Palestine Journal of Mathematics

Vol. 12(3)(2023) , 74–84

A note on type 2 poly-Bernoulli polynomials of the second kind

Waseem A. Khan[1] and M. Kamarujjama[2]

Communicated by Jose Luis Lopez-Bonilla

MSC 2010 Classifications: 11B83, 05A19.

Keywords and phrases: polyexponential functions, type 2 poly-Bernoulli polynomials of the second kind, unipoly functions.

Abstract Recently, Kim-Kim [4] introduced the type 2 poly-Bernoulli polynomials and numbers arising from the polyexponential function. In this paper, we study the type 2 poly-Bernoulli polynomials of the second kind arising from polyexponential function and derive their explicit expressions and some identity involving them. Also, we derive the type 2 unipoly-Bernoulli polynomials of the second kind and discuss some properties of them.

1 Introduction

It is well known, the ordinary Bernoulli polynomials of the second are defined by the generating function

$$\frac{t}{\log(1+t)}(1+t)^x = \sum_{n=0}^{\infty} b_n(x)\frac{t^n}{n!}, \text{(see [2, 3, 9, 14])}. \tag{1.1}$$

When $x = 0$, $b_n = b_n(0)$ are called the Bernoulli numbers of the second kind.

In (2019), Kim-Kim [7] introduced the polyexponential function, as an inverse to the polylogarithm function to be

$$\text{Ei}_k(x) = \sum_{n=1}^{\infty} \frac{x^n}{(n-1)!n^k}, (k \in \mathbb{Z}). \tag{1.2}$$

For $k = 1$, (1.2) gives

$$\text{Ei}_1(x) = \sum_{n=1}^{\infty} \frac{x^n}{n!} = e^x - 1. \tag{1.3}$$

For $k \in \mathbb{Z}$, Kim-Kim considered the type 2 poly-Bernoulli polynomials are defined by means of the following generating function

$$\frac{\text{Ei}_k\left(\log(1+t)\right)}{e^t - 1} e^{xt} = \sum_{n=0}^{\infty} B_n^{(k)}(x)\frac{t^n}{n!}, \text{(see [7])}. \tag{1.4}$$

When $x = 0$, $B_n^{(k)} = B_n^{(k)}(0)$ are called the type 2 poly-Bernoulli numbers. Note that $B_n^{(1)}(x) = B_n(x)$ are called the ordinary Bernoulli polynomials.

For $k \in \mathbb{Z}$, the polylogarithm function is defined by

$$\text{Li}_k(x) = \sum_{n=1}^{\infty} \frac{x^n}{n^k}, (\mid x \mid < 1), \text{(see, [1, 8])}. \tag{1.5}$$

Note that

$$\text{Li}_1(x) = \sum_{n=1}^{\infty} \frac{x^n}{n} = -\log(1-x). \tag{1.6}$$

Kim *et al.* [9] introduced the poly-Bernoulli polynomials of the second given by

$$\frac{\mathrm{Li}_k(1-e^{-t})}{\log(1+t)}(1+t)^x = \sum_{n=0}^{\infty} b_n^{(k)}(x)\frac{t^n}{n!}. \tag{1.7}$$

When $x = 0$, $b_n^{(k)} = b_n^{(k)}(0)$ are called the poly-Bernoulli numbers of the second kind.

The Daehee polynomials are defined by

$$\frac{\log(1+t)}{t}(1+t)^x = \sum_{n=0}^{\infty} D_n(x)\frac{t^n}{n!}, \text{(see [10, 16])}. \tag{1.8}$$

When $x = 0$, $D_n = D_n(0)$ are called the Daehee numbers.

For $n \geq 0$, the Stirling numbers of the first kind are defined by

$$(x)_n = \sum_{l=0}^{n} S_1(n,l)x^l, \text{(see [4, 5, 6])}, \tag{1.9}$$

where $(x)_0 = 1$, and $(x)_n = x(x-1)\cdots(x-n+1), (n \geq 1)$. From (1.9), it is easily to see that

$$\frac{1}{k!}(\log(1+t))^k = \sum_{n=k}^{\infty} S_1(n,k)\frac{t^n}{n!}, \quad (k \geq 0), \text{(see [11, 12, 13])}. \tag{1.10}$$

In the inverse expression to (1.9), the Stirling numbers of the second kind are defined by

$$x^n = \sum_{l=0}^{n} S_2(n,l)(x)_l. \tag{1.11}$$

From (1.11), it is easily to see that

$$\frac{1}{k!}(e^t-1)^k = \sum_{n=l}^{\infty} S_2(n,l)\frac{t^n}{n!}, \text{(see [15, 17])}. \tag{1.12}$$

In this paper, we introduce type 2 poly-Bernoulli polynomials and numbers of the second kind by arising from polyexponential functions. We investigate some fundamental properties of these polynomials. Furthermore, we consider type 2 unipoly-Bernoulli polynomials and numbers of the second kind and derive explicit expressions for these polynomials.

