Palestine Journal of Mathematics

Vol. 12(3)(2023) , 691–701

Results on the T- Stability of Picard Iteration in Partial E-Cone Metric Spaces

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Communicated by Thabet Abdeljawad

MSC 2010 Classifications: Primary 20M99, 13F10; Secondary 13A15, 13M05.—

Keywords and phrases: E-metric space, partial E-cone metric space, fixed point, T-stable.

Abstract The purpose of this paper is to introduce the concept of partial E-cone metric spaces as a generalization of partial metric and E-metric spaces. Analog to Banach contraction principle, some fixed point results are proved investigating the T-stability of Picard's iteration as well as a Hardy-Rogers type mapping in such cone metric spaces.

1 Introduction

In 1912 Luitzen E. Brouwer [15] published a famous paper discussing results on fixed point theory. Many of its proofs were later adapted in a topological sense. Banach, 1922, [12], presented a method for finding the fixed point of a self operator in complete metric spaces in a systematic manner. Later, a great deal of work on variants and generalizations was published to improve the Banach contraction principle by modifying the topology of the space or acting on the contraction requirement, (see, e.g., [1] – [4], [7], [8], [11], [13], [19], [21], [30], [35], [36]).

In some works on non-convex analysis, the authors define an order specially in ordered normed spaces by using a cone in a vector space, [18], [23] – [25]. In this manner Huang and Zhang, [20] presented the notion of cone metric space with a fresh point of view in which Cauchy and convergent sequences are analyzed in terms of the interior points with respect to the cone partial ordering. Many mathematicians followed Huang's lead and focused on fixed point problems in such spaces (see, [9], [23]-[27]) and the references therein).

In this sequel, Mehmood et al. [24] introduced the concept of E-metric spaces as generalization of metric spaces. They proved the contraction mapping principle in E-metric spaces that generalized the famous Banach contraction principle in such spaces.

On the other hand, Matthews, [22] introduced the notion of partial metric space as a part of the study of denotational semantics of dataflow network. In partial metric spaces, the distance of a point in the self may not be zero. Introducing partial metric space, Matthews proved the Banach fixed point theorem in the setting of partial metric. Many authors considered such spaces and introduced interesting fixed point results and generalizations (see, e.g., [8], [13], [36]).

In solving some real life problems, for example in Physics, finance, or transportation, numerical procedures that compute an iterative sequence are considered to be useful if they posses some convergence and stability properties, (see, e.g., [5], [6], [17], [27], [31]-[33]). In 1991, Rhoades [29], provided a survey paper on the stability of some iteration techniques used to obtain fixed points for maps satisfying certain contractive conditions. While Asadi et al. [10], and Yousefi [34], proved and investigated T-stability iteration procedure in cone metric spaces. For more on stability of Picard's iteration, see [27], [28].

In this paper, we generalize both E-metric and partial cone metric by introducing the partial E-cone metric space. We give some basic properties in E-metric spaces with regard to cones containing semi-interior points. Furthermore, as we cape the existence and uniqueness of fixed points for Hardy-Rogers type mapping, we established the T-stability of Picard's iteration and the equivalence between two distinct e-sequences in such spaces. Throughout this paper, $\mathbb{N}$ and $\mathbb{R}$ denote the set of all natural numbers and the set of all real numbers respectively. The following definitions and results will be needed in this paper.

Definition 1.1. [20] An ordered space E is a vector space over the real numbers, with a partial order relation " $\preceq$ " such that

(i) for all x, y and $z \in E$, $x \preceq z$ implies $x + y \preceq z + y$

(ii) for all $\alpha \in \mathbb{R}^+$ and for all $x \in E$ with $x \succeq 0_E$, $\alpha x \succeq 0_E$, where 0_E stands for the zero of E.

Moreover, if E is equipped with a norm ||.||, then E is called normed ordered space.

Definition 1.2. [18] Let E be a real normed space, E^+ be a non-empty closed and convex subset of E, and 0_E be a zero element in E. Then E^+ is called a positive cone if it satisfies

(i) $x \in E^+$ and $a \geq 0$ imply $ax \in E^+$;

(ii) $x \in E^+$ and $-x \in E^+$ imply $x = 0_E$.

Definition 1.3. [18] The positive cone E^+ of a normal ordered space E is called normal if there exists a constant $K > 0$, such that

$$0 \preceq y \preceq x \text{ implies } \|y\| \leq K \|x\| \qquad \text{for all such } y, x \in E.$$

Equivalently the cone E^+ is regular if and only if every decreasing sequence which is bounded from below is convergent. It is well known that a regular cone is a normal cone.

The following definition of an E-metric space defined in [24].

Definition 1.4. Let X be a non-empty set and let E be an ordered space over the real scalars. An ordered E-metric on X is an E-valued function $d^E : X \times X \to E$ such that for all $x, y, z \in X$, we have

(i) $0_E \preceq d^E(x, y)$, $d^E(x, y) = 0_E$ if and only if $x = y$;

(ii) $d^E(x, y) = d^E(y, x)$;

(iii) $d^E(x, y) \preceq d^E(x, z) + d^E(z, y)$.

