Palestine Journal of Mathematics

Vol. 12(3)(2023) , 94–102

Spectral Method for the Nonhomogeneous Wave Equation with Axial Symmetry

Amor Boutaghou

Communicated by Suheil Khoury

MSC 2010 Classifications: Primary 65D05, 65D32; Secondary 65D32, 65N35.

Keywords : Polynomials, Spectral method, Stability estimation, Convergence, Figures illustrations.

Acknowledgments : The author wish thanks referees for their valuable comments, corrections and suggestions.

Abstract : In this paper we analyse an orthogonal pseudospectral collocation semidiscrete discretization the r-variable of the nonhomogeneous wave axial symmetric and continuous in the variable t. The analysis is based on an approximation property converges to the Gauss-Radau interpolation operator. We investigate the stability and the convergence.

1 Introduction

Spectral methods are techniques for approximating the solutions of partial differential equations. Their main characteristic is that the discrete solutions are sought in high degree polynomial spaces. In this sense, the precision of its methods is limited only by the regularity of the approximate function, unlike other types of approximation such as finite differences or finite elements. We know that the distance of an analytical function from a polynomial space of degree $\leq N$ decreases with the parameter N. In this paper we present a novel approach, based on an orthogonal pseudo spectral collocation semidiscrete discretization of the r-variable, for the numerical solution of the problem proposed and continuous in the variable t. In this method the order of the matrix is less than the order of the matrix in the other methods.

We consider the nonhomogeneous wave equation with axial symmetric in a finite domain Λ with the general initial conditions

$$\begin{cases} u_{tt} - \alpha(u_{rr} + \frac{1}{r}u_r) = f(r,t), & \text{in } \Lambda \times (0,\infty) \\ u(r,t) = u_0(r), & \\ u_t(r,t) = u_1(r), & \text{in } \Lambda, \quad t = 0 \end{cases} \tag{1.1}$$

where $\Lambda = [0,1]$, α is a positive real number, $f \in L^2_1(\Lambda)$, where :

$$L^2_1(\Lambda) = \left\{ f : \ \Lambda \to R \text{ measurable } / \int_\Lambda (f(r))^2 \, r dr < \infty \right\}, \tag{1.2}$$

u_0 and u_1 are smooth on Λ , we denote by $\|.\|_{L^2_1(\Lambda)}$ and $(.,.)$ the norm and inner product in $L^2_1(\Lambda)$ and the standard Sobolev space $H^1_1(\Lambda)$.

In this work we construct approximate solution to the problem (1.1) in the form

$$u_N(r,t) = \sum_{n=0}^{N} a_n(t) l_n(r), \tag{1.3}$$

where the Lagrangian interpolates $l_n(r)$, $0 \leq n \leq N$, are defined at the points $r_j \in \Lambda$, $0 \leq j \leq N$. The grid made by $r_j, 0 \leq j \leq N$, is denoted by $\sum_{N+1}$.

The choice of the form (1.3) for the solution, added to some technics lead to a linear system of second-order ordinary differential equations which can be written in a matricial form as $\Gamma D^2 a + Aa = F$, where A is a square symmetric positive defined matrix and Γ is a diagonal invertible matrix and the operator $D^2 = \frac{d^2}{dt^2}$. We write $a = Pv$ where P is an orthogonal matrix such that $P^{-1}\left(\Gamma^{-1}A\right)P = C$ is a diagonal matrix, then we obtain a system of N second-order ordinary differential equations $D^2 v_j(t) + c_j v_j(t) = h_j(t),\ j = \overline{0,N}$, we can use Lagrange's method of undetermined parameters to solve for each component $v_j(t)$ of v , finally we conclude the expressions of functions $a_n(t)$ and for which we obtain the approximation solution u_N.[4, 10].