2 Type 2 poly-Bernoulli polynomials of the second kind

For $k \in \mathbb{Z}$, and by means of the polyexponential functions, we define the type 2 poly-Bernoulli polynomials of the second kind by

$$\frac{\mathrm{Ei}_k\left(\log(1+t)\right)}{\log(1+t)}(1+t)^x = \sum_{n=0}^{\infty} Pb_n^{(k)}(x)\frac{t^n}{n!}. \tag{2.1}$$

When $x = 0$, $Pb_n^{(k)} = Pb_n^{(k)}(0)$ are called the type 2 poly-Bernoulli numbers of the second kind.

For $k = 1$, by using (1.3) and (2.1), we see that

$$\frac{\mathrm{Ei}_1\left(\log(1+t)\right)}{\log(1+t)}(1+t)^x = \frac{t}{\log(1+t)}(1+t)^x = \sum_{n=0}^{\infty} b_n(x)\frac{t^n}{n!}, \tag{2.2}$$

where $b_n(x)$ are called the ordinary Bernoulli polynomials of the second kind.

By (2.1) and (2.2), we get

$$Pb_n^{(1)}(x) = b_n(x), (n \geq 0).$$

First, we note that

$$\begin{aligned}
\mathrm{Ei}_k\left(\log(1+t)\right) &= \sum_{m=1}^{\infty} \frac{(\log(1+t))^m}{(m-1)!m^k} \\
&= \sum_{m=0}^{\infty} \frac{(\log(1+t))^{m+1}}{m!(m+1)^k} \\
&= \sum_{m=0}^{\infty} \frac{1}{(m+1)^{k-1}} \frac{1}{(m+1)!}(\log(1+t))^{m+1} \\
&= \sum_{m=0}^{\infty} \frac{1}{(m+1)^{k-1}} \sum_{l=m+1}^{\infty} S_1(l, m+1)\frac{t^l}{l!}. \qquad (2.3)
\end{aligned}$$

By (2.3), we see that (2.1) is equal to

$$\begin{aligned}
\frac{t}{\log(1+t)}(1+t)^x \mathrm{Ei}_k\left(\log(1+t)\right) &= \frac{t}{\log(1+t)}(1+t)^x \sum_{m=0}^{\infty} \frac{1}{(m+1)^{k-1}} \sum_{l=m}^{\infty} \frac{S_1(l+1, m+1)}{l+1}\frac{t^l}{l!} \\
&= \sum_{n=0}^{\infty} b_n(x)\frac{t^n}{n!} \sum_{m=0}^{\infty} \frac{1}{(m+1)^{k-1}} \sum_{l=m}^{\infty} \frac{S_1(l+1, m+1)}{l+1}\frac{t^l}{l!} \\
&= \sum_{n=0}^{\infty} \left(\sum_{l=0}^{n} \binom{n}{l} \sum_{m=0}^{l} \frac{1}{(m+1)^{k-1}} \frac{S_1(l+1, m+1)}{l+1} b_{n-l}(x) \right) \frac{t^n}{n!}. \qquad (2.4)
\end{aligned}$$

Therefore, by (2.3) and (2.4), we obtain the following theorem.

Theorem 2.1. For $k \in \mathbb{Z}$ and $n \geq 0$, we have

$$Pb_n^{(k)}(x) = \sum_{l=0}^{n} \binom{n}{l} \sum_{m=0}^{l} \frac{1}{(m+1)^{k-1}} \frac{S_1(l+1, m+1)}{l+1} b_{n-l}(x).$$

Corollary 2.1. For $n \geq 0$, we have

$$Pb_n(x) = \sum_{l=0}^{n} \binom{n}{l} \sum_{m=0}^{l} \frac{S_1(l+1, m+1)}{l+1} b_{n-l}(x).$$

Moreover,

$$\sum_{l=0}^{n} \binom{n}{l} \sum_{m=0}^{l} \frac{S_1(l+1, m+1)}{l+1} b_{n-l}(x) = 0, (n \geq 1).$$

Let $k \geq 1$, be an integer. For $s \in \mathbb{C}$, we define the function $\eta_k(s)$ as

$$\eta_k(s) = \frac{1}{\Gamma(s)} \int_0^{\infty} \frac{t^{s-1}}{\log(1+t)} \mathrm{Ei}_k\left(\log(1+t)\right) dt. \qquad (2.5)$$

From (2.5), we have

$$\begin{aligned}
\eta_1(s) &= \frac{1}{\Gamma(s)} \int_0^{\infty} \frac{t^{s-1}}{\log(1+t)} \mathrm{Ei}_1\left(\log(1+t)\right) dt = \frac{1}{\Gamma(s)} \int_0^{\infty} \frac{t^s}{\log(1+t)} dt \\
&= \frac{s}{s\Gamma(s)} \int_0^{\infty} \frac{t^s}{\log(1+t)} dt = \frac{s}{\Gamma(s+1)} \int_0^{\infty} \frac{t^s}{\log(1+t)} dt = s\zeta(s+1), \qquad (2.6)
\end{aligned}$$

where

$$\zeta(s)=\sum_{n=1}^{\infty}\frac{1}{n^s}.$$

is the Rieman zeta function.