Then the pair d^E (X, d) is called E-metric space.

2 Partial E-cone metric space

Rezapour and Hamlbarani, [26] extended the notion of K-metric spaces and convergence in an ordered Banach space X with a solid cone E without normality assumption. Most fixed point issues in cone metric spaces are embedded in solid cones, which are cones with non empty interior. Unfortunately, there were just a few results that took non-solid cones into account, [14, 24].

Fortunately, by embedding non-solid cones that contain semi-interior points in E-metric spaces, Basile et al. [14] established the concept of the semi-interior point and took fixed point results in E-metric spaces into consideration. Embedding such cones in the setting of E-metric spaces, Mehmood et al. [24] and Huang, [18], obtained some fixed Theorems in 2019.

As examples in [14] illustrate, the class of cones with semi-interior point and empty interior is wider than the one with non empty interior, fixed points results for ordered normed spaces also hold for this wider class of cones with semi-interior points, which is quite fascinating.

Let E be an ordered normed space ordered by the positive cone E^+, we shall denote by 0_E the zero of E. The set

$$B = \{x \in E : \|x\| \leq 1\} \text{ is the closed unit ball of } E,$$

and that

$$B_+ = B \cap E^+ \text{ is the positive part of } B.$$

The point $x_0 \in E^+$ is called a semi-interior point of E^+ if there exists a real number $\lambda > 0$ such that

$$x_0 - \lambda B_+ \subseteq E^+.$$

Hereafter, we denote by $(E^+)^{\oslash}$ the set of all semi-interior points of E^+.
Now, let E be a normed space ordered by its positive cone E^+.
For $x, y \in E^+$, $x \preceq y$ if and only if $y - x \in E^+$ and $x \lll y$ if and only if $y - x \in (E^+)^{\oslash}$.
It is easy to see that

$$x \in \left(E^+\right)^{\oslash} \text{ if and only if } 0_E \lll x.$$

Proposition 2.1. *Let $x, y, z \in E$. Then $0_E \preceq z$, $x \preceq y - z$ implies that $x \preceq y$.*

Proof. Let $x, y, z \in E$, $0_E \preceq z$ and $x \preceq y - z$. Then

$$0_E \preceq z, y - z - x \in E^+.$$

Noting that E^+ is a positive cone, it follows that

$$y - x = (y - z - x) + z \in E^+.$$

Thus, $y - x \in E^+$, that is $x \preceq y$. □

The following topological properties relevant to semi-interior points in E-metric spaces defined in [18].

Proposition 2.2. *If $x, y \in E$, then $y \lll x$ implies $y \preceq x$.*

Proposition 2.3. *If $0_E \preceq u \lll e$ holds for any $e \in (E^+)^{\oslash}$, then $u = 0_E$.*

Proposition 2.4. *[18] Let $0 \le \lambda < 1$ be a constant, $u \in E^+$ and $u \preceq \lambda u$. Then $u = 0_E$.*

Definition 2.5. [18] A sequence (y_n) in E^+ is said to be an e-sequence if for each $0_E \lll e$, there exists $k \in \mathbb{N}$ such that for all $n > k$, $y_n \lll e$.

It is easy to see that (y_n) is an e-sequence if (y_n) converges to 0_E as n goes to infinity.

Proposition 2.6. *Let $0 \le \lambda < 1$ be a constant, (x_n) and (y_n) be two sequences in E^+ satisfying*

$$x_{n+1} \preceq \lambda x_n + y_n. \tag{2.1}$$

Then (x_n) is an e-sequence if (y_n) is an e-sequence.

We now state the following definition of partial E-cone metric space.

Definition 2.7. Let $X \neq \phi$ and E be a ordered normed space with assumption that $(E^+)^{\oslash}$ is non-empty. A partial E-cone metric on the set X is a function $p^E : X \times X \to E^+$ such that for all $x, y, z \in X$;

(p$_1$) $0_E \preceq p^E(x, x) \preceq p^E(x, y)$,

(p$_2$) $x = y \Longleftrightarrow p^E(x, x) = p^E(x, y) = p^E(y, y)$,

(p$_3$) $p^E(x, y) = p^E(y, x)$,

(p$_4$) $p^E(x, y) \preceq p^E(x, z) + p^E(x, y) - p^E(z, z)$.

A partial E-cone metric space is a pair (X, p^E) such that X is non-empty set and p^E is a partial E-cone metric on X.

It is clear that, if $p^E(x, y) = 0$, then from (p_1) and (p_2), $x = y$. But if $x = y$, $p^E(x, y)$ may not be equal to 0_E.

Now we define the e-convergence and the e-Cauchy convergence criteria in the ordered normed space E, with non-solid cone E^+.

