Palestine Journal of Mathematics

Vol. 12(4)(2023) , 100–105

ON COMMUTATIVITY OF BANACH ALGEBRAS WITH ENDOMORPHISMS

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Communicated by Ayman Badawi

MSC 2010 Classifications: 47L10, 16U80, 08A35.

Keywords and phrases: Banach algebras, Commutativity, Endomorphism.

Abstract: This article focuses on decomposing a Banach algebra $\mathcal{X}$ via its endomorphisms. In particular, we show that if a Banach algebra $\mathcal{X}$ has an injective continuous endomorphism f such that $f([x^n, y^m])$ is in the center of $\mathcal{X}$, $Z(\mathcal{X})$, for two integers $n = n(x,y), m = m(x,y)$ and sufficiently many x, y, then for all x in $\mathcal{X}$ either x in $Z(\mathcal{X})$ or $f(x)$ in $Z(\mathcal{X})$. To demonstrate the importance of our theorem assumptions, we will provide several examples.

1 Introduction

Let $\mathcal{X}$ be a Banach algebra with center $Z(\mathcal{X})$. Recall that $\mathcal{X}$ is prime, if for any $x, y \in \mathcal{X}$, $x\mathcal{X}y = 0$ implies either $x = 0$ or $y = 0$. The Lie product and Jordan product of $x, y \in \mathcal{X}$ are noted by $[x, y]$ and $x \circ y$ respectively, where $[x, y] = xy - yx$ and $x \circ y = xy + yx$. A derivation is an additive mapping, denoted as d, defined on the set $\mathcal{X}$. It satisfies the property $d(xy) = d(x)y + xd(y)$ for all $x, y \in \mathcal{X}$. If $d(x) = [a, x]$ for all $x \in \mathcal{X}$, then d is called an inner derivation induced by an element a in $\mathcal{X}$. For more examples, please refer to sources such as [[3], [5]]. In the case of Banach algebras, Yood [12] proved that if a semiprime Banach algebra $\mathcal{X}$ having two nonvoid open subsets $\mathcal{H}_1$ and $\mathcal{H}_2$ verify for all $(x, y) \in \mathcal{H}_1 \times \mathcal{H}_2$ there is $(n, m) \in \mathbb{N}^* \times \mathbb{N}^*$ such that $[x^n, y^m] = 0$, then $\mathcal{X}$ must be commutative. Inspired by Yood's result, Mohamed Moumen, Lahcen Taoufiq, and Lahcen Oukhtite [10] proved that if a prime Banach algebra, denoted by $\mathcal{X}$, has a continuous derivation d and satisfies the condition $d(x^n y^m) + [x^n, y^m] \in Z(\mathcal{A})$ for integers n and m determined by x and y, and for a sufficiently large number of x and y, then $\mathcal{X}$ is commutative (see [6], [7], [8] and [9] for further information and examples).

Motivated by these results, the purpose of this article is to establish the results with a similar conclusion, but with other identities. For example, we have proven that a prime Banach algebra $\mathcal{X}$ is equal to $Z(\mathcal{X}) \cup \{x \in \mathcal{X} \mid f(x) \in Z(\mathcal{X})\}$ under certain conditions. These conditions include the existence of two non-empty open subsets $\mathcal{H}_1$ and $\mathcal{H}_2$ and an injective continuous endomorphism f which verifies for any pair (x, y) of $\mathcal{H}_1 \times \mathcal{H}_2$, there exist two strictly positive integers n and m such that $f([x^n, y^m])$ in $Z(\mathcal{X})$. In this context, other similar results have been found.

In this article, we will utilize the following commonly-known results without specifically mentioning them.

Remark. Let $\mathcal{X}$ be a prime Banach algebra.
1. If $x \in Z(\mathcal{X})$ and $xy \in Z(\mathcal{X})$, then $x = 0$ or $y \in Z(\mathcal{X})$.
2. $Z(\mathcal{X})$ does not admit any zero divisors.
3. If d is a non zero derivation of $\mathcal{X}$ such that $[d(x), x] \in Z(\mathcal{X})$ for all $x \in \mathcal{X}$ (in particular if $d(\mathcal{X}) \subset Z(\mathcal{X})$), then $\mathcal{X}$ is commutative.