We see from (2.6) that $\eta_k(s)$ is holomorphic for $\Re(s)>0$, since $\mathrm{Ei}_k(\log(1+t))\leq \mathrm{Ei}_1(\log(1+t))$, for $t\geq 0$. From (2.6), we note that

$$\begin{aligned}\eta_k(s)&=\frac{1}{\Gamma(s)}\int_0^{\infty}\frac{t^{s-1}}{\log(1+t)}\mathrm{Ei}_k(\log(1+t))\,dt\\&=\frac{1}{\Gamma(s)}\int_0^{1}\frac{t^{s-1}}{\log(1+t)}\mathrm{Ei}_k(\log(1+t))\,dt\\&+\frac{1}{\Gamma(s)}\int_1^{\infty}\frac{t^{s-1}}{\log(1+t)}\mathrm{Ei}_k(\log(1+t))\,dt.\end{aligned}\tag{2.7}$$

The second integral converges absolutely for any $s\in\mathbb{C}$ and hence, the second term on the right hand side vanishes at non-positive integers. That is,

$$\lim_{s\to -m}\left|\frac{1}{\Gamma(s)}\int_1^{\infty}\frac{t^{s-1}}{\log(1+t)}\mathrm{Ei}_k(\log(1+t))\,dt\right|\leq\frac{1}{\Gamma(-m)}M=0.\tag{2.8}$$

On the other hand, for $\Re(s)>0$, the first integral in (2.7) can be written as

$$\frac{1}{\Gamma(s)}\sum_{l=0}^{\infty}\frac{Pb_l^{(k)}}{l!}\frac{1}{s+l},$$

which defines an entire function of s. Thus, we may include that $\eta_k(s)$ can be continued to an entire function of s.

Further, from (2.7) and (2.8), we obtain

$$\begin{aligned}\eta_k(-m)&=\lim_{s\to -m}\frac{1}{\Gamma(s)}\int_0^{1}\frac{t^{s-1}}{\log(1+t)}\mathrm{Ei}_k(\log(1+t))\,dt\\&=\lim_{s\to -m}\frac{1}{\Gamma(s)}\int_0^{1}t^{s-1}\sum_{l=0}^{\infty}\frac{Pb_l^{(k)}t^l}{l!}dt=\lim_{s\to -m}\frac{1}{\Gamma(s)}\sum_{l=0}^{\infty}\frac{Pb_l^{(k)}}{s+l}\frac{1}{l!}\\&=\cdots+0+\cdots+0+\lim_{s\to -m}\frac{1}{\Gamma(s)}\frac{1}{s+m}\frac{Pb_m^{(k)}}{m!}+0+0+\cdots\\&=\lim_{s\to -m}\frac{\left(\frac{\Gamma(1-s)\sin\pi s}{\pi}\right)}{s+m}\frac{Pb_m^{(k)}}{m!}=\Gamma(1+m)\cos(\pi m)\frac{Pb_m^{(k)}}{m!}\\&=(-1)^m Pb_m^{(k)}.\end{aligned}\tag{2.9}$$

Therefore, by (2.9), we obtain the following theorem.

Theorem 2.2. Let $k\geq 1$ and $m\in\mathbb{N}\bigcup\{0\}$, $s\in\mathbb{C}$, we have

$$\eta_k(-m)=(-1)^m Pb_m^{(k)}.$$

Using (1.2), we observe that

$$\frac{d}{dx}\mathrm{Ei}_k(\log(1+x))=\frac{d}{dx}\sum_{n=1}^{\infty}\frac{(\log(1+x))^n}{n^k(n-1)!}$$

$$= \frac{1}{(1+x)\log(1+x)} \sum_{n=1}^{\infty} \frac{(\log(1+x))^n}{n^{k-1}(n-1)!} = \frac{1}{(1+x)\log(1+x)} \mathrm{Ei}_{k-1}(\log(1+x)). \quad (2.10)$$

Thus, by (2.10), for $k \geq 2$, we get

$$\mathrm{Ei}_k(\log(1+x)) = \int_0^x \frac{1}{(1+t)\log(1+t)} \mathrm{Ei}_{k-1}(\log(1+t))dt$$

$$= \int_0^x \underbrace{\frac{1}{(1+t)\log(1+t)} \int_0^t \cdots \frac{1}{(1+t)\log(1+t)} \int_0^t \frac{1}{(1+t)\log(1+t)}}_{(k-2)-\text{times}} dt \cdots dt$$

$$\times \mathrm{Ei}_1(\log(1+t))dt \cdots dt$$

$$= \int_0^x \underbrace{\frac{1}{(1+t)\log(1+t)} \int_0^t \cdots \frac{1}{(1+t)\log(1+t)} \int_0^t \frac{1}{(1+t)\log(1+t)}}_{(k-2)-\text{times}} dt \cdots dt. \quad (2.11)$$