Definition 2.8. Let E be an ordered normed space with the assumption that $(E^+)^{\oslash}$ is non-empty and $\left(X, p^E\right)$ be a partial E-cone metric. Let (x_n) be a sequence in X and $x \in X$. Then

(i) A sequence (x_n) is said to be e-converges to x if for every $0_E \lll e$, there exists a natural number n_0 such that

$$p^E(x_n, x) \lll e, \quad \text{for all } n \geq n_0.$$

In this case, we write $\lim\limits_{n\to\infty} x_n = x$ or $x_n \xrightarrow{e} x$.

(ii) A sequence (x_n) is said to be e-Cauchy sequence if for every $0_E \lll e$, there exists a natural number n_0 such that

$$p^E(x_n, x_m) \lll e, \quad \text{for all } n, m \geq n_0.$$

(iii) (X, p^E) is e-complete if every e-Cauchy sequence is e-convergent.

3 FIXED POINT THEOREMS

In this section, we recall some known definitions and give some applications to fixed point theory with respect to Hardy-Rogers type mappings on partial E-cone metric space.

Definition 3.1. [18] Let (X, p^E) be an e-complete partial E-cone metric space with closed positive cone E^+such that $(E^+)^{\oslash} \neq \emptyset$, (z_n) an e-sequence and T a self-map on X. Let $x_0 \in X$ and $x_{n+1} = Tx_n$ be the Picard's iteration in X. The iteration procedure $x_{n+1} = Tx_n$ is said to be T-stable with respect to T if (x_n) e-converges to a fixed point x of T, and $\left(p^E(x_{n+1}, Tx_n)\right)$ is an e-sequence, then (z_n) e-converges to x.

Definition 3.2. [18] Let (X, p^E) be a partial E-cone metric space with closed positive cone E^+such that $(E^+)^{\oslash} \neq \emptyset$. The mapping $T : X \to X$ is called Hardy-Rogers type on X if

$$\begin{aligned} p^E(Tx, Ty) &\preceq \alpha_1 p^E(x, y) + \alpha_2 p^E(x, Tx) + \alpha_3 p^E(y, Ty) \\ &\quad + \alpha_4 p^E(x, Ty) + \alpha_5 p^E(y, Tx). \end{aligned} \tag{3.1}$$

for all $x, y \in X$, where $\alpha_i \geq 0\,(i = 1, 2, 3, 4, 5)$ are constants and $0 \leq \sum\limits_{i=1}^{5} \alpha_i < 1$.

We begin with a simple, but useful theorem.

Theorem 3.3. *Let (X, p^E) be an e-complete partial E-cone metric space with closed positive cone E^+such that $(E^+)^{\oslash} \neq \emptyset$ and (x_n) a sequence in X satisfying*

$$p^E(x_n, x_{n+1}) \preceq \lambda p^E(x_{n-1}, x_n) \quad (n = 1, 2, ...),$$

where $0 \leq \lambda < 1$ is a constant. Then (x_n) is an e-Cauchy sequence in X.

Proof. Suppose that (x_n) is a contractive sequence in X. Then for some real number $\lambda \in [0, 1)$, we have

$$p^E(x_n, x_{n+1}) \preceq \lambda p^E(x_{n-1}, x_n) \preceq \lambda^2 p^E(x_{n-2}, x_{n-1}) \preceq ... \preceq \lambda^n p^E(x_0, x_1).$$

For any $n, m \in \mathbb{N}$, we have

$$\begin{aligned} p^E(x_m, x_n) &\preceq p^E(x_m, x_{m-1}) + p^E(x_{m-1}, x_{m-2}) + ... + p^E(x_{n+1}, x_n) \\ &\quad - \sum_{r=1}^{m-n-1} p^E(x_{m-r}, x_{m-r}). \end{aligned}$$

Using Proposition 2.1, we have

$$\begin{aligned} p^E(x_m, x_n) &\preceq p^E(x_m, x_{m-1}) + p^E(x_{m-1}, x_{m-2}) + ... + p^E(x_{n+1}, x_n) \\ &\preceq \left(\lambda^{m-1} + \lambda^{m-2} + ... + \lambda^n\right) p^E(x_0, x_1) \\ &\preceq \lambda^m \left(\frac{1 - \lambda^{n-m}}{1 - \lambda}\right) p^E(x_1, x_0). \end{aligned}$$

Let $0_E \lll e$ be given, choose $\rho > 0$ such that $e - \rho B_+ \subseteq E^+$ and a natural number k_1 such that $\lambda^m \left(\frac{1-\lambda^{n-m}}{1-\lambda}\right) p^E(x_1, x_0) \in \frac{\rho}{2} B_+$ for any $m, n \geq k_1$. Therefore

$$e - \lambda^m \left(\frac{1-\lambda^{n-m}}{1-\lambda}\right) p^E(x_1, x_0) - \frac{\rho}{2} B_+ \subseteq e - \rho B_+ \subseteq E^+.$$