2 Main results

Our main results rely heavily on the lemma presented by Bonsall and Duncan in their work [2].

Lemma 2.1. *Let $\mathcal{X}$ be a real or complex Banach algebra and $S(t) = \sum_{i=0}^{n} t^i s_i$ a polynomial in the real variable t with coefficients in $\mathcal{X}$. If for an infinite set of real values of t, $P(t) \in C$, where C is a closed linear subspace of $\mathcal{X}$, then every s_i lies in C.*

Theorem 2.2. *Let f be an injective and continuous endomorphism of a prime Banach algebra $\mathcal{X}$, such that:*

$$(\forall x \in \mathcal{H}_1)\ (\forall y \in \mathcal{H}_2)\ (\exists n \in \mathbb{N}^*)\ (\exists m \in \mathbb{N}^*) \text{ such that } f([x^n, y^m]) \in Z(\mathcal{X})$$

Then, $\mathcal{X} = Z(\mathcal{X}) \cup f^{-1}(Z(\mathcal{X}))$(where $\mathcal{H}_1$ and $\mathcal{H}_2$ are two non-void open subsets of $\mathcal{X}$).

Proof. For any pair of natural numbers (n, m), we establish the following set:

$O_{n,m} = \{(x, y) \in \mathcal{X}^2 \mid f([x^n, y^m]) \notin Z(\mathcal{X})\}$ and $F_{n,m} = \{(x, y) \in \mathcal{X}^2 \mid f([x^n, y^m]) \in Z(\mathcal{X})\}$.

We assert that every $F_{n,m}$ is a closed set in $\mathcal{X} \times \mathcal{X}$. To prove this, we examine a sequence $((x_k, y_k))_{k\in\mathbb{N}} \subset F_{n,m}$ that converges to $(x, y) \in \mathcal{X} \times \mathcal{X}$. Given that $((x_k, y_k))_{k\in\mathbb{N}} \subset F_{n,m}$, it follows that

$$f([(x_k)^n, (y_k)^m]) \in Z(\mathcal{X}) \text{ for all } k \in \mathbb{N}.$$

Since $([(x_k)^n, (y_k)^m])_{k\in\mathbb{N}}$ converges to $[x^n, y^m]$ and f is continuous, we can conclude that $f([x^n, y^m]) \in Z(\mathcal{X})$. This means that $F_{n,m}$ is a closed set and $O_{n,m}$ is open. Assuming that $O_{n,m}$ is dense for all (n, m), the Baire category theorem states that their intersection must also be dense. However, this contradicts the fact that $(\cap O_{n,m}) \cap (\mathcal{H}_1 \times \mathcal{H}_2)$ is empty. Therefore, we can conclude that there exists p and q in $\mathbb{N}^*$ such that $O_{p,q}$ is not a dense set. Furthermore, there exists a non-empty open subset $O \times O'$ in $F_{p,q}$ where $f([x^p, y^q]) \in Z(\mathcal{X})$ for all $x \in O$ and $y \in O'$. Now, we consider $y_0 \in O$ and $y \in \mathcal{X}$ we have $y_0 + ty \in O$ for all sufficiently small real $t \in \mathbb{R}$ and $f([x^p, (y_0 + ty)^q]) \in Z(\mathcal{X})$.
The expression $(y_0 + ty)^q$ can be written as:

$$(y_0 + ty)^q = A_{q,0}(y_0, y) + tA_{q-1,1}(y_0, y) + ... + t^q A_{0,q}(y_0, y).$$

While

$$[x^p, (y_0 + ty)^q] = [x^p, A_{q,0}(y_0, y)] + t[x^p, A_{q-1,1}(y_0, y)] + ... + t^q[x^p, A_{0,q}(y_0, y)] \in Z(\mathcal{X})$$

and

$$f([x^p, (y_0+ty)^q]) = f([x^p, A_{q,0}(y_0, y)])+tf([x^p, A_{q-1,1}(y_0, y)])+...+t^q f([x^p, A_{0,q}(y_0, y)]) \in Z(\mathcal{X}).$$