From (2.1) and (2.11), we get

$$\sum_{n=0}^{\infty} Pb_n^{(k)} \frac{x^n}{n!} = \frac{\mathrm{Ei}_k\left(\log(1+x)\right)}{\log(1+x)} = \frac{1}{\log(1+x)}$$

$$\times \int_0^x \underbrace{\frac{1}{(1+t)\log(1+t)} \int_0^t \frac{1}{(1+t)\log(1+t)} \cdots}_{(k-2)-\text{times}} \int_0^t \frac{t}{(1+t)\log(1+t)} dt \cdots dt. \quad (2.12)$$

$$= \frac{1}{\log(1+x)} \sum_{m=0}^{\infty} \sum_{m_1+\cdots+m_{k-1}=m} \binom{m}{m_1, \cdots, m_{k-1}}$$

$$\times \frac{B_{m_1}^{(m_1)}(0)}{m_1+1} \frac{B_{m_2}^{(m_2)}(0)}{m_1+m_2+1} \cdots \frac{B_{m_{k-1}}^{(m_{k-1})}(0)}{m_1+\cdots+m_{k-1}+1} \frac{x^n}{n!}$$

$$= \sum_{n=0}^{\infty} \sum_{m=0}^{n} \binom{n}{m} \sum_{m_1+\cdots+m_{k-1}=m} \binom{m}{m_1, \cdots, m_{k-1}} b_{n-m}$$

$$\times \frac{B_{m_1}^{(m_1)}(0)}{m_1+1} \frac{B_{m_2}^{(m_2)}(0)}{m_1+m_2+1} \cdots \frac{B_{m_{k-1}}^{(m_{k-1})}(0)}{m_1+\cdots+m_{k-1}+1} \frac{x^n}{n!}. \quad (2.13)$$

Therefore, by (2.13), we obtain the following theorem.

Theorem 2.3. For $n \in \mathbb{N}$ and $k \in \mathbb{Z}$, we have

$$Pb_n^{(k)} = \sum_{m=0}^{n} \binom{n}{m} \sum_{m_1+\cdots+m_{k-1}=m} \binom{m}{m_1, \cdots, m_{k-1}} b_{n-m}$$

$$\times \frac{B_{m_1}^{(m_1)}(0)}{m_1+1} \frac{B_{m_2}^{(m_2)}(0)}{m_1+m_2+1} \cdots \frac{B_{m_{k-1}}^{(m_{k-1})}(0)}{m_1+\cdots+m_{k-1}+1}.$$

Corollary 2.2. For $n \geq 0$, we have

$$Pb_n^{(2)} = \sum_{m=0}^{n} \binom{n}{m} \frac{B_{m+1}^{(m)}(0)}{l+1} b_{n-m}.$$

Replacing t by $e^t - 1$ in (2.1), we get

$$\sum_{m=0}^{\infty} Pb_m^{(k)}(x) \frac{(e^t-1)^m}{m!} = \frac{\mathrm{Ei}_k\left(t\right)}{t} e^{xt}$$

$$= \sum_{n=0}^{\infty} \frac{x^n t^n}{n!} \sum_{l=0}^{\infty} \frac{t^l}{(l+1)^k l!} = \sum_{n=0}^{\infty} \left(\sum_{l=0}^{n} \binom{n}{l} \frac{x^{n-l}}{(l+1)^k} \right) \frac{t^n}{n!}. \tag{2.14}$$

On the other hand,

$$\sum_{m=0}^{\infty} Pb_m^{(k)}(x) \frac{(e^t-1)^m}{m!} = \sum_{m=0}^{\infty} Pb_m^{(k)}(x) \sum_{n=m}^{\infty} S_2(n,m) \frac{t^n}{n!}$$

$$= \sum_{n=0}^{\infty} \left(\sum_{m=0}^{n} Pb_m^{(k)}(x) S_2(n,m) \right) \frac{t^n}{n!}. \tag{2.15}$$

Therefore, by equations (2.14) and (2.15), we obtain the following theorem.