2 Discretization of the problem

2.1 Continuous problem, weak form.

The variational formulation of problem (1.1), it is written :

$$\left\{\text{find } u \in H_1^1(\Lambda), \text{ such that } \forall v \in H_1^1(\Lambda),\ (u_{tt}, v) + a_1(u,v) = (f,v) \quad \right\}. \tag{2.1}$$

where the bilinear form $a(.,.)$ is given by:

$$a_1(u,v) = \int_0^1 \partial_r u \partial_r v r dr, \tag{2.2}$$

see[8].Where the pivot space of the problem (1.1) is the space $L_1^2(\Lambda)$, the variational space is

$$H_1^1(\Lambda) = \left\{v/\ v, \partial_r v \in L_1^2(\Lambda)\right\}, \tag{2.3}$$

the corresponding norm is defined as

$$\|v\|^2_{L_1^2(\Lambda)} = \int_\Lambda v^2 r dr dt, \tag{2.4}$$

the semi norm is defined as

$$|v|^2_{H_1^1(\Lambda)} = \int_\Lambda ((\partial_r v)^2 r dr dt,$$

and the norm of the variational space

$$\|v\|^2_{H_1^1(\Lambda)} = \|v\|^2_{L_1^2(\Lambda)} + |v|^2_{H_1^1(\Lambda)}.$$

2.2 Discrete spaces

The approximate spaces is essentially generated by the finite dimensional subspace of $L_1^2(\Lambda)$, $IP_N(\Lambda)$ is the approximate spaces of the space $H_1^1(\Lambda)$.In this work we consider the quadrature formula and introduce a bilinear form a_{1N} which approach the form a_1 and we approximate $(.,.)_N$ for $(.,.)$, where N represent in spectral method the degree of polynomials. Here, the discrete product is defined for all functions g and h continuous on Λ by

$$(g,h)_N = \sum_{k=0}^{N} g(r_k)h(r_k)\omega_k. \tag{2.5}$$

2.3 Discrete problem, variational formulation

The variational formulation (2.1) is written :

$$\left\{\begin{array}{l} \quad \text{find } u_N \in IP_N(\Lambda) \text{ such that} \\ \forall v_N \in IP_N(\Lambda),\ (u_{ttN}, v_N)_N + a_{1N}(u_N, v_N) = (f_N, v_N)_N \end{array}\right. \tag{2.6}$$

where u_{0N} and $\partial_t u_{0N}$ are the interpolating polynomial of the initial conditions u and $\partial_t u$ at the Gauss-Radau nodes and the bilinear form $a_{1N}(.,.)$ is given by:

$$a_{1N}(u_N, v_N) = (\partial_r u_N, \partial_r v_N)_N = \sum_{k=0}^{N} (\partial_r u_N \partial_r v_N)(r_k)\omega_k, \tag{2.7}$$

$$(u_{ttN}, v_N)_N = \sum_{k=0}^{N} (u_{ttN} v_N)(r_k)\omega_k. \tag{2.8}$$

The equation is now what are the necessary tools to insure the existence and uniqueness of the approximate solution which verify the variational formulation (2.6).

2.4 The spectral method

In this section we describe the spectral element method applied to subsection 2.3 of the algorithm given in the introduction. The spectral method is based on a weak formulation of the considered problem. The approximate solution representation then is given by :

$$u_N(r,t) = \sum_{n=0}^{N} a_n(t) l_n(r), \quad l_n(r) \in IP_N(\Lambda),$$

the Lagrangian interpolates $l_n(r), 0 \leq n \leq N$ are defined on the interval Λ with $r \in \Lambda$,where the points r_k are the roots of the polynomial $A_N(r)$, $A_N(r) = \frac{2}{N!}\frac{d^N}{dr^N}((r-1)(r^2-r))^N$.

2.5 Orthogonal polynomials

We work in the interval $\Lambda = [0,1]$ and we use the polynomials

$$O_n(r) = \frac{2}{n!}\frac{d^n}{dr^n}((r^2-r)^n), \quad n \geq 0,$$

occur from the Legendre polynomials with change of variable, each polynomial O_n has the degree n and in $L_1^2(\Lambda)$ satisfies the following property:

$$\int_0^1 O_n^2(r) r dr = \frac{2}{2n+1}, \tag{2.9}$$

also we use the polynomials

$$A_n(r) = \frac{2}{n!}\frac{d^n}{dr^n}((r-1)(r^2-r)^n), \quad n \geq 0 \tag{2.10}$$

with the degree $n+1$, which satisfies in $L_1^2(\Lambda)$ the following property:

