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Turán-type integral mean inequalities of the polar derivative of a polynomial

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Abstract If $p(z)$ is a polynomial of degree n having all its zeros in $|z| \leq k, k \geq 1$, then for any real $r \geq 1$, Aziz [J. Approx. Theory, 55(1988), 232-239] proved the integral mean inequality

$$\left\{\int_0^{2\pi} \left|1+k^n e^{i\theta}\right|^r d\theta\right\}^{\frac{1}{r}} \max_{|z|=1}|p'(z)| \geq n\left\{\int_0^{2\pi} \left|p\left(e^{i\theta}\right)\right|^r d\theta\right\}^{\frac{1}{r}}.$$

In this paper, we obtain an improved extension of the above inequality by using the polar derivative instead of the ordinary derivative and involving the leading coefficient and the constant term of the polynomial.

1 Introduction and preliminaries

The problem of the extremal properties of polynomials piqued the interest of famous chemist Mendeleev in the second half of the nineteenth century, who was looking for an upper bound of $\max\limits_{-1\leq x\leq 1}|p'(x)|$, where $p(x)$ was a quadratic polynomial of real variable x with real coefficients. He was able to prove that $\max\limits_{-1\leq x\leq 1}|p'(x)| \leq 4$, if $-1 \leq p(x) \leq 1$ for $-1 \leq x \leq 1$.

While working on a problem in Approximation Theory, Bernstein needed an upper bound estimate of maximum modulus of the derivative of a complex polynomial $p(z)$ in terms of the maximum modulus of $p(z)$, which is an analogue of above Mendeleev's problem in the complex domain. In fact, he obtained his famous inequality known as Bernstein's inequality as an immediate consequence of an inequality concerning trigonometric polynomials proved by him [3]. On the other hand, Paul Turán [18] was the first who estimated the lower bound for the maximum modulus of the derivative of a polynomial in terms of the maximum modulus of the polynomial. In fact, he proved that if $p(z)$ is a polynomial of degree n having all its zeros in $|z| \leq 1$, then

$$\max_{|z|=1}|p'(z)| \geq \frac{n}{2}\max_{|z|=1}|p(z)|. \tag{1.1}$$

Let $p(z)$ be a polynomial of degree n over the set of complex numbers and for a real number $r > 0$, we define the integral mean of $p(z)$ by

$$\|p\|_r = \left\{\frac{1}{2\pi}\int_0^{2\pi} \left|p(e^{i\theta})\right|^r d\theta\right\}^{\frac{1}{r}}. \tag{1.2}$$

It should be observed that for $1 \leq r < \infty$, $\|\cdot\|_r$ is a norm for Hardy space H^p (see Duren [6]). For our convenience, we will retain the norm notation even if $0 < r < 1$ when $\|\cdot\|_r$ is not a genuine norm. If we take the limit as $r \to \infty$ in (1.2) and make use of the well-known fact from the analysis [16, 17] that

$$\lim_{r\to\infty}\left\{\frac{1}{2\pi}\int_0^{2\pi} \left|p(e^{i\theta})\right|^r d\theta\right\}^{\frac{1}{r}} = \max_{|z|=1}|p(z)|,$$

we can suitably denote

$$\|p\|_\infty = \max_{|z|=1}|p(z)|.$$

Similarly, we can define

$$\|p\|_0 = exp\left\{\frac{1}{2\pi}\int_0^{2\pi}\log|p(e^{i\theta})|d\theta\right\},$$

and it follows easily that $\lim_{r\to 0^+}\|p\|_r = \|p\|_0$.

It would be of further interest that by taking the limit as $r \to 0^+$, the stated results on the integral mean inequalities holding for $r > 0$, hold for $r = 0$ as well.