Theorem 2.4. For $k \in \mathbb{Z}$ and $n \geq 0$, we have

$$\sum_{m=0}^{n} Pb_m^{(k)}(x) S_2(n,m) = \sum_{l=0}^{n} \binom{n}{l} \frac{x^{n-l}}{(l+1)^k}.$$

From (2.1), we get

$$\sum_{n=0}^{\infty} \left[b_n^{(k)}(x+1) - b_n^{(k)}(x) \right] \frac{t^n}{n!} = \frac{\mathrm{Ei}_k\left(\log(1+t)\right)}{\log(1+t)} (1+t)^{x+1} - \frac{\mathrm{Ei}_k\left(\log(1+t)\right)}{\log(1+t)} (1+t)^x$$

$$= \frac{t\mathrm{Ei}_k\left(\log(1+t)\right)}{\log(1+t)} (1+t)^x = \left(\frac{t}{\log(1+t)} (1+t)^x \right) \left(\mathrm{Ei}_k\left(\log(1+t)\right) \right)$$

$$= \left(\sum_{n=0}^{\infty} b_n(x) \frac{t^n}{n!} \right) \left(\sum_{m=1}^{\infty} \frac{(\log(1+t))^m}{(m-1)! m^k} \right)$$

$$= \left(\sum_{n=0}^{\infty} b_n(x) \frac{t^n}{n!} \right) \left(\sum_{m=0}^{\infty} \frac{(\log(1+t))^{m+1}}{m!(m+1)^k} \right)$$

$$= \left(\sum_{n=0}^{\infty} b_n(x) \frac{t^n}{n!} \right) \left(\sum_{m=0}^{\infty} \frac{1}{(m+1)^{k-1}} \frac{1}{(m+1)!} (\log(1+t))^{m+1} \right)$$

$$= \left(\sum_{n=0}^{\infty} b_n(x) \frac{t^n}{n!} \right) \left(\sum_{m=0}^{\infty} \frac{1}{(m+1)^{k-1}} \sum_{l=m+1}^{\infty} S_1(l, m+1) \frac{t^l}{l!} \right)$$

$$= \left(\sum_{n=0}^{\infty} b_n(x) \frac{t^n}{n!} \right) \left(\sum_{l=0}^{\infty} \sum_{m=0}^{l} \frac{1}{(m+1)^{k-1}} \frac{S_1(l+1, m+1)}{l+1} \frac{t^l}{l!} \right)$$

$$= \sum_{n=0}^{\infty} \left(\sum_{l=0}^{n} \binom{n}{l} \sum_{m=0}^{l} \frac{1}{(m+1)^{k-1}} \frac{S_1(l+1, m+1)}{l+1} b_n(x) \right) \frac{t^n}{n!}. \tag{2.16}$$

Therefore, by comparing the coefficients of $\frac{t^n}{n!}$ in both sides, we get the following theorem.

Theorem 2.5. For $n \geq 0$, we have

$$b_n^{(k)}(x+1) - b_n^{(k)}(x) = \sum_{l=0}^{n} \binom{n}{l} \sum_{m=0}^{l} \frac{1}{(m+1)^{k-1}} \frac{S_1(l+1, m+1)}{l+1} b_n(x).$$

From (2.1), we have

$$\sum_{n=0}^{\infty} b_n^{(k)}(x+y) \frac{t^n}{n!} = \frac{\mathrm{Ei}_k\left(\log(1+t)\right)}{\log(1+t)} (1+t)^{x+y}$$

$$= \frac{\mathrm{Ei}_k\left(\log(1+t)\right)}{\log(1+t)}(1+t)^x(1+t)^y = \left(\sum_{n=0}^{\infty} b_n^{(k)}(x)\frac{t^n}{n!}\right)\left(\sum_{m=0}^{\infty}(y)_m\frac{t^m}{m!}\right)$$

$$\sum_{n=0}^{\infty}\left(\sum_{m=0}^{n}\binom{n}{m}b_{n-m}^{(k)}(x)(y)_m\right)\frac{t^n}{n!}. \tag{2.17}$$

Therefore, by equation (2.17), we get the following theorem.

Theorem 2.6. For $n \geq 0$, we have

$$b_n^{(k)}(x+y) = \sum_{m=0}^{n}\binom{n}{m}b_{n-m}^{(k)}(x)(y)_m.$$

By, (2.1), we have

$$\frac{\mathrm{Ei}_k\left(\log(1+t)\right)}{\log(1+t)} = \sum_{n=0}^{\infty} b_n^{(k)}\frac{t^n}{n!}$$

$$\mathrm{Ei}_k\left(\log(1+t)\right) = \log(1+t)\sum_{n=0}^{\infty} b_n^{(k)}\frac{t^n}{n!}$$

$$\frac{\mathrm{Ei}_k\left(\log(1+t)\right)}{t} = \frac{\log(1+t)}{t}\sum_{n=0}^{\infty} b_n^{(k)}\frac{t^n}{n!}$$

$$= \left(\sum_{m=0}^{\infty} D_m\frac{t^m}{m!}\right)\left(\sum_{n=0}^{\infty} b_n^{(k)}\frac{t^n}{n!}\right)$$

$$= \sum_{n=0}^{\infty}\left(\sum_{m=0}^{n}\binom{n}{m}b_{n-m}^{(k)}D_m\right)\frac{t^n}{n!}. \tag{2.18}$$