$$\int_0^1 A_n^2(r) r dr = \frac{n+1}{4(n+1)^2-1}, \quad n \geq 0, \tag{2.11}$$

also we have

$$\int_0^1 A_n'(r) A_n'(r) r dr = 2n+2. \tag{2.12}$$

2.6 Weighted quadrature formula

Proposition 2.1. *Gauss-Radau weigthed quadrature formula for the Lebesgue measure. There exists a unique set of* $N+1$ *nodes* $r_n, n = \overline{0,N}$ *in* Λ *, and* $N+1$ *positive real numbers* $\omega_n, n = \overline{0,N}$ *such that the following exactness property holds :*

$$\forall \varphi \in IP_{2N-1}(\Lambda), \int_0^1 \varphi(r) r dr = \sum_{n=0}^{N} \varphi(r_n)\omega_n,$$

where $r_n, n = \overline{0,N}$ *are the roots of polynomial* $A_N(r)$ *and the weights are given by:*

$$\omega_n = \frac{r_n}{(n+1)^2 O_N^2(r_n)}, ..., n = \overline{0,N}$$

see[5]

Proposition 2.2. *The polynomial* q_{2N-1} *with degree* $(2N-1)$ *has the form*

$$q_{2N-1}(r) = A_{N-1}^2(r) + \alpha(N) O_{N-1}(r) A_N(r),$$

where $\alpha(N) = -\frac{(N+1)(2N-1)}{N(2N+1)}$.

Lemma 2.3. *The polynomial $A_{N-1}(r) \in IP_N(\Lambda)$ verify the double inequality,*

$$\|A_{N-1}(r)\|^2_{L^2_1(\Lambda)} \leq (A_{N-1}(r), A_{N-1}(r))_N \leq 2\,\|A_{N-1}(r)\|^2_{L^2_1(\Lambda)} \tag{2.13}$$

using (2.9) and (2.11) we find

$$I_1 = \int_0^1 A^2_{N-1}(r)rdr = \frac{N}{4N^2-1}, \tag{2.14}$$

$$I_2 = \int_0^1 O_{N-1}(r)A_N(r)rdr = -\frac{N}{(4N^2-1)}, \tag{2.15}$$

using the exact quadrature formula we can write:

$$\begin{aligned}\int_0^1 q_{2N-1}(r)rdr &= \sum_{n=0}^{N} q_{2N-1}(r_n)w_n \\ &= I_1 + \alpha(N)I_2 \\ &= \sum_{n=0}^{N} A^2_{N-1}(r_n)w_n \\ &= (A_{N-1}(r), A_{N-1}(r))_N,\end{aligned}$$

$\alpha(N)I_2 = \frac{N+1}{4N^2+1} \leq I_1$ *that's give the result (2.13).*

Proposition 2.4. *By using (2.10) we can write the solution in the following form*

$$U_N(r,t) = \sum_{n=0}^{N-1} a_n(t)A_n(r),$$

and by using (2.11) we find

$$\|U_N(r,t)\|^2_{L^2_1(\Lambda)} \leq C\left(\ln(4N^2-1)\right), \tag{2.16}$$

see[2].

2.7 Existence and uniqueness of solution

Proposition 2.5. *The bilinear form $a_{1N}(\cdot,\cdot)$ satisfies the following properties of continuity*:

$$\forall v_N \in P_N(\Lambda), \forall u_N \in P_N(\Lambda), |a_{1N}(u_N,v_N)| \leq |u_N|_{H^1_1(\Lambda)} \cdot |v_N|_{H^1_1(\Lambda)}, \tag{2.17}$$

and ellipticity

$$\forall u_N \in P_N(\Lambda), \quad |a_{1N}(u_N,v_N)| \geq |u_N|^2_{H^1_1(\Lambda)}. \tag{2.18}$$

Proof. $a_{1N}(u_N,v_N) = (\partial_r u_N, \partial_r v_N)_N$ the degree of polynomials $\partial_r u_N, \partial_r v_N$ is less than or equal to $2N-2$ then $(\partial_r u_N, \partial_r v_N)_N = \int_0^1 \partial_r u_N \partial_r v_N rdr$, by the Schwartz inequality we obtain the desired results. □