Inequality (1.1) of Turán [18] has been of considerable interest and applications, and it would be of interest to seek its generalization for polynomials having all their zeros in $|z| \le k, k > 0$. The case when $0 < k \le 1$ was settled by Malik [12] and proved

$$\|p'\|_\infty \ge \frac{n}{1+k}\|p\|_\infty, \tag{1.3}$$

while the case when $k \ge 1$ by Govil [8] and obtained

$$\|p'\|_\infty \ge \frac{n}{1+k^n}\|p\|_\infty. \tag{1.4}$$

Equalities in (1.3) and (1.4) hold respectively for $p(z) = (z+k)^n, 0 < k \le 1$ and $p(z) = z^n + k^n, k \ge 1$.

Inequality (1.4) has certain drawbacks; as we can see, its bound depends only on the zero with maximum modulus, consider the polynomials $p_1(z) = (z-2)^6$ and $p_2(z) = z^5(z-2)$ then inequality (1.4) gives same bounds for $p_1(z)$ and $p_2(z)$ as

$$\max_{|z|=1}|p_1'(z)| \ge \frac{6}{1+2^6}\max_{|z|=1}|p_1(z)| \quad \text{and} \quad \max_{|z|=1}|p_2'(z)| \ge \frac{6}{1+2^6}\max_{|z|=1}|p_2(z)|,$$

even though $p_2(z)$ has only one zero on $|z| = 2$ while $p_1(z)$ has all its zeros on $|z| = 2$. Also the extremal polynomial for inequality (1.4) is $p(z) = z^n + k^n, k \ge 1$ and it should be possible to obtain sharper bound when some coefficients of the polynomial are non-zero.

For the first time in 1984, Malik [11] extended inequality (1.1) into an integral mean version by proving

$$\|1+z\|_r\|p'\|_\infty \ge n\|p\|_r, \tag{1.5}$$

where $r > 0$.

The result is sharp and equality holds for $p(z) = (z+1)^n$.

In 1988, Aziz [1] established an integral mean extension of inequality (1.3) and proved that if $p(z)$ is a polynomial of degree n having all its zeros in $|z| \le k, 0 < k \le 1$, then for each $r > 0$,

$$\|1+kz\|_r\|p'\|_\infty \ge n\|p\|_r. \tag{1.6}$$

Equality in (1.6) holds for the polynomial $p(z) = (\alpha z + \beta k)^n$, where $|\alpha| = |\beta|$.

It is worth noting that if we take the limit as $r \to \infty$, we obtain inequality (1.3). Further, in the same paper [1] the author also obtained the integral mean extension of inequality (1.4) by proving the following result.

Theorem 1.1. *If $p(z)$ is a polynomial of degree n having all its zeros in $|z| \le k, k \ge 1$, then for each $r \ge 1$,*

$$\|1+k^n z\|_r\|p'\|_\infty \ge n\|p\|_r. \tag{1.7}$$

The result is sharp and equality holds for $p(z) = \alpha z^n + \beta k^n, |\alpha| = |\beta|$.

For a polynomial $p(z)$ of degree n and a complex number α, let

$$D_\alpha p(z) = np(z) + (\alpha - z)p'(z)$$

denote the polar derivative of the polynomial $p(z)$ with respect to α.
Note that $D_\alpha p(z)$ is a polynomial of degree at most $n-1$, and it generalizes the ordinary derivative in the sense that

$$\lim_{\alpha\to\infty}\frac{D_\alpha p(z)}{\alpha}=p^{'}(z).$$

Aziz and Rather [2] first extended inequality (1.3) to the polar derivative version and proved that if $p(z)$ is a polynomial of degree n having all its zeros in $|z|\le k, 0<k\le 1$ and for every complex number α with $|\alpha|\ge k$

$$\|D_\alpha p(z)\|_\infty \ge n\left(\frac{|\alpha|-k}{1+k}\right)\|p(z)\|_\infty. \tag{1.8}$$

Further, in the same paper [2], they also proved a polar derivative extension of (1.4) with the same assumption on $p(z)$ and obtained

$$\|D_\alpha p(z)\|_\infty \ge n\left(\frac{|\alpha|-k}{1+k^n}\right)\|p(z)\|_\infty, \tag{1.9}$$

where α is any complex number with $|\alpha|\ge k$.
Recently, Govil and Kumar [7] proved a generalization and improvement of inequality (1.9) by incorporating the leading coefficient and the coefficient of the lowest degree term of the polynomial.

