \mathrm{End}_{A_n \times G}(W^{\otimes k})$,
(ii) $A_n \times G$ *generates* $\mathrm{End}_{P_k(n,G)}(W^{\otimes k})$.

Proof. (i). Recall that for $n \geq 2k$, $P_k(x,G) \cong End_{S_n \times G}(W^{\otimes k})$. Then $End_{S_n \times G}(W^{\otimes k})$ is a subalgebra of $End_{A_n \times G}(W^{\otimes k})$; under proposition 5.2 the subdivision (iii) validates that $dim\ End_{S_n \times G}(W^{\otimes k}) = dim\ End_{A_n \times G}(W^{\otimes k})$ for $n \geq 2k+2$. Thus $End_{S_n \times G}(W^{\otimes k}) = End_{A_n \times G}(W^{\otimes k})$ for $n \geq 2k+2$. Specifically, $P_k(x,G) \cong End_{S_n \times G}(W^{\otimes k}) = End_{A_n \times G}(W^{\otimes k})$ for $n \geq 2k+2$.

Proof of (ii). The double centralizer Theorem and from the proof of subdivision (i), the result of subdivision (ii) holds. □

6 ALGEBRA $End_{A_n}(W^{\otimes k})$

Let A_n signify the group of even permutations on n elements and W represent the permutation module of the symmetric group $S_{n|G|}$. Similar to Section 3.2, S_n acts diagonally on $W^{\otimes k}$, and this action restricted to $A_n \subset S_n$. We formulate a basis for the centralizer algebra $End_{A_n}(W^{\otimes k})$. Clearly, $End_{S_n}(W^{\otimes k}) \subseteq End_{A_n}(W^{\otimes k})$. We calculate the dimension of $End_{A_n}(W^{\otimes k})$ and describe when $End_{A_n}(W^{\otimes k})$ is simply the Extended G- vertex colored partition algebra $P_k(x,G)$.

Lemma 6.1. $A \in \mathrm{End}_{A_n}(W^{\otimes k}) \;\Leftrightarrow\; A_I^J = A_{\pi(I)}^{\pi(J)}$ for all $\pi \in A_n$

Proof.

$$\begin{aligned} We\ have\ A \in \mathrm{End}_{A_n}(W^{\otimes k}) \quad &\Leftrightarrow \quad \pi A = A\pi \ \ \forall \ \ \pi \in A_n, \\ &\Leftrightarrow \quad \pi A(v_J) = A\pi(v_J) \ \ \forall \ \ v_J \\ . \quad &\Leftrightarrow \quad \pi \sum_I A_I^J(v_I) = A(v_{\pi(J)}) \\ &\Leftrightarrow \quad \sum_I A_I^J \pi(v_I) = \sum_I A_I^{\pi(J)}(v_I) \\ &\Leftrightarrow \quad \sum_I A_I^J(v_{\pi(I)}) = \sum_I A_{\pi(I)}^{\pi(J)}(v_{\pi(I)}). \end{aligned}$$

Since the action of S_n is permutation representation. Hence, it holds from linear independence and equating the scalars. □

$\mathbb{S} = [n] \times G$ The action of S_n on $\mathbb{S}$ defined by

$$\pi(i,g) = (\pi(i), g)$$

Hence, we can describe a basis of $End_{A_n}(W^{\otimes} k)$ by describing A_n-orbits on

$$\mathbb{S}^{2k} = (I,J) = \{[^{((i_1,g_1),(i_2,g_2),\ldots,(i_k,g_k))}_{((i_{k+1},g_{k+1}),(i_{k+2},g_{k+2}),\ldots,(i_{2k},g_{2k}))}] : 1 \leq i_1, i_2, \ldots, i_{2k} \leq n\}$$

The action of S_n on $\mathbb{S}^{2k}$ by $\pi(I,J) = (\pi(I), \pi(J))$. As we saw in Section 3, the number of S_n-orbits on $\mathbb{S}^{2k}$ gives the dimension of $End_{S_n}(W^{\otimes k})$.

Each orbit corresponds to a basis element T of $End_{S_n}(W^{\otimes k})$ (From Equation 3.2). In Section 3, we obtained that the S_n-orbits are in one-to-one correspondence with the set partitions of $1, 2, \ldots, 2k$ and a $2k$-tuple $(e, g_2', g_3', \ldots, g_{2k}')$ and vice versa . Whereas we know that the A_n-orbits are not necessarily in $1-1$ correspondence with the equivalence relations on $1, 2, \ldots, 2k$

and a $2k-$tuple $(e, g_2', g_3', \ldots, g_{2k}')$. Because what was an entire S_n-orbits can be regarded as the disjoint union of more than one A_n-orbits.