Lemma 2.1 implies that the coefficient $f([x^p, y^q])$ of t^q in this polynomial belongs to $Z(\mathcal{X})$.
Consequently, for all $(x, y) \in O \times \mathcal{X}$

$$f([x^p, y^q]) \in Z(\mathcal{X}).$$

Now, fix $y \in \mathcal{X}$, if we continue with the same method, we discover that $f([x^p, y^q])$ belongs to $Z(\mathcal{X})$ for all values of x and y in $\mathcal{X}$.
Assuming that x belongs to the set $\mathcal{X}$, when we substitute z with x^p, we get $f([z, (y + tz)^q])$ belonging to $Z(\mathcal{X})$ for any y in $\mathcal{X}$ and any t in the real numbers. Since

$$P(t) = f([z, (y + tz)^q]) = \sum_{k=0}^{q} t^k f([z, A_{q-k,k}(z, y)])$$

where $A_{q-k,k}(z, y)$ denotes the sum of all terms in which y appears exactly $q - k$ times and z appears exactly k times. By Lemma 2.1 we have $f([z, A_{q-k,k}(z, y)]) \in Z(\mathcal{X})$ for all $0 \leq k \leq q$. The coefficient of t in this polynomial is $f([z, A_{q-1,1}(z, y)])$, where $A_{q-1,1}(z, y) = \sum_{k=0}^{q-1} z^{q-1-k} y z^k$, then $[z, A_{q-1,1}(z, y)] = \sum_{k=0}^{q-1} [z, z^{q-1-k} y z^k] = [z^q, y]$.
Therefore, for all $x, y \in \mathcal{X}$, we have

$$f([x^{pq}, y]) \in Z(\mathcal{X}).$$

We have two cases:
▼ If $x^{pq} \in Z(\mathcal{X})$ for all $x \in \mathcal{X}$, we will show that $\mathcal{X}$ is commutative. For this:
Let $x \in \mathcal{X}$ and a be non-zero element of $Z(\mathcal{X})$ we have $(a+tx)^{pq} \in Z(\mathcal{X})$ for all $t \in \mathbb{R}$. Then $(a+tx)^{pq} = \sum_{k=0}^{pq} \binom{pq}{k} t^k a^{pq-k} x^k \in Z(\mathcal{X})$ (because $a \in Z(\mathcal{X})$). By using Lemma 2.1, we conclude that $a^{pq-k}x^k \in Z(\mathcal{X})$ for all $0 \leq k \leq pq$. In particular, for $k = pq-1$ we have $xa^{pq-1} \in Z(\mathcal{X})$, by Remark 1, we have $x \in Z(\mathcal{X})$ because $a^{pq-1} \in Z(\mathcal{X}) \setminus \{0\}$. Then $\mathcal{X}$ is commutative.
▼ If there is $x \in \mathcal{X}$ such that $x^{pq} \notin Z(\mathcal{X})$, we shall prove that $f(x) \in Z(\mathcal{X})$. We have $f([x^{pq}, y]) \in Z(\mathcal{X})$ for all $y \in \mathcal{X}$, we replace y by xy and we obtain $f(x)f([x^{pq}, y]) \in Z(\mathcal{X})$. By Remark 1, $f([x^{pq}, y]) = 0\ \forall y \in \mathcal{X}$ or $f(x) \in Z(\mathcal{X})$. Suppose that $f(x) \notin Z(\mathcal{R})$, then $f([x^{pq}, y]) = 0$ for all $y \in \mathcal{X}$. Since f is injective, then $[x^{pq}, y] = 0$ for all $y \in \mathcal{X}$ that is $d(\mathcal{X}) \in Z(\mathcal{X})$ where d is the inner derivation associated by x^{pq}. According to Remark 3 we conclude that $\mathcal{X}$ must be commutative, contradiction. Hence $f(x) \in Z(\mathcal{R})$. □

Theorem 2.3. *Consider a prime Banach algebra $\mathcal{X}$ that can be either real or complex. Let f denote an injective continuous endomorphism. Suppose that*

$$(\forall (x,y) \in \mathcal{H}_1 \times \mathcal{H}_2)\ (\exists (n,m) \in \mathbb{N}^* \times \mathbb{N}^*) \text{ such that } f(x^n y^m) \in Z(\mathcal{X}),$$

where $\mathcal{H}_1$ and $\mathcal{H}_2$ are two non void open subsets of $\mathcal{X}$. Then $\mathcal{X} = Z(\mathcal{X}) \cup f^{-1}(Z(\mathcal{X}))$.