Theorem 1.2. *If $p(z)=z^s\left(\sum_{j=0}^{n-s}c_jz^j\right), 0\le s\le n$, is a polynomial of degree n having all its zeros in $|z|\le k, k\ge 1$, then for every complex number α with $|\alpha|\ge k$,*

$$\|D_\alpha p\|_\infty \ge \frac{|\alpha|-k}{1+k^n}\left\{n+s+\frac{k^{n-s}|c_{n-s}|-|c_0|}{k^{n-s}|c_{n-s}|+|c_0|}\right\}\|p\|_\infty. \tag{1.10}$$

In literature, there exist several generalizations and improvements of inequality (1.6) concerning polar derivative of a polynomial (see [4], [20] and [19]). A typical example is that of Dewan et al. [4] where they proved the following integral mean extension of (1.8).

Theorem 1.3. *If $p(z)$ is a polynomial of degree n having all its zeros in $|z|\le k, 0<k\le 1$, then for every complex number α with $|\alpha|\ge k$ and each $r>0$*

$$\|1+kz\|_r\|D_\alpha p\|_\infty \ge n(|\alpha|-k)\|p\|_r. \tag{1.11}$$

The result is sharp and equality holds for $p(z)=(z-k)^n$.

For about 19 years, the integral mean extension of inequality (1.9) had not been in the literature of polynomial inequalities, untill Rather and Bhat [14] obtained the corresponding integral form of inequality (1.9). In this paper, we are able to prove the following result which not only extends Theorem 1.2 to integral mean, but also gives an improvement as well as generalization of Theorem 1.1 concerning polar derivative. More precisely, we prove.

Theorem 1.4. *If $p(z)=z^s\left(\sum_{j=0}^{n-s}c_jz^j\right), 0\le s\le n$ is a polynomial of degree n having all its zeros in $|z|\le k, k\ge 1$, then for every complex number α with $|\alpha|\ge k$ and for each real number $r>0$,*

$$\|D_\alpha p\|_r \ge \frac{|\alpha|-k}{E_r}A\|p\|_r, \tag{1.12}$$

where

$$A=\frac{1}{2}\left\{n+s+\frac{k^{n-s}|c_{n-s}|-|c_0|}{k^{n-s}|c_{n-s}|+|c_0|}\right\},$$

and

$$E_r=\frac{\left\{\int_0^{2\pi}|1+k^ne^{i\theta}|^rd\theta\right\}^{\frac{1}{r}}}{\left\{\int_0^{2\pi}|1+e^{i\theta}|^rd\theta\right\}^{\frac{1}{r}}}.$$

Remark 1.5. If we let $r \to \infty$ in (1.12) and note that $E_r \to \frac{1+k^n}{2}$ as $r \to \infty$, then (1.12) reduces to (1.10) of Theorem 1.2.
Further, dividing both sides of (1.12) by $|\alpha|$ and letting $|\alpha| \to \infty$, we get the following interesting result.

Corollary 1.6. *If* $p(z) = z^s \left(\sum_{j=0}^{n-s} c_j z^j \right), 0 \leq s \leq n$, *is a polynomial of degree* n *having all its zeros in* $|z| \leq k, k \geq 1$, *then for each* $r > 0$,

$$\|p'\|_r \geq \frac{A}{E_r} \|p\|_r, \tag{1.13}$$

where A *and* E_r *are as defined in Theorem 1.4.*

If we let $r \to \infty$ in (1.13) and note the simple fact that $E_r \to \frac{1+k^n}{2}$ as $r \to \infty$, we get the following improvement and generalization of (1.4), which, in fact, is a result obtained by Govil and Kumar ([7], Corollary 1.2).