Let $\sim_l$ denote an equivalence relation with l classes on $1, 2, \ldots, 2k$ and a $2k-$tuple $(e, g_2', g_3', \ldots, g_{2k}')$. There are $|G|^{2k}S(2k,l)$ such equivalence relations; then corresponding to $\sim_l$ are $|G|^{2k}S(2k,l)$ basis elements T of $End_{S_n}(W^{\otimes k})$. Now $T_{\sim_l}$ represent a basis element of $End_{S_n}(W^{\otimes k})$ from the nature of the partition $\sim_l$, we conclude that $T_{\sim_l}$ is of type $|G|^{2k}S(2k,l)$. Once again $T_{\sim_l}$ be a basis element of type $|G|^{2k}S(2k,l)$ in $End_{S_n}(W^{\otimes k})$, such that the entries of $T_{\sim_l}$ have indices partitioned according to $\sim_l$. Define

$$\mathbb{S}^{2k}(T_{\sim_l}){=}\{[^{((i_1,g_1),(i_2,g_2),\ldots,(i_k,g_k))}_{((i_{k+1},g_{k+1}),(i_{k+2},g_{k+2}),\ldots,(i_{2k},g_{2k}))}] : 1 \leq i_1, i_2, \ldots, i_{2k} \leq n,$$
$$where\ i_1, i_2, \ldots, i_{2k}\ are\ partitioned\ according\ to\ \sim_l\}$$

then $\mathbb{S}^{2k}(T_{\sim_l})$ represents the positions of the nonzero entries in $T_{\sim_l}$. When $\alpha \,\epsilon\, \mathbb{S}^{2k}(T_{\sim_l})$ and O_α represent the A_n-orbit of α. Now we observe $|\mathbb{S}^{2k}(T_{\sim_l})|{=}n!/(n{-}l)!$ along with $|O_\alpha|{=}n!/2|(A_n\times G)_\alpha|$, where $(A_n)_\alpha \subseteq A_n$ is the stabilizer of α. If $\mathbb{S}^{2k}(T_{\sim_l})$ is the disjoint union of two A_n-orbits, then it is concluded that $T_{\sim_l} \in End_{S_n}(W^{\otimes k})$ splits when lifted to $End_{A_n}(W^{\otimes k})$.

Proposition 6.2. *i) For* $2 < n \leq 2k$, *we obtain*

$$dim\ End_{A_n}(W^{\otimes k}) = |G|^{2k}(\sum_{l=1}^{n-2} S(2k,l) + 2S(2k,n-1) + 2S(2k,n))$$

ii) For $n = 2k+1$, *we obtain*

$$dim\ End_{A_n}(W^{\otimes k}) = |G|^{2k}(\sum_{l=1}^{2k-1} S(2k,l) + 2S(2k,2k)) = |G|^{2k}(B(2k)+1)$$

iii) For $n \geq 2k+2$, *we obtain*

$$dim\ End_{A_n}(W^{\otimes k}) = |G|^{2k}\sum_{l=1}^{2k} S(2k,l) = |G|^{2k}B(2k)$$

Proof. i) As observed from lemma 3.6, dim $End_{S_n}(W^{\otimes k}) = |G|^{2k}\sum_{l=1}^{n} S(2k,l)$. Observe that $End_{A_n}(W^{\otimes k}) = |G|^{2k}\sum_{l=1}^{n} c_l S(2k,l)$ in this c_l is the number of disjoint A_n-orbits that $\mathbb{S}^{2k}(T_{\sim_l})$ comprises when $T_{\sim_l}$ is of type $|G|^{2k}S(2k,l)$ (c_l is independent of $\sim$, and only depends on l). We will see that c_l is either 1 or 2. In other words, $T_{\sim_l} \in End_{S_n}(W^{\otimes k})$ of type $|G|^{2k}S(2k,l)$ either splits as $T_{\sim_l} = T^{-}_{\sim_l} + T^{+}_{\sim_l}$ into a sum of two basis elements of $End_{A_n}(W^{\otimes k})$ when lifted to $End_{S_n}(W^{\otimes k})$, or remains a basis element of type $|G|^{2k}S(2k,l)$ in $End_{S_n}(W^{\otimes k})$. Let $\alpha \in \mathbb{S}^{2k}(T_{\sim_l})$ where $T_{\sim_l} \in End_{S_n}(W^{\otimes k})$ is of type $|G|^{2k}S(2k,n)$. Then, it can be easily deduced that the identity is the only element of A_n that fixes the entire n different entries of α. Hence $|O_\alpha| = n!/2$ as $\mathbb{S}^{2k}(T_{\sim_l}) = n!$, $c_n = 2$. Following the above discussion, we also obtain for $c_{n-1} = 2$. For $\alpha \in \mathbb{S}^{2k}(T_{\sim_l})$, $(T_{\sim_l}) \in End_{S_n}(W^{\otimes k})$ is of the form of $|G|^{2k}S(2k,l)$, $1 \leq l \leq n-2$. We infer $(A_n)_\alpha \cong A_{n-l}$. Therefore $|O_\alpha| = ((n!/2)/(n-l)!/2) = n!/(n-l)! = \mathbb{S}^{2k}(T_{\sim_l})$, so $c_l = 1$, $1 \leq l \leq n-2$, hence the condition follows.