Hence

$$p^E(x_m, x_n) \preceq \lambda^m \left(\frac{1-\lambda^{n-m}}{1-\lambda}\right) p^E(x_1, x_0) \lll e, \quad \text{for all } n, m \geq k_1.$$

It follows that (x_n) is an e-Cauchy sequence in X. □

Theorem 3.4. *Let (X, p^E) be an e-complete partial E-cone metric space with closed positive cone E^+such that $(E^+)^{\oslash} \neq \emptyset$. Let $T : X \to X$ be a Hardy-Rogers type mapping on X. Then T has a unique fixed in X, and for each $x \in X$, the iterative sequence $\{T^n x\}_{n \geqslant 0}$ $e-$converges to the unique fixed point.*

Proof. We choose $x_0 \in X$ and $n \geq 1$, consider the iterative sequence

$$x_{n+1} = Tx_n = T^{n+1} x_0.$$

Using (3.1), on one hand, we have

$$\begin{aligned} p^E(x_n, x_{n+1}) &= p^E(Tx_{n-1}, Tx_n) \\ &\preceq \alpha_1 p^E(x_{n-1}, x_n) + \alpha_2 p^E(x_{n-1}, Tx_{n-1}) + \alpha_3 p^E(x_n, Tx_n) \\ &\quad + \alpha_4 p^E(x_{n-1}, Tx_n) + \alpha_5 p^E(x_n, Tx_{n-1}). \\ &\preceq (\alpha_1 + \alpha_2) p^E(x_{n-1}, x_n) + \alpha_3 p^E(x_n, x_{n+1}) + \alpha_4 p^E(x_{n-1}, x_{n+1}) \\ &\quad + \alpha_5 p^E(x_n, x_n) \\ &\preceq (\alpha_1 + \alpha_2) p^E(x_{n-1}, x_n) + \alpha_3 p^E(x_n, x_{n+1}) + \alpha_5 p^E(x_n, x_n) \\ &\quad + \alpha_4 \left[p^E(x_{n-1}, x_n) + p^E(x_n, x_{n+1}) - p^E(x_n, x_n)\right]. \\ &\preceq (\alpha_1 + \alpha_2 + \alpha_4) p^E(x_{n-1}, x_n) + (\alpha_3 + \alpha_4) p^E(x_n, x_{n+1}) \\ &\quad + (\alpha_5 - \alpha_4) p^E(x_n, x_n). \end{aligned} \tag{3.2}$$

On the other hand, we obtain that

$$\begin{aligned} p^E(x_n, x_{n+1}) &= p^E(Tx_{n-1}, Tx_n) \\ &\preceq \alpha_1 p^E(x_{n-1}, x_n) + \alpha_2 p^E(x_n, Tx_n) + \alpha_3 p^E(x_{n-1}, Tx_{n-1}) \\ &\quad + \alpha_4 p^E(x_n, Tx_{n-1}) + \alpha_5 p^E(x_{n-1}, Tx_n) \\ &\preceq (\alpha_1 + \alpha_3) p^E(x_{n-1}, x_n) + \alpha_2 p^E(x_n, x_{n+1}) + \alpha_4 p^E(x_n, x_n) \\ &\quad + \alpha_5 \left[p^E(x_{n-1}, x_n) + p^E(x_n, x_{n+1}) - p^E(x_n, x_n)\right]. \\ &\preceq (\alpha_1 + \alpha_3 + \alpha_5) p^E(x_{n-1}, x_n) + (\alpha_2 + \alpha_5) p^E(x_n, x_{n+1}) \\ &\quad + (\alpha_4 - \alpha_5) p^E(x_n, x_n). \end{aligned} \tag{3.3}$$

Adding up (3.2) and (3.3) yields

$$\begin{aligned} 2p^E(x_n, x_{n+1}) &\preceq (2\alpha_1 + \alpha_2 + \alpha_3 + \alpha_4 + \alpha_5) p^E(x_{n-1}, x_n) \\ &\quad + (\alpha_2 + \alpha_3 + \alpha_4 + \alpha_5) p^E(x_n, x_{n+1}). \end{aligned}$$

Hence

$$p^E(x_n, x_{n+1}) \preceq \frac{2\alpha_1 + \alpha_2 + \alpha_3 + \alpha_4 + \alpha_5}{2 - \alpha_2 - \alpha_3 - \alpha_4 - \alpha_5} p^E(x_{n-1}, x_n). \tag{3.4}$$

Since $0 \leq \sum_{i=1}^{5} \alpha_i < 1$, we have $\alpha = \frac{2\alpha_1+\alpha_2+\alpha_3+\alpha_4+\alpha_5}{2-\alpha_2-\alpha_3-\alpha_4-\alpha_5} < 1$.