On the other hand,

$$\frac{\mathrm{Ei}_k\left(\log(1+t)\right)}{t} = \frac{1}{t}\sum_{m=1}^{\infty}\frac{(\log(1+t))^m}{(m-1)!m^k}$$

$$= \frac{1}{t}\sum_{m=0}^{\infty}\frac{(\log(1+t))^{m+1}}{m^k!(m+1)}$$

$$= \frac{1}{t}\sum_{m=0}^{\infty}\frac{1}{(m+1)^{k-1}}\frac{1}{(m+1)!}(\log(1+t))^{m+1}$$

$$= \sum_{n=0}^{\infty}\left(\sum_{m=0}^{n}\frac{1}{(m+1)^{k-1}}\frac{S_1(n+1,m+1)}{n+1}\right)\frac{t^n}{n!}. \tag{2.19}$$

Therefore, by equations (2.18) and (2.19), we obtain the following theorem.

Theorem 2.7. For $n \geq 0$, we have

$$\sum_{m=0}^{n}\binom{n}{m}b_{n-m}^{(k)}D_m = \sum_{m=0}^{n}\frac{1}{(m+1)^{k-1}}\frac{S_1(n+1,m+1)}{n+1}.$$

3 The unipoly-Bernoulli polynomials of the second kind

Let p be any arithmetic function which is a real or complex valued function defined on the set of positive integers $\mathbb{N}$. Kim-Kim [7] defined the unipoly function attached to polynomials $p(x)$ by

$$u_k(x|p) = \sum_{n=1}^{\infty} \frac{p(n)}{n^k} x^n, (k \in \mathbb{Z}). \tag{3.1}$$

Moreover,

$$u_k(x|1) = \sum_{n=1}^{\infty} \frac{x^n}{n^k} = \mathrm{Li}_k(x), (\text{see [8]}), \tag{3.2}$$

is the ordinary polylogarithm function.

Now, we define the unipoly-Bernoulli polynomials of the second kind attached to polynomials $p(x)$ by

$$\frac{1}{\log(1+t)} u_k\left(\log(1+t)|p\right)(1+t)^x = \sum_{n=0}^{\infty} Pb_{n,p}^{(k)}(x) \frac{t^n}{n!}. \tag{3.3}$$

When $x = 0$, $Pb_n^{(k,p)} = PB_n^{(k,p)}(0)$ are called the unipoly-Bernoulli numbers of the second attached to p, (see [4]).

If we take $p(n) = \frac{1}{\Gamma(n)}$. Then, we have

$$\begin{aligned} \sum_{n=0}^{\infty} Pb_{n,\frac{1}{\Gamma}}^{(k)}(x) \frac{t^n}{n!} &= \frac{1}{\log(1+t)} (1+t)^x u_k\left(\log(1+t)|\frac{1}{\Gamma}\right) \\ &= \frac{1}{\log(1+t)} (1+t)^x \sum_{m=1}^{\infty} \frac{(\log(1+t))^m}{m^k (m-1)!}. \end{aligned} \tag{3.4}$$

In particular, for $k = 1$, we obtain

$$\sum_{n=0}^{\infty} Pb_{n,\frac{1}{\Gamma}}^{(1)}(x) \frac{t^n}{n!} = \frac{1}{\log(1+t)} (1+t)^x \sum_{m=1}^{\infty} \frac{(\log(1+t))^m}{m!} = \frac{t}{\log(1+t)} (1+t)^x. \tag{3.5}$$

Therefore by (3.3) and (3.5), we have

$$Pb_{n,\frac{1}{\Gamma}}^{(k)}(x) = b_n(x). \tag{3.6}$$

From (3.3), we have

$$\begin{aligned} \sum_{n=0}^{\infty} Pb_{n,p}^{(k)}(x) \frac{t^n}{n!} &= \frac{1}{\log(1+t)} (1+t)^x u_k\left(\log(1+t)|p\right) \\ &= \frac{1}{\log(1+t)} (1+t)^x \sum_{m=1}^{\infty} \frac{p(m)}{m^k} (\log(1+t))^m \\ &= \frac{1}{\log(1+t)} (1+t)^x \sum_{m=0}^{\infty} \frac{p(m+1)}{(m+1)^k} (\log(1+t))^{m+1} \\ &= \frac{1}{\log(1+t)} (1+t)^x \sum_{m=0}^{\infty} \frac{p(m+1)(m+1)!}{(m+1)^k} \sum_{l=m+1}^{\infty} S_1(l, m+1) \frac{t^l}{l!} \\ &= \frac{1}{\log(1+t)} (1+t)^x \sum_{m=0}^{\infty} \frac{p(m+1)(m+1)!}{(m+1)^k} \sum_{l=m}^{\infty} \frac{S_1(l+1, m+1)}{l+1} \frac{t^l}{l!} \end{aligned}$$