Corollary 1.7. *If* $p(z) = z^s \left(\sum_{j=0}^{n-s} c_j z^j \right), 0 \leq s \leq n$ *is a polynomial of degree* n *having all its zeros in* $|z| \leq k, k \geq 1$, *then*

$$\|p'\|_\infty \geq \frac{1}{1+k^n} \left\{ n + s + \frac{k^{n-s}|c_{n-s}| - |c_0|}{k^{n-s}|c_{n-s}| + |c_0|} \right\} \|p\|_\infty. \tag{1.14}$$

The result is sharp and equality in (1.14) holds for $p(z) = z^n + k^n$.

Putting $s = 0$, Corollary 1.7 further reduces to the following interesting result, which improves (1.4) under the same hypotheses.

Corollary 1.8. *If* $p(z) = \sum_{j=0}^{n} c_j z^j$ *is a polynomial of degree* n *having all its zeros in* $|z| \leq k$, $k \geq 1$, *then*

$$\|p'\|_\infty \geq \frac{1}{1+k^n} \left\{ n + \frac{k^n|c_n| - |c_0|}{k^n|c_n| + |c_0|} \right\} \|p\|_\infty. \tag{1.15}$$

The result is best possible and equality in (1.15) holds for $p(z) = z^n + k^n$.

Remark 1.9. It may be noted that as the polynomial $p(z)$ is of degree $n \geq 1$, the leading coefficient c_n can not be zero. In fact, it is obvious that the inequalities (1.14) and (1.15) would give improvements over the bound given by inequality (1.4). It is of interest to notice that if at least one zero of the polynomial does not lie on $|z| = k$, then $|c_n|k^n - |c_0| > 0$. It is worth noting that for larger values of $s > 0$, inequality (1.14) provides a better bound than (1.15).

Example 1.10. Consider the polynomial $p(z) = z^3 \left(z^2 - 9\right)$ having all its zeros in $|z| \leq k = 3$. For this polynomial, we have $\max_{|z|=1} |p(z)| = 10$. Then it can be easily obtained that inequality (1.4) gives $\max_{|z|=1} \left|p^{'}(z)\right| \geq \frac{25}{122}$, while inequality (1.14) gives $\max_{|z|=1} \left|p^{'}(z)\right| \geq \frac{40}{122}$ and we have an improvement of 60% over the bound obtained from inequality (1.4). Also by inequality (1.15), $\max_{|z|=1} \left|p^{'}(z)\right| \geq \frac{30}{122}$ and an improvement of 20% over the bound given by (1.4) is obtained.

2 Lemmas

We shall need the following lemmas to prove Theorem 1.4. For a polynomial $p(z)$ of degree n, we will use $\tilde{p}(z) = z^n \overline{p\left(\frac{1}{\bar{z}}\right)}$. The first lemma is due to Rather et al. [15].

Lemma 2.1. *If* $(x_j)_{j=1}^{\infty}$ *is a sequence of real numbers such that* $0 \leq x_j \leq 1, j \in \mathbb{N}$, *then*

$$\sum_{j=1}^{n} \frac{1-x_j}{1+x_j} \geq \frac{1-\prod_{j=1}^{n} x_j}{1+\prod_{j=1}^{n} x_j}, \ \forall\, n \in \mathbb{N}.$$

The following lemma is due to Malik [12].

Lemma 2.2. *If* $p(z)$ *is a polynomial having all its zeros in* $|z| \leq k, 0 < k \leq 1$, *then for* $|z| = 1$,

$$|\tilde{p}^{'}(z)| \leq k|p^{'}(z)|.$$

Lemma 2.3. *If* $p(z)$ *is a polynomial of degree* n, *then for every* $R \geq 1$ *and* $r > 0$,

$$\left\{\int_0^{2\pi} \left|p\left(Re^{i\theta}\right)\right|^r d\theta\right\}^{\frac{1}{r}} \leq R^n \left\{\int_0^{2\pi} \left|p\left(e^{i\theta}\right)\right|^r d\theta\right\}^{\frac{1}{r}}. \tag{2.1}$$