Using equation(3.4) and Theorem 3.3 , then (x_n) is an e-Cauchy sequence in X.
Since $\left(X, p^E\right)$ is an e-complete, then there exists $x \in X$ such that (x_n) e-converges to x.
In the following, we show that x is a fixed point of T.
Indeed, using (3.1) , we have

$$\begin{aligned}
p^E(Tx,x) &\preceq p^E(Tx,x_n) + p^E(x_n,x) - p^E(x_n,x_n)\\
&= p^E(Tx,Tx_{n-1}) + p^E(x_n,x)\\
&\preceq \alpha_1 p^E(x,x_{n-1}) + \alpha_2 p^E(x,Tx) + \alpha_3 p^E(x_{n-1},Tx_{n-1})\\
&\quad +\alpha_4 p^E(x,Tx_{n-1}) + \alpha_5 p^E(x_{n-1},Tx) + p^E(x_n,x)\\
&\preceq \alpha_1 p^E(x,x_{n-1}) + \alpha_2 p^E(x,Tx) + \alpha_3 \begin{pmatrix} p^E(x_{n-1},x) + p^E(x,Tx_{n-1}) \\ -p^E(x,x) \end{pmatrix}\\
&\quad +\alpha_4 p^E(x,x_n) + \alpha_5\left[p^E(x_{n-1},x) + p^E(x,Tx) - p^E(x,x)\right] + p^E(x_n,x)\\
&\preceq \alpha_1 p^E(x,x_{n-1}) + \alpha_2 p^E(x,Tx) + \alpha_3\left[p^E(x_{n-1},x) + p^E(x,Tx_{n-1})\right]\\
&\quad +\alpha_4 p^E(x,x_n) + \alpha_5\left[p^E(x_{n-1},x) + p^E(x,Tx)\right] + p^E(x_n,x)\\
&\preceq (\alpha_1+\alpha_3+\alpha_5)\, p^E(x_{n-1},x) + (\alpha_2+\alpha_5)\, p^E(Tx,x)\\
&\quad +(1+\alpha_3+\alpha_4)\, p^E(x_n,x)\,.
\end{aligned}$$

That means

$$p^E(Tx,x) \preceq \frac{\alpha_1+\alpha_3+\alpha_5}{1-\alpha_2-\alpha_5} p^E(x_{n-1},x) + \frac{1+\alpha_3+\alpha_4}{1-\alpha_2-\alpha_5} p^E(x_n,x) \triangleq \beta_n, \tag{3.5}$$

where $k_1 = \frac{\alpha_1+\alpha_3+\alpha_5}{1-\alpha_2-\alpha_5}$ and $k_2 = \frac{1+\alpha_3+\alpha_4}{1-\alpha_2-\alpha_5}$ are positive numbers.

Since (x_n) e-converges to x, then $\left(p^E(x_n,x)\right)$ e-converges to 0_E. Thus, (β_n) also e-converges to 0_E. Therefore by (3.5) for any $e \ggg 0_E$, there exists $n_1 \in \mathbb{N}$ such that for all $n \geqslant n_1$, ones have

$$p^E(Tx,x) \lll e. \tag{3.6}$$

By Proposition 2.3, $p^E(Tx,x) = 0_E$, i.e, x is a fixed point of T.

To prove that the fixed point x is unique, let y be another fixed of T. Then using (3.1), it follows that

$$\begin{aligned}
p^E(x,y) &= p^E(Tx,Ty)\\
&\preceq \alpha_1 p^E(x,y) + \alpha_2 p^E(y,Ty) + \alpha_3 p^E(x,Tx)\\
&\quad +\alpha_4 p^E(y,Tx) + \alpha_5 p^E(x,Ty)\\
&= (\alpha_1+\alpha_4+\alpha_5)\, p^E(x,y)\,.
\end{aligned}$$

As $0 \leq \alpha_1+\alpha_4+\alpha_5 \leq \sum_{i=1}^{5} \alpha_i < 1$, by Proposition 2.4, we get

$$p^E(x,y) = 0_E.$$

Hence, $x = y$. □

Theorem 3.5. *Let $\left(X, p^E\right)$ be an e-complete partial E-cone metric space with closed positive cone E^+such that $\left(E^+\right)^{\oslash} \neq \emptyset$. If $T : X \to X$ is a Hardy-Rogers type mapping on X, then the Picard's iteration is T-stable.*

Proof. Let z be any fixed point of T and let (x_n) be a sequence in X such that $\left(p^E(x_{n+1},Tx_n)\right)$ is an e-sequence.

Using (3.1), we have

$$\begin{aligned} p^E(Tx_n,z) &= p^E(Tx_n,Tz) \\ &\preceq \alpha_1 p^E(x_n,z)+\alpha_2 p^E(x_n,Tx_n)+\alpha_3 p^E(z,Tz) \\ &\quad +\alpha_4 p^E(x_n,Tz)+\alpha_5 p^E(z,Tx_n) \\ &= \alpha_1 p^E(x_n,z)+\alpha_2 p^E(x_n,x_{n+1})+\alpha_3 p^E(z,z) \\ &\quad +\alpha_4 p^E(x_n,z)+\alpha_5 p^E(z,x_{n+1}) \\ &\preceq \alpha_1 p^E(x_n,z)+\alpha_2\left[p^E(x_n,z)+p^E(z,x_{n+1})-p^E(z,z)\right] \\ &\quad +\alpha_3 p^E(z,z)+\alpha_4 p^E(x_n,z)+\alpha_5 p^E(z,x_{n+1}) \\ &\preceq (\alpha_1+\alpha_2+\alpha_4)\,p^E(x_n,z)+(\alpha_2+\alpha_5)\,p^E(z,x_{n+1}) \\ &\quad +(\alpha_3-\alpha_2)\,p^E(z,z). \end{aligned} \tag{3.7}$$