2.8 Stability estimation

Proposition 2.6. *The solution u_N in the $IP_N(\Lambda)$ satisfies the inequality of stability*

$$(f_N, u_N)_N \leq C_1\left(\ln(4N^2-1)\right)^{\frac{1}{2}} \|f_N\|_{L^2_1(\Lambda)}. \tag{2.19}$$

Proof. Using(2.6), (2.13),(2.16) and (2.14), yields the desired results. □

3 Numerical experiment

The variables r and t play different role, to separate these variables we consider the solution $u(r,t)$ and $f(r,t)$ as functions of the variable t, its values are in the function space defined in Λ, we consider u defined by

$$\begin{aligned} u \quad &: \quad [0,\infty) \rightarrow H_1^2(\Lambda) \\ & \qquad t \quad \rightarrow \quad u(t) \end{aligned}$$

then we can not $u(r,t) = u(t)(r)$, the variational formulation can be written as

$$\left\{ \begin{array}{l} \quad \text{find } \; u_N \text{ in } IP_N(\Lambda), \text{ such that} \\ \forall v_N \in IP_N(\Lambda), \; \left(\frac{d^2}{dt^2} u_N(t), v_N \right) + a_{1N}(u_N(t), v_N) = (f_N, v_N)_N \end{array} \right. \tag{3.1}$$

the formulation (3.1) is true for all $v_N \in IP_N(\Lambda)$, then it is true for $v_m(r) = l_m(r)$, $m = \overline{0,N}$ where $(l_m(r))_{0 \leq m \leq N}$ form a basis to the polynomial space $IP_N(\Lambda)$, the degree of the polynomial $u_N(t)v_N$ is $2N$ and the degree of the polynomial $\partial_r u_N(t)\partial_r v_N$ is $2N-2$,with respect the variable r, then we can write (3.1) as:

$$\left\{ \begin{array}{l} \text{find } \; u_N \in IP_N(\Lambda), \text{ such that} \\ \forall l_m \in IP_N(\Lambda), \; \sum\limits_{k=0}^{N} \left(\sum\limits_{n=0}^{N} a_n''((t) l_n(r_k) l_m(r_k)\omega_k \right) + \sum\limits_{k=0}^{N} \left(\sum\limits_{n=0}^{N} a_n(t) l_n'(r_k) l_m'(r_k)\omega_k \right) \\ \quad = \sum\limits_{k=0}^{N} f_N(t, r_k) l_m(r_k))\omega_k, \; m = \overline{0,N} \end{array} \right. \tag{3.2}$$

(3.2) is equivalent to,

$$\left\{ \begin{array}{l} a_m''(t)\omega_m + \sum\limits_{n=1}^{N+1} \left(\sum\limits_{k=0}^{N} l_n'(r_k) l_m'(r_k)\omega_k \right) a_n(t) = f_N(r_m, t)\omega_m, \; m = \overline{1,N} \quad \text{in } \; \Lambda \cap \sum_{N+1} \times IR^{*+} \\ \left\{ \begin{array}{l} u_N(r,t) = u_{N0}(r) \\ u_t(r,t) = u_{N1}(r) \end{array} \right. \quad r \in \Lambda \, , t = 0 \end{array} \right. \tag{3.3}$$

We obtain a linear system, then we can write this system in a matricial form:

$$\Gamma D^2 a + Aa = F, \tag{3.4}$$

Where A is a symmetric positive defined matrix with order $N+1$, its elements have the form:

$$\alpha_{mn} = \sum_{k=0}^{N} l_n'(r_k) l_m'(r_k)\omega_k, \; n = \overline{0,N} \, \}, m = \overline{0,N}$$

Γ is a diagonal invertible matrix, its elements are define as:

$$\gamma_{mn} = \left\{ \begin{array}{ll} w_m, & n = m \\ 0, & n \neq m \end{array} \right. \quad , \; m,n = \overline{0,N}$$