ii) As observed from lemma 3.6, dim $End_{S_n}(W^{\otimes k}) = |G|^{2k}\Sigma_{l=1}^{2k}S(2k,l) = |G|^{2k}B(2k)$. Again, we use dim $End_{A_n}(W^{\otimes k}) = |G|^{2k}\Sigma_{l=1}^{2k}c_l S(2k,l)$ and from (i), the only $T_{\sim_l} \in End_{S_n}(W^{\otimes k})$ that can split are of type $S(2k,n)$ or $S(2k,n-1)$. As $c_n = c_{2k+1}$ is absent in the above mentioned sum, only $T_{\sim_l} \in End_{S_n}(W^{\otimes k})$ of type $S(2k,n-1) = S(2k,2k)$ splits. Hence $c_{2k} = 2$ and $c_l = 1, 1 \leq l \leq n-2 = 2k-1$. To point out $S(2k,2k) = 1$, so dim $End_{A_n}(W^{\otimes k}) = |G|^{2k}(\Sigma_{l=1}^{2k-1}S(2k,l) + 2S(2k,2k)) = |G|^{2k}(\Sigma_{l=1}^{2k}S(2k,l) + S(2k,2k)) = |G|^{2k}(B(2k)+1)$.

iii) Similarly, dim $End_{S_n}(W^{\otimes k}) = |G|^{2k}\Sigma_{l=1}^{2k}S(2k,l) = |G|^{2k}B(2k)$. As $n \geq 2k+2$, c_n and c_{n-1} are not present in the sum dim $End_{A_n}(W^{\otimes k}) = |G|^{2k}\Sigma_{l=1}^{2k}c_l S(2k,l)$ and $c_l = 1$, $1 \leq l \leq 2k(\leq n-2)$. Hence the condition holds. □

$End_{A_n}(W^{\otimes k})$ always remains similar for higher values of n, and theorem 6.3 provides the necessary proof.

Theorem 6.3. *When $n \geq 2k+2$, $\mathbb{C}[A_n]$ and $\widehat{P_k}(x,G)$ generate full centralizers of each other in* $\mathrm{End}(W^{\otimes k})$. *Mathematically, these are represented as*
(i) $\widehat{P_k}(x,G) \cong \mathrm{End}_{A_n}(W^{\otimes k})$,
(ii) $A_n \times G$ *generates* $\mathrm{End}_{\widehat{P_k}(x,G)}(W^{\otimes k})$.

Proof. (i). Recall that for $n \geq 2k$, $\widehat{P_k}(x,G) \cong End_{S_n}(W^{\otimes k})$. Then $End_{S_n}(W^{\otimes k})$ is a subalgebra of $End_{A_n}(W^{\otimes k})$; under proposition 6.2 subdivision (iii) validates that $dim\ End_{S_n}(W^{\otimes k}) = dim\ End_{A_n}(W^{\otimes k})$ for $n \geq 2k+2$. Hence, $End_{S_n}(W^{\otimes k}) = End_{A_n}(W^{\otimes k})$ for $n \geq 2k+2$. Specifically, $\widehat{P_k}(x,G) \cong End_{S_n}(W^{\otimes k}) = End_{A_n}(W^{\otimes k})$ for $n \geq 2k+2$.

(ii). The double centralizer Theorem and from the proof of subdivision (i), the result of subdivision (ii) holds. □

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