On the other hand we have

$$\begin{aligned} p^E(Tx_n,z) &= p^E(z,Tx_n)=p^E(Tz,Tx_n) \\ &\preceq \alpha_1 p^E(z,x_n)+\alpha_2 p^E(z,Tz)+\alpha_3 p^E(x_n,Tx_n) \\ &\quad +\alpha_4 p^E(z,Tx_n)+\alpha_5 p^E(x_n,Tz) \\ &= \alpha_1 p^E(x_n,z)+\alpha_2 p^E(z,z)+\alpha_3 p^E(x_n,Tx_n) \\ &\quad +\alpha_4 p^E(z,x_{n+1})+\alpha_5 p^E(x_n,z) \\ &\preceq \alpha_1 p^E(x_n,z)+\alpha_2 p^E(z,z) \\ &\quad +\alpha_3\left[p^E(x_n,z)+p^E(z,x_{n+1})-p^E(z,z)\right] \\ &\quad +\alpha_4 p^E(z,x_{n+1})+\alpha_5 p^E(x_n,z) \\ &\preceq (\alpha_1+\alpha_3+\alpha_5)\,p^E(x_n,z)+(\alpha_3+\alpha_4)\,p^E(z,x_{n+1}) \\ &\quad +(\alpha_2-\alpha_3)\,p^E(z,z). \end{aligned} \tag{3.8}$$

Adding up (3.7) and (3.8), yields

$$\begin{aligned} 2p^E(Tx_n,z) &\preceq (2\alpha_1+\alpha_2+\alpha_3+\alpha_4+\alpha_5)\,p^E(x_n,z) \\ &\quad +(\alpha_2+\alpha_3+\alpha_4+\alpha_5)\,p^E(z,x_{n+1}). \end{aligned}$$

That means

$$p^E(Tx_n,z)\preceq \frac{2\alpha_1+\alpha_2+\alpha_3+\alpha_4+\alpha_5}{2-\alpha_2-\alpha_3-\alpha_4-\alpha_5}p^E(x_n,z),$$

Now, since $0\le\sum_{i=1}^{5}\alpha_i<1$, we have $\alpha=\frac{2\alpha_1+\alpha_2+\alpha_3+\alpha_4+\alpha_5}{2-\alpha_2-\alpha_3-\alpha_4-\alpha_5}<1$, and

$$p^E(Tx_n,z)\preceq \alpha p^E(x_n,z).$$

Setting $u_n=p^E(x_n,z)$ and $w_n=p^E(x_{n+1},Tx_n)$, we get

$$\begin{aligned} u_{n+1} &= p^E(x_{n+1},z)\preceq p^E(x_{n+1},Tx_n)+p^E(Tx_n,z)-p^E(Tx_n,Tx_n) \\ &\preceq p^E(x_{n+1},Tx_n)+p^E(Tx_n,Tz) \\ &\preceq p^E(x_{n+1},Tx_n)+\alpha p^E(x_n,z) \\ &= \alpha u_n+w_n. \end{aligned}$$

Since (w_n) is an e-sequence, using Proposition 2.6, we deduce that (u_n) is an e-sequence. Thus, (x_n) converges to z as $n\to\infty$. This implies that the Picard's iteration is T-stable. □

Theorem 3.6. *Let (X,p^E) be an e-complete partial E-cone metric space with closed positive cone E^+such that $(E^+)^{\varnothing}\neq\emptyset$. If $T:X\to X$ is a Hardy-Rogers type mapping on X, then $(p^E(x_n,Tx_n))$ is an e-sequence if and only if $(p^E(x_{n+1},Tx_n))$ is an e-sequence.*

Proof. Suppose (x_n) is a sequence in X. Put $v_n = p^E(x_n, Tx_n)$ and $w_n = p^E(x_{n+1}, Tx_n)$.