F is a known vector where:

$$\begin{aligned} F &= (g_0(t), g_1(t), g_2(t), ..., g_{N-1}(t), g_N(t))^t \\ h_m(t) &= f(r_m, t)\omega_m, \; m = \overline{0,N} \end{aligned}$$

and the vector a is an unknown vector where

$$a = (a_0(t), a_1(t), a_2(t), ..., a_{N-1}(t), a_N(t))^t$$

where the operator,

$$D = \frac{d^2}{dt^2},$$

multiplying (3.4) by the invertible matrix Γ^{-1} of Γ , then we obtain

$$Da^2 + \Gamma^{-1}Aa = \Gamma^{-1}F, \tag{3.5}$$

the matrix $\Gamma^{-1}A$ has positive eigenvalues and there exists an orthogonal invertible matrix P such that,

$$P^{-1}\left(\Gamma^{-1}A\right)P = C,$$

where C is a diagonal matrix, the elements of the diagonal are the eigenvalues $\alpha_m, m = \overline{0, N}$ of the matrix $\Gamma^{-1}A$, if we consider the vector v such that

$$a = Pv,$$

then the system (3.5) becomes

$$PD^2v + (\Gamma^{-1}A)Pv = \Gamma^{-1}F, \tag{3.6}$$

multiplying (3.6) by the matrix P^{-1} we obtain,

$$D^2v + Cv = P^{-1}\Gamma^{-1}F, \tag{3.7}$$

The matricial form (3.7) has $N+1$ linear equations defined as

$$\begin{aligned} v_m^{''}(t) + \alpha_m v_m(t) &= h_m(t), \quad \alpha_m > 0, \\ \text{where} \quad h_m(t) &= \sum_{j=0}^{N} p^{-1}(m,j)\,\Gamma^{-1}(m,j)\,g_j(t),\ 0 \le m \le N, \end{aligned} \tag{3.8}$$

$p^{-1}(m,j)$ are the elements of the inverse matrix P^{-1}. To solve the equations (3.8), we may write the solution in the closed form :

$$v_m(t) = \int_0^t \sin(\alpha_m(t-s))h_m(s)ds + \sin(\alpha_m t + \gamma_m),\ \ 0 \le m \le N, \tag{3.9}$$

or $\sin(\alpha_m t + \gamma_m) = c_m \cos(\alpha_m t) + d_m \sin(\alpha_m t)$, where $c_m = \sin(\gamma_m)$ and $d_m = \cos(\gamma_m)$ are constants to be determined, using the initial conditions then (3.9) may be written in the following form:

$$\begin{aligned} &v_m(t) = \textstyle\int_0^t \sin(\alpha_m(t-s))h_m(s)ds + \sin(\alpha_m t + \gamma_m), \\ &\sin(\gamma_m) = \left(\sum_{j=0}^{N} p^{-1}(m,j)\,u_0(r_j)\right),\ \ \cos(\gamma_m) = \left(\sum_{j=0}^{N} p^{-1}(m,j)\,u_1(r_j)\right), \end{aligned} \tag{3.10}$$

Finally we obtain the functions,

$$a_m(t) = \sum_{j=0}^{N} p_{mj}\left(\int_0^t \sin(\alpha_m(t-s))h_m(s)ds + \sin(\alpha_m t + \gamma_m)\right),\ \ 0 \le m \le N,$$

where p_{nj} , $0 \le n, j \le N$ are the elements of the matrix P and the approximation solution is

$$u_N(r,t) = \sum_{m=0}^{N}\left(\sum_{j=0}^{N} p_{mj}\left(\int_0^t \sin(\alpha_m(t-s))h_m(s)ds + \sin(\alpha_m t + \gamma_m)\right)\right) l_m(r),$$

see[1, 7].

3.1 Error estimation

Definition 3.1. The polynomial space $P_N(\Omega)$ dense in the space of continuous functions on Ω hence in $L_1^2(\Omega)$ then any function $u \in L_1^2(\Omega)$ admits the expansion

$$u(r,t) = \sum_{n=0}^{\infty} a_n(t) A_n(r) \ , \tag{3.11}$$

Where $\Omega = \Lambda \times IR^{*+}$.