As far as Lemma 2.3 is concerned, it is difficult to trace its origin. It was deduced from a well-known result of Hardy [9], according to which for every function $f(z)$ analytic in $|z| < t_0$, and for every $r > 0$,

$$\left\{\int_0^{2\pi} \left|f\left(e^{i\theta}\right)\right|^r d\theta\right\}^{\frac{1}{r}}$$

is a non-decreasing function of t for $0 < t < t_0$. If $p(z)$ is a polynomial of degree n, then $f(z) = z^n \overline{p\left(\frac{1}{\bar{z}}\right)}$ is again a polynomial, that is, an entire function and by Hardy's result for $r > 0$,

$$\left\{\int_0^{2\pi} \left|f\left(te^{i\theta}\right)\right|^r d\theta\right\}^{\frac{1}{r}} \leq \left\{\int_0^{2\pi} \left|f\left(e^{i\theta}\right)\right|^r d\theta\right\}^{\frac{1}{r}},$$

for $t = \frac{1}{R} < 1$. This is equivalent to (2.1).

Lemma 2.4. *If* $p(z) = z^s \left(\sum_{j=0}^{n-s} c_j z^j\right), 0 \leq s \leq n$, *is a polynomial of degree* n *having all its zeros in* $|z| \leq 1$, *then for* $|z| = 1$,

$$\left|p^{'}(z)\right| \geq \frac{1}{2}\left\{n + s + \frac{|c_{n-s}| - |c_0|}{|c_{n-s}| + |c_0|}\right\} |p(z)|.$$

The above result can be obtained from a theorem of Dubinin [5]. Here, we present an alternative proof by using Lemma 2.1.

Proof of Lemma 2.4. Let $p(z) = z^s \sum_{j=0}^{n-s} c_j z^j = z^s c_{n-s} \prod_{j=0}^{n-s} (z - z_j), 0 \leq s \leq n$ where $|z_j| \leq 1, j = 0, \cdots n-s$. Then on $|z| = 1$, for which $p(z) \neq 0$,

$$\Re\left(\frac{zp^{'}(z)}{p(z)}\right) = s + \sum_{j=0}^{n-s} \Re\left(\frac{z}{z - z_j}\right). \tag{2.2}$$

Since $|z_j| \leq 1, j = 0, \cdots, n-s$ and straight forward calculations give

$$\Re\left(\frac{z}{z - z_j}\right) \geq \frac{1}{1 + |z_j|}, j = 0, \cdots, n-s. \tag{2.3}$$

Combining inequalities (2.2) and (2.3), we get

$$\begin{aligned} \Re\left(\frac{zp^{'}(z)}{p(z)}\right) &\geq s + \sum_{j=0}^{n-s} \frac{1}{1+|z_j|} \\ &= \frac{1}{2}\left(n + s + \sum_{j=0}^{n-s} \frac{1-|z_j|}{1+|z_j|}\right). \end{aligned} \tag{2.4}$$

On applying Lemma 2.1, inequality (2.4) gives

$$\begin{aligned}\Re\left(\frac{zp^{'}(z)}{p(z)}\right) &\geq \frac{1}{2}\left(n+s+\frac{1-\prod_{j=0}^{n-s}|z_j|}{1+\prod_{j=0}^{n-s}|z_j|}\right)\\ &= \frac{1}{2}\left(n+s+\frac{|c_{n-s}|-|c_0|}{|c_{n-s}|+|c_0|}\right),\end{aligned}$$

and hence

$$\left|\frac{zp^{'}(z)}{p(z)}\right| \geq \Re\left(\frac{zp^{'}(z)}{p(z)}\right) \geq \frac{1}{2}\left\{n+s+\frac{|c_{n-s}|-|c_0|}{|c_{n-s}|+|c_0|}\right\},$$

which is equivalent to

$$\left|p^{'}(z)\right| \geq \frac{1}{2}\left\{n+s+\frac{|c_{n-s}|-|c_0|}{|c_{n-s}|+|c_0|}\right\}|p(z)|, \quad \text{for } |z|=1, p(z)\neq 0. \tag{2.5}$$

Also, inequality (2.5) is trivially satisfied for $p(z)=0$, and hence Lemma 2.4 is proved.