If (v_n) is an e-sequence, then

$$\begin{aligned} w_n &= p^E(x_{n+1}, Tx_n) \preceq p^E(x_{n+1}, Tx_{n+1}) + p^E(Tx_{n+1}, Tx_n) - p^E(Tx_{n+1}, Tx_{n+1}) \\ &\preceq p^E(x_{n+1}, Tx_{n+1}) + p^E(Tx_{n+1}, Tx_n) \\ &\preceq p^E(x_{n+1}, Tx_{n+1}) + \alpha_1 p^E(x_{n+1}, x_n) + \alpha_2 p^E(x_{n+1}, Tx_{n+1}) \\ &\quad + \alpha_3 p^E(x_n, Tx_n) + \alpha_4 p^E(x_{n+1}, Tx_n) + \alpha_5 p^E(x_n, Tx_{n+1}) \\ &\preceq p^E(x_{n+1}, Tx_{n+1}) + \alpha_1\left(p^E(x_{n+1}, Tx_n) + p^E(x_n, Tx_n) - p^E(Tx_n, Tx_n)\right) \\ &\quad + \alpha_2 p^E(x_{n+1}, Tx_{n+1}) + \alpha_3 p^E(x_n, Tx_n) + \alpha_4 p^E(x_{n+1}, Tx_n) \\ &\quad + \alpha_5 \begin{pmatrix} p^E(x_n, Tx_n) + p^E(Tx_n, x_{n+1}) + p^E(x_{n+1}, Tx_{n+1}) \\ -p^E(x_{n+1}, x_{n+1}) - p^E(Tx_n, Tx_n) \end{pmatrix} \\ &\preceq p^E(x_{n+1}, Tx_{n+1}) + \alpha_1\left(p^E(x_{n+1}, Tx_n) + p^E(x_n, Tx_n)\right) \\ &\quad + \alpha_2 p^E(x_{n+1}, Tx_{n+1}) + \alpha_3 p^E(x_n, Tx_n) + \alpha_4 p^E(x_{n+1}, Tx_n) \\ &\quad + \alpha_5\left(p^E(x_n, Tx_n) + p^E(Tx_n, x_{n+1}) + p^E(x_{n+1}, Tx_{n+1})\right) \\ &= (1 + \alpha_2 + \alpha_5) p^E(x_{n+1}, Tx_{n+1}) + (\alpha_1 + \alpha_3 + \alpha_5) p^E(x_n, Tx_n) \\ &\quad + (\alpha_1 + \alpha_4 + \alpha_5) p^E(x_{n+1}, Tx_n) \\ &= (1 + \alpha_2 + \alpha_5) v_{n+1} + (\alpha_1 + \alpha_3 + \alpha_5) v_n + (\alpha_1 + \alpha_4 + \alpha_5) w_n. \end{aligned}$$

It is obvious that

$$w_n \preceq \frac{1 + \alpha_2 + \alpha_5}{1 - \alpha_1 - \alpha_4 - \alpha_5} v_{n+1} + \frac{\alpha_1 + \alpha_3 + \alpha_5}{1 - \alpha_1 - \alpha_4 - \alpha_5} v_n \triangleq \gamma_n,$$

where $k_3 = \frac{1+\alpha_2+\alpha_5}{1-\alpha_1-\alpha_4-\alpha_5}$ and $k_4 = \frac{\alpha_1+\alpha_3+\alpha_5}{1-\alpha_1-\alpha_4-\alpha_5}$ are positive numbers.

Since (v_n) is an e-sequence, it follows that (v_n) e-converges to 0_E and so (γ_n) e-converges to 0_E. Thus, given $e \ggg 0_E$ there exists $K > 0$ such that $w_n \preceq \gamma_n \lll e$. Hence (w_n) e-converges to 0_E and (w_n) is an e-sequence.

Conversely, if (w_n) is an e-sequence, then for one thing, we have

$$\begin{aligned} v_n &= p^E(x_n, Tx_n) \preceq p^E(x_n, Tx_{n-1}) + p^E(Tx_{n-1}, Tx_n) - p^E(x_n, x_n) \\ &\preceq p^E(x_n, Tx_{n-1}) + p^E(Tx_n, Tx_{n-1}) \\ &\preceq p^E(x_n, Tx_{n-1}) + \alpha_1 p^E(x_n, x_{n-1}) + \alpha_2 p^E(x_n, Tx_n) \\ &\quad + \alpha_3 p^E(x_{n-1}, Tx_{n-1}) + \alpha_4 p^E(x_n, Tx_{n-1}) + \alpha_5 p^E(x_{n-1}, Tx_n) \\ &\preceq p^E(x_n, x_n) + \alpha_1\left(p^E(x_n, Tx_{n-1}) + p^E(Tx_{n-1}, x_{n-1}) - p^E(Tx_{n-1}, Tx_{n-1})\right) \\ &\quad + \alpha_2 p^E(x_n, x_{n+1}) + \alpha_3 p^E(x_{n-1}, x_n) + \alpha_4 p^E(x_n, x_n) \\ &\quad + \alpha_5 \begin{pmatrix} p^E(x_{n-1}, Tx_{n-1}) + p^E(Tx_{n-1}, x_n) + p^E(x_n, Tx_n) \\ -p^E(Tx_{n-1}, Tx_{n-1}) - p^E(x_n, x_n) \end{pmatrix} \\ &= (\alpha_2 + \alpha_5) p^E(x_n, x_{n+1}) + (\alpha_1 + \alpha_3 + \alpha_5) p^E(x_{n-1}, x_n) \\ &\quad + (1 + \alpha_1 + \alpha_4 - \alpha_5) p^E(x_n, x_n) \\ &= (\alpha_2 + \alpha_5) v_n + (\alpha_1 + \alpha_3 + \alpha_5) v_{n-1} + (1 + \alpha_4 - \alpha_5) w_{n-1}. \end{aligned}$$