Proposition 3.2. *The following estimate holds between the exact solution u in $H_1^1(\Lambda)$ and the approximation solution $u_N \in P_N(\Lambda)$ verify,*

$$\|u - u_N\|_{L_1^2(\Omega)} \leq \left(\frac{1}{2(2N+1)(2N+3) - C} \right) \|f - f_N\|_{L_1^2(\Omega_2)} \tag{3.12}$$

Proposition 3.3. *Suppose that the functions $a_n^{(l)}(t)$, $l = 0, 1, 2$ and $n \in IN$ are bounded on IR^+, that is there exists a real positive number M such that $\left| a_n^{(l)}(t) \right| \leq M$, for $t \in IR^+$, then there exists a real positive number C such that:*

$$\|u_{tt} - u_{Ntt})\|_{L_1^2(\Omega)} \leq C \|u - u_N\|_{L_1^1(\Omega)} \ , \tag{3.13}$$

where $\Omega = \Lambda \times IR^+$.

Proof. Using the ellipticity condition (2.1) and (2.2)we can write

$$(u_{tt} - u_{ttN}, u - u_N) + a_1(u - u_N, u - u_N) = (f - f_N, u - u_N),$$

$$\int_\Lambda (\ u_r - u_{rN})^2 r dr = \int_\Lambda ((\ u_{tt} - u_{ttN}) \, (u_N - u)) r dr + \int_\Lambda ((f - f_N) \, (u - u_N)) r dr, \tag{3.14}$$

using Schwartz inequality in the right hand side of (3.14) then we find,

$$\|u_r - u_{rN}\|_{L_1(\Omega)}^2 \leq \Big((\ u_{tt} - u_{ttN}) + \|f - f_N)\|_{L_1^2(\Omega_2)} \Big) \|u - u_N\|_{L_1(\Omega)} \ , \tag{3.15}$$

using (2.11), (2.12) and (3.13) then we obtain,

$$\begin{aligned} 2(2N+1)(2N+3) \|u - u_N\|_{L_1^2(\Omega)}^2 &\leq \Big(C \|u - u_N\|_{L_1^1(\Omega)} + C \|f - f_N\|_{L_1^2(\Omega)} \Big) \|u - u_N\|_{L_1^2(\Omega)} \\ \|u - u_N\|_{L_1^2(\Omega)} &\leq \frac{1}{2(2N+1)(2N+3) - C} \|f - f_N\|_{L_1^2(\Omega)} \end{aligned}$$

□

3.2 Figures illustrations

The figures 1 and 2 present the true and the approximate solution u and u_N respectively, and figures 3 and 4 present the true and the approximate function $b_k = u(r,t), r = r(k)$, $k = \overline{0, N}$ and a_k ,$k = \overline{0, N}$ respectively, these plots occur when $N = 5$ and the test function is : $u(r,t) = -\cos(\pi r)) \sin(\pi t)$.

These plots are occurred when $N = 5$ and $(r,t) \in [0,1] \times [0,3]$.

Figure1:The true solution u(r,t)

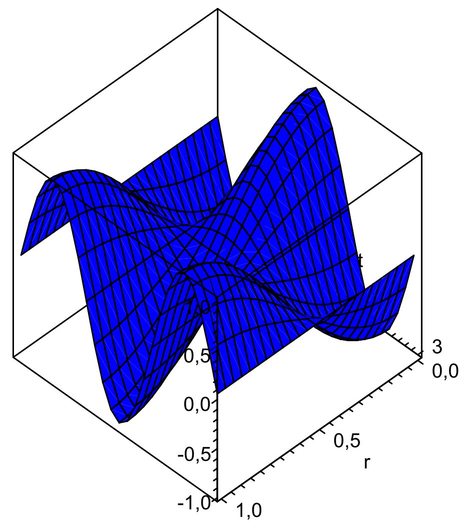


Figure2:The approximate solution un(r,t)