Lemma 2.5. *If $p(z)$ is a polynomial of degree n having no zero in $|z|<1$, then for every $R\geq 1$ and $r>0$,*

$$\left\{\int_0^{2\pi}\left|p(Re^{i\theta})\right|^r d\theta\right\}^{\frac{1}{r}} \leq E_r\left\{\int_0^{2\pi}\left|p(e^{i\theta})\right|^r d\theta\right\}^{\frac{1}{r}}, \tag{2.6}$$

where

$$E_r = \frac{\left\{\int_0^{2\pi}|1+R^n e^{i\theta}|^r d\theta\right\}^{\frac{1}{r}}}{\left\{\int_0^{2\pi}|1+e^{i\theta}|^r d\theta\right\}^{\frac{1}{r}}}. \tag{2.7}$$

This lemma was proved by Boas and Rahman [10] for $r\geq 1$. Later, Rahman and Schmeisser [13] showed the validity for $0<r<1$ as well.

3 Proof of the Theorem

Proof of Theorem 1.4. By hypothesis, $p(z)$ has all its zeros in $|z|\leq k, k\geq 1$, then the polynomial $R(z)=p(kz)$ has all its zeros in $|z|\leq 1$.
It is easy to see that for $|z|=1$

$$|\tilde{R}^{'}(z)| = \left|nR(z)-zR^{'}(z)\right|, \tag{3.1}$$

where

$$\tilde{R}(z) = z^n\overline{R\left(\frac{1}{\bar{z}}\right)}.$$

Applying Lemma 2.2 to $R(z)$, we have for $|z|=1$

$$\left|\tilde{R}^{'}(z)\right| \leq \left|R^{'}(z)\right|. \tag{3.2}$$

Using (3.1) and (3.2), we have for $\left|\frac{\alpha}{k}\right|\geq 1$ and $|z|=1$,

$$\begin{aligned}\left|D_{\frac{\alpha}{k}}R(z)\right| &= \left|nR(z)+\left(\frac{\alpha}{k}-z\right)R^{'}(z)\right|\\ &\geq \left|\frac{\alpha}{k}\right||R^{'}(z)|-\left|nR(z)-zR^{'}(z)\right|\\ &= \left|\frac{\alpha}{k}\right||R^{'}(z)|-|\tilde{R}^{'}(z)|\\ &\geq \left(\left|\frac{\alpha}{k}\right|-1\right)|R^{'}(z)|.\end{aligned} \tag{3.3}$$

Applying Lemma 2.4 to $R(z)$, we have for $|z|=1$

$$\left|R^{'}(z)\right| \geq \frac{1}{2}\left\{n+s+\frac{k^{n-s}|c_{n-s}|-|c_0|}{k^{n-s}|c_{n-s}|+|c_0|}\right\}|R(z)|. \tag{3.4}$$

Combining (3.3) and (3.4), we get

$$\left|D_{\frac{\alpha}{k}}R(z)\right| \geq \frac{|\alpha|-k}{2k}\left\{n+s+\frac{k^{n-s}|c_{n-s}|-|c_0|}{k^{n-s}|c_{n-s}|+|c_0|}\right\}|R(z)|.$$

Replacing $R(z)$ by $p(kz)$ in the above inequality, we obtain

$$\left|np(kz)+\left(\frac{\alpha}{k}-z\right)kp^{'}(kz)\right| \geq \frac{|\alpha|-k}{k}A|p(kz)|, \tag{3.5}$$

where

$$A=\frac{1}{2}\left\{n+s+\frac{k^{n-s}|c_{n-s}|-|c_0|}{k^{n-s}|c_{n-s}|+|c_0|}\right\}.$$