This implies

$$(1 - \alpha_2 - \alpha_5) v_n \preceq (\alpha_1 + \alpha_3 + \alpha_5) v_{n-1} + (1 + \alpha_1 + \alpha_4 - \alpha_5) w_{n-1}. \tag{3.9}$$

While

$$\begin{aligned} v_n &= p^E(x_n,Tx_n) \preceq p^E(x_n,Tx_{n-1}) + p^E(Tx_{n-1},Tx_n) - p^E(Tx_{n-1},Tx_{n-1}) \\ &\preceq p^E(x_n,Tx_{n-1}) + p^E(Tx_{n-1},Tx_n) \\ &\preceq p^E(x_n,Tx_{n-1}) + \alpha_1 p^E(x_{n-1},x_n) + \alpha_2 p^E(x_{n-1},Tx_{n-1}) + \alpha_3 p^E(x_n,Tx_n) \\ &\quad + \alpha_4 p^E(x_{n-1},Tx_n) + \alpha_5 p^E(x_n,Tx_{n-1}) \\ &\preceq p^E(x_n,x_n) + \alpha_1\left(p^E(x_{n-1},Tx_{n-1}) + p^E(Tx_{n-1},x_n) - p^E(Tx_{n-1},Tx_{n-1})\right) \\ &\quad + \alpha_2 p^E(x_{n-1},x_n) + \alpha_3 p^E(x_n,x_{n+1}) + \alpha_5 p^E(x_n,x_n) \\ &\quad + \alpha_4 \begin{pmatrix} p^E(x_{n-1},Tx_{n-1}) + p^E(Tx_{n-1},x_n) +_4 p^E(x_n,Tx_n) \\ -p^E(x_n,x_n) - p^E(Tx_{n-1},Tx_{n-1}) \end{pmatrix} \\ &= (\alpha_3+\alpha_4)\, p^E(x_n,x_{n+1}) + (\alpha_1+\alpha_2+\alpha_4)\, p^E(x_{n-1},x_n) + (1+\alpha_5-\alpha_4)\, p^E(x_n,x_n) \\ &= (\alpha_3+\alpha_4)\, v_n + (\alpha_1+\alpha_2+\alpha_4)\, v_{n-1} + (1+\alpha_5-\alpha_4)\, w_{n-1}. \end{aligned}$$

That means

$$(1-\alpha_3-\alpha_4)\, v_n \preceq (\alpha_1+\alpha_2+\alpha_4)\, v_{n-1} + (1+\alpha_5-\alpha_4)\, w_{n-1}. \tag{3.10}$$

Adding (3.9) to (3.10) we get

$$v_n \preceq \frac{2\alpha_1+\alpha_2+\alpha_3+\alpha_4+\alpha_5}{2-\alpha_2-\alpha_3-\alpha_4-\alpha_5} v_{n-1} + \frac{2}{2-\alpha_2-\alpha_3-\alpha_4-\alpha_5} w_{n-1},$$

Now since $0 \le \alpha = \frac{2\alpha_1+\alpha_2+\alpha_3+\alpha_4+\alpha_5}{2-\alpha_2-\alpha_3-\alpha_4-\alpha_5} < 1$ using Proposition 2.6 we get that (v_n) is an e-sequence. □

Theorem 3.7. *Let (X, p^E) be an e-complete partial E-cone metric space with closed positive cone E^+such that $(E^+)^{\oslash} \neq \emptyset$. Let $T : X \to X$ be a mapping satisfying*

$$p^E(Tx,Ty) \preceq \alpha p^E(x,y),$$

for all $x, y \in X$ and some $\alpha \in [0,1)$. Then T has a unique fixed point in X, and for each $x \in X$, the iterative sequence $(T^n x)_{n\geq 0}$ converges to the fixed point of T.

Proof. The result follows from Theorem if $\alpha_2 = \alpha_3 = \alpha_4 = \alpha_5 = 0$. □

Remark 3.8. Theorem 3.4 greatly generalizes the main theorems of [24]. As a matter of fact, if we take $\alpha_1 = \alpha_4 = \alpha_5 = 0$ and $\alpha_2 = \alpha_3 = \alpha \in \left[0, \frac{1}{2}\right)$ in (3.1) then give us Theorem of Kannan fixed point theorem; if take $\alpha_1 = \alpha_2 = \alpha_3 = 0$ and $\alpha_4 = \alpha_5 = \alpha \in \left[0, \frac{1}{2}\right)$in (3.1) , then give us Theorem Chatterjea fixed point theorem in partial E-cone metric space.

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Received: June 13, 2022.
Accepted: January 28, 2023.