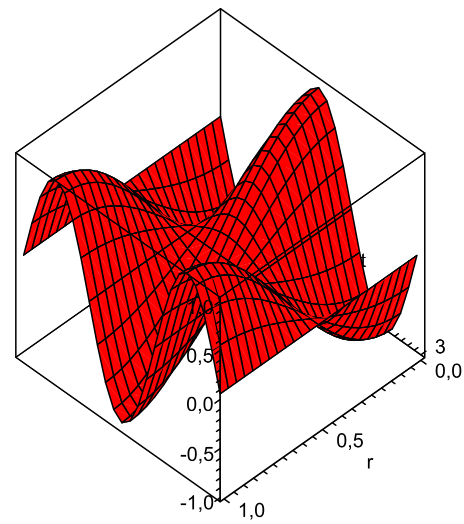


Figure3: The fuctions b(n,t), n=1,N

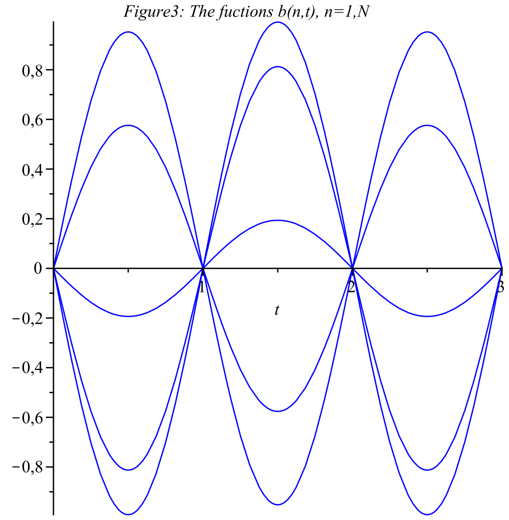
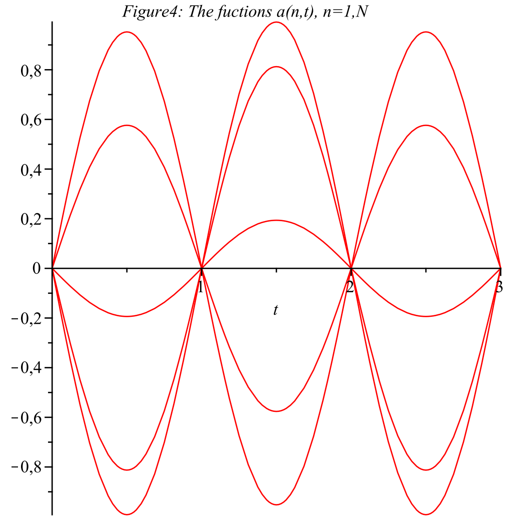


Figure4: The fuctions a(n,t), n=1,N



References

[1] A. Boutaghou, On Finite Spectral Method for Helmholtz Problem, Int.J. of Appl. Math and Mech. 7(3): 70-92, (2011).

[2] A. Boutaghou and F.Z. Nouri, On Finite Spectral Method for Axi-symmetric Elliptic problem, Journal of Analysis and Applications, Vol. 4 (2006), No 3, pp.149-168. SAS International Publications.

[3] A. Boutaghou, Spectral method for Mixed Initial-Boundary Value Problem, Paletine Journal of Mathematics, Vol 4(1)(2015) 12-20.

[4] C. Bernardy and Y-Maday, Approximation spectrale de problemes aux limites elliptiques, 1992, Springer Verlag France, Paris.

[5] C. Bernardy, M. Dauge and Y. Maday, Numerical algorithms and tests due to Mejdi Azaiez, 1999 Spectal Methods for Axisymmetric Domains, Editions scientifiques et médicales Elsevier France.

[6] D. Richards, Advanced Mathematical Methods with Maple, 2002, Cambridge University Press.

[7] G. Allaire, Analyse Numérique et Optimisation. Une introduction à la modélisation mathématique et à la simulation numérique.

[8] J. de Frutos, R. Munoz-Sola, Chebyshev pseudospectral collocation for parabolic probléms with nonconstant coefficients. Appear in Houston J. Math (Proccedings ICOSAHOM 95).

[9] Maple V, Release 4, Waterloo Maple Inc., 1995.

[10] T.N. Phillips and A.R. Davies, On semi-infinite spectral elements for Poisson problems with re-entrant boundary singularities, 1988, Journal of Computationl and Apllied Mathematics 21 pp. 173-188.

[11] R. G. Owens and T.N. Phillips, Steady Viscoelastic Flow past a sphere using Spectral Elements. 1996, Int. J. Numer. Meth. Engng., 39: 1517–1534.

Author information

Amor Boutaghou, Ecole Nationale Supérieure d'Hydraulique (ENSH), Blida, ALGERIA.
E-mail: boutaghou_a@yahoo.com

Received: 2021-07-02
Accepted: 2023-03-24