$$= \sum_{n=0}^{\infty} b_n(x)\frac{t^n}{n!}\sum_{l=0}^{\infty}\left(\sum_{m=0}^{l}\frac{p(m+1)(m+1)!}{(m+1)^k}\frac{S_1(l+1,m+1)}{l+1}\right)\frac{t^l}{l!}$$

$$= \sum_{n=0}^{\infty}\left(\sum_{l=0}^{n}\sum_{m=0}^{l}\binom{n}{l}\frac{p(m+1)(m+1)!}{(m+1)^k}\frac{S_1(l+1,m+1)}{l+1}b_{n-l}(x)\right)\frac{t^n}{n!}. \tag{3.7}$$

Therefore, by comparing the coefficients on both sides of (3.7), we obtain the following theorem.

Theorem 3.1. For $n \geq 0$ and $k \in \mathbb{Z}$, we have

$$Pb_{n,p}^{(k)}(x) = \sum_{l=0}^{n}\sum_{m=0}^{l}\binom{n}{l}\frac{p(m+1)(m+1)!}{(m+1)^k}\frac{S_{1,\lambda}(l+1,m+1)}{l+1}b_{n-l}(x).$$

Moreover,

$$Pb_{n,\frac{1}{\Gamma}}^{(k)}(x) = \sum_{l=0}^{n}\sum_{m=0}^{l}\binom{n}{l}\frac{1}{(m+1)^{k-1}}\frac{S_1(l+1,m+1)}{l+1}b_{n-l}(x).$$

From (3.3), we have

$$\sum_{n=0}^{\infty} Pb_{n,p}^{(k)}(x)\frac{t^n}{n!} = \frac{1}{\log(1+t)}u_k(\log(1+t)|p)(1+t)^x$$

$$= \frac{u_k\left(\log_\lambda(1+t)|p\right)}{e_\lambda(t)-1}\sum_{n=0}^{\infty}(x)_n\frac{t^n}{n!}$$

$$= \sum_{m=0}^{\infty} Pb_{m,p}^{(k)}\frac{t^m}{m!}\sum_{n=0}^{\infty}(x)_n\frac{t^n}{n!}$$

$$= \sum_{n=0}^{\infty}\left(\sum_{m=0}^{n}\binom{n}{m}Pb_{m,p}^{(k)}(x)_{n-m}\right)\frac{t^n}{n!}. \tag{3.8}$$

By comparing the coefficients on both sides of (3.8), we get the following theorem.

Theorem 3.2. For $n \geq 0$ and $k \in \mathbb{Z}$, we have

$$Pb_{n,p}^{(k)}(x) = \sum_{m=0}^{n}\binom{n}{m}Pb_{m,p}^{(k)}(x)_{n-m}.$$

From (3.3), we have

$$\sum_{n=0}^{\infty} Pb_{n,p}^{(k)}\frac{t^n}{n!} = \frac{1}{\log(1+t)}u_k\left(\log(1+t)|p\right)$$

$$= \frac{1}{\log(1+t)}\sum_{m=1}^{\infty}\frac{p(m)}{m^k}(\log(1+t))^m$$

$$= \sum_{m=0}^{\infty}\frac{p(m+1)}{(m+1)^k}(\log(1+t))^{m+1}$$

$$= \frac{t}{\log(1+t)}\frac{\log(1+t)}{t}\sum_{m=0}^{\infty}\frac{p(m+1)m!}{(m+1)^k}\frac{(\log(1+t))^m}{m!}$$

$$= \sum_{n=0}^{\infty} D_n\frac{t^n}{n!}\sum_{j=0}^{\infty} b_j\frac{t^j}{j!}\sum_{m=0}^{\infty}\frac{p(m+1)m!}{(m+1)^k}\sum_{l=m}^{\infty} S_{1,\lambda}(l,m)\frac{t^l}{l!}$$

$$=\sum_{n=0}^{\infty}D_n\frac{t^n}{n!}\sum_{j=0}^{\infty}b_j\frac{t^j}{j!}\sum_{l=0}^{\infty}\sum_{m=0}^{l}\frac{p(m+1)m!}{(m+1)^k}S_1(l,m)\frac{t^l}{l!}$$

$$=\sum_{n=0}^{\infty}\sum_{j=0}^{n}\binom{n}{j}D_{n-j}b_j\frac{t^n}{n!}\sum_{l=0}^{\infty}\sum_{m=0}^{l}\frac{p(m+1)m!}{(m+1)^k}S_1(l,m)\frac{t^l}{l!}$$

$$=\sum_{n=0}^{\infty}\left(\sum_{l=0}^{n}\sum_{m=0}^{l}\sum_{j=0}^{n-l}\binom{n-l}{j}\binom{n}{l}D_{n-j-l}b_j\frac{p(m+1)m!}{(m+1)^k}S_1(l,m)\right)\frac{t^n}{n!}. \tag{3.9}$$

Therefore, by comparing the coefficients on both sides of (3.9), we obtain the following theorem.