Inequality (3.5) is equivalent to

$$\left|np(kz)+(\alpha-kz)\,kp^{'}(kz)\right| \geq \frac{|\alpha|-k}{k}A|p(kz)|,$$

therefore for any $r>0$, we have

$$\left|D_\alpha p\left(ke^{i\theta}\right)\right|^r \geq \left(\frac{|\alpha|-k}{k}A\right)^r \left|p(ke^{i\theta})\right|^r, 0\leq\theta<2\pi,$$

and hence

$$\left\{\int_0^{2\pi}\left|D_\alpha p\left(ke^{i\theta}\right)\right|^r d\theta\right\}^{\frac{1}{r}} \geq \frac{|\alpha|-k}{k}A\left\{\int_0^{2\pi}\left|p(ke^{i\theta})\right|^r d\theta\right\}^{\frac{1}{r}}. \tag{3.6}$$

Since $R(z)$ has all its zeros in $|z|\leq 1$, $\tilde{R}(z)$ is a polynomial of degree at most n having no zero in $|z|<1$. Applying Lemma 2.5 to $\tilde{R}(z)$, we get

$$\left\{\int_0^{2\pi}\left|\tilde{R}\left(ke^{i\theta}\right)\right|^r d\theta\right\}^{\frac{1}{r}} \leq E_r\left\{\int_0^{2\pi}\left|\tilde{R}(e^{i\theta})\right|^r d\theta\right\}^{\frac{1}{r}}, \tag{3.7}$$

where $E_r=\frac{\left\{\int_0^{2\pi}|1+k^ne^{i\theta}|^r d\theta\right\}^{\frac{1}{r}}}{\left\{\int_0^{2\pi}|1+e^{i\theta}|^r d\theta\right\}^{\frac{1}{r}}}$.
Now it can be easily obtained that $\left|\tilde{R}(ke^{i\theta})\right|=k^n\left|p\left(e^{i\theta}\right)\right|$ and $\left|\tilde{R}\left(e^{i\theta}\right)\right|=\left|p\left(ke^{i\theta}\right)\right|$.
With the above two relations, (3.7) gives

$$k^n\left\{\int_0^{2\pi}\left|p\left(e^{i\theta}\right)\right|^r d\theta\right\}^{\frac{1}{r}} \leq E_r\left\{\int_0^{2\pi}\left|p\left(ke^{i\theta}\right)\right|^r d\theta\right\}^{\frac{1}{r}}. \tag{3.8}$$

Since $D_\alpha p(z)$ is a polynomial of degree at most $n-1$, applying Lemma 2.3 to $D_\alpha p(z)$ with $R=k\geq 1$, we have

$$\frac{1}{k^{n-1}}\left\{\int_0^{2\pi}\left|D_\alpha p\left(ke^{i\theta}\right)\right|^r d\theta\right\}^{\frac{1}{r}} \leq \left\{\int_0^{2\pi}\left|D_\alpha p\left(e^{i\theta}\right)\right|^r d\theta\right\}^{\frac{1}{r}}. \tag{3.9}$$

Using (3.9) in (3.6), we get

$$k^{n-1}\left\{\int_0^{2\pi}\left|D_\alpha p\left(e^{i\theta}\right)\right|^r d\theta\right\}^{\frac{1}{r}} \geq \frac{|\alpha|-k}{k}A\left\{\int_0^{2\pi}\left|p(ke^{i\theta})\right|^r d\theta\right\}^{\frac{1}{r}}. \tag{3.10}$$

Combining (3.8) and (3.10), we have

$$\left\{\int_0^{2\pi}\left|D_\alpha p\left(e^{i\theta}\right)\right|^r d\theta\right\}^{\frac{1}{r}} \geq \frac{|\alpha|-k}{E_r}A\left\{\int_0^{2\pi}\left|p\left(e^{i\theta}\right)\right|^r d\theta\right\}^{\frac{1}{r}}.$$

This completes the proof of Theorem 1.4.

Conclusion

For the class of polynomials having all its zeros in $|z| \leq k, 0 < k \leq 1$, there have been integral mean extensions of Turán-type inequalities concerning polar derivative as well. In this paper, for the same class of polynomials with $k \geq 1$, we obtain integral analogue of inequality (1.2) recently proved by Govil and Kumar [7] and our result implicates various existing known results in the literature and gives the techniques for further extensions of related Turán-type inequalities.
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