Theorem 3.3. For $n \geq 0$ and $k \in \mathbb{Z}$, we have

$$Pb_{n,p}^{(k)}=\sum_{l=0}^{n}\sum_{m=0}^{l}\sum_{j=0}^{n-l}\binom{n-l}{j}\binom{n}{l}D_{n-j-l}b_j\frac{p(m+1)m!}{(m+1)^k}S_1(l,m).$$

Acknowledgement. The author Waseem A. Khan thanks to Prince Mohammad bin Fahd University, Saudi Arabia for providing facilities and support.

References

[1] Eastham, M.S.P. On Polylogarithms. Proc. Glasgow Math. Assoc. **1964**, 6, 169-171.

[2] Khan, W.A.; Ghayasuddin, M.; Shadab, M. Multiple poly-Bernoulli polynomials of the second kind associated with Hermite polynomials. Fasciculi Mathematici. **2017**, 58, 97-112.

[3] Khan, W.A.; Ghayasuddin, M.; Khan, N.; Choi, J. The 3-variable Hermite poly-Bernoulli polynomials of the second kind. Nonlinear Func. Anal. Appl. **2017**, 22(5), 1013-1027.

[4] Khan, W.A. A new class of degenerate Frobenius-Euler-Hermite polynomials. Adv. Stud. Contemp. Math. (Kyungshang). **2018**, 28(4), 567-576.

[5] Khan, W.A.; Acikgoz, M.; Duran, U. Note on the type 2 degenerate multi-poly-Euler polynomials. Symmetry. **2020**, 12, 1-10.

[6] Khan, W.A.; Ali, R.; Alzobydi, K.A.H.; Ahmed, N. A new family of degenerate poly-Genocchi polynomials with its certain properties. J. Function Spaces. **2021**, Volume 2021, Article ID 6660517, 8 pages.

[7] Kim, D.S.; Kim, T. A note on polyexponential and unipoly functions. Russ. J. Math. Phys. bf 2019, 26(1), 40-49.

[8] Kaneko, M. Poly-Bernoulli numbers. J. Thé́or Nombres Bordeaux. **1997**, 9(1), 221-228.

[9] Kim, T.; Kwon, H.-I.; Lee, S.-H; Seo, J.I. A note on poly-Bernoulli numbers and polynomials of the second kind. Adv. Diiffe. Eqn. **2014**, 2014:219.

[10] Kim, D. S. and Kim, T., Daehee numbers and polynomials, Appl. Math. Sciences, **2013**, 7(120), 5969-5976.

[11] Muhiuddin, G.; Khan, W.A.; Duran, U. Two variable type 2 Fubini polynomials. Mathematics. **2021**, 9(281) 1-13.

[12] Muhiuddin, G.; Khan, W.A.; Duran, U.; Kadi, D.-Al. Some identities of the multi poly-Bernoulli polynomials of complex variable. J. Function Spaces. Volume 2021, Article ID 7172054, 8 pages.

[13] Muhiuddin, G.; Khan, W.A.; Duran, U.; and Al-Kadi, D.; A new class of higher-order Hypergeometric Bernoulli polynomials associated with Lagrange-Hermite polynomials. Symmetry. **2021**, 13, 648. https://doi.org/10.3390/sym13040648

[14] Roman, S. The umbral calculus. Pure App. Math. Vol. 111. Academic Press, Inc., New York, **1984**.

[15] Sharma, S.K.; Khan, W.A.; Ryoo, C.S. A parametric kind of the degenerate Fubini numbers and polynomials. Mathematics. **2020**, 8,405;
doi:10.3390/math8030405.

[16] Sharma, S.K.; Khan, W.A.; Araci, S.; Ahmed, S.S. New type of degenerate Daehee polynomials of the second kind. Adv. Differ. Equ. **2020**, 2020:428, 14pp.

[17] Sharma, S.K.; Khan, W.A.; Araci, S.; Ahmed, S.S. New construction of the type 2 of degenerate central Fubini polynomials with their certain properties. Adv. Differ. Equ. **2020**, 2020:587, 11pp.

Author information

Waseem A. Khan[1] and M. Kamarujjama[2], [1]Department of Mathematics and Natural Sciences, Prince Mohammad Bin Fahd University, P.O Box 1664, Al Khobar 31952, Saudi Arabia, [2]Department of Applied Mathematics, Faculty of Engineering, Aligarh Muslim University, Aligarh-202002,, India.
E-mail: wkhan1@pmu.edu.sa, mdkamarujjama@rediffmail.com

Received: 2021-04-18
Accepted: 2021-07-16