Proof. We define the following sets for all $n, m \in \mathbb{N}^*$

$$O_{n,m} = \{(x,y) \in \mathcal{X}^2 \mid f(x^n y^m) \notin Z(\mathcal{X})\} \text{ and } F_{n,m} = \{(x,y) \in \mathcal{X}^2 \mid f(x^n y^m) \in Z(\mathcal{X})\}.$$

Using the Baire category theorem on the sets $O_{n,m}$, we can conclude, as we did before, that there exist two integers p and q (excluding zero) such that:

$$f(x^p y^q) \in Z(\mathcal{X}) \text{ for all } (x,y) \in \mathcal{X}^2.$$

By substituting x with x^q and y with y^p in the final expressions, we can derive:

$$f(x^{pq} y^{pq}) \in Z(\mathcal{X}) \text{ for all } (x,y) \in \mathcal{X}^2.$$

We also have

$$f(y^{pq} x^{pq}) \in Z(\mathcal{X}) \text{ for all } (x,y) \in \mathcal{X}^2.$$

As a result

$$f([x^{pq}, y^{pq}]) \in Z(\mathcal{X}) \text{ for all } (x,y) \in \mathcal{X}^2.$$

Therefore, according to Theorem 2.2, we reach the desired conclusion. □

Theorem 2.4. *Let $\mathcal{X}$ be a real or complex prime Banach algebra and $\mathcal{H}_1$ and $\mathcal{H}_2$ two non-void open subsets. If an injective and continuous endomorphism f satisfies: $(\forall x \in \mathcal{H}_1)$ $(\forall y \in \mathcal{H}_2)$ $(\exists n \in \mathbb{N}^*)$ $(\exists m \in \mathbb{N}^*)$ such that $f(x^n \circ y^m) \in Z(\mathcal{X})$*

then, $\mathcal{X} = Z(\mathcal{X}) \cup f^{-1}(Z(\mathcal{X}))$.

Proof. The proof for this outcome follows a similar approach to Theorem 2.2. □

Corollary 2.5. *Let $\mathcal{X}$ be a real or complex prime Banach algebra and $\mathcal{H}_1$ and $\mathcal{H}_2$ two non-void open subsets of it. If one of the following conditions is true:*
1. $(\forall (x,y) \in \mathcal{H}_1 \times \mathcal{H}_2)$ $(\exists (n,m) \in \mathbb{N}^* \times \mathbb{N}^*)$ *such that* $x^n y^m \in Z(\mathcal{X})$
2. $(\forall (x,y) \in \mathcal{H}_1 \times \mathcal{H}_2)$ $(\exists (n,m) \in \mathbb{N}^* \times \mathbb{N}^*)$ *such that* $[x^n, y^m] \in Z(\mathcal{X})$
3. $(\forall (x,y) \in \mathcal{H}_1 \times \mathcal{H}_2)$ $(\exists (n,m) \in \mathbb{N}^* \times \mathbb{N}^*)$ *such that* $x^n \circ y^m \in Z(\mathcal{X})$
then $\mathcal{X}$ must be commutative.

Proof. We can take $f = I$ where I is the identical application of $\mathcal{X}$. □

The next example proves that $\mathcal{X}$ must be prime in the assumption of Theorem 2.2.

Example 2.6. Let $\mathcal{X} = \left\{ \begin{pmatrix} 0 & a \\ 0 & 0 \end{pmatrix} \mid a \in \mathbb{R} \right\}$. It is noteworthy that $\mathcal{X}$ is a Banach algebra when its norm is defined as $\|M\| = |a|$, where $M = \begin{pmatrix} 0 & a \\ 0 & 0 \end{pmatrix}$. However, it is not a prime algebra, as shown by the equation:

$$\begin{pmatrix} 0 & 2 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} 0 & a \\ 0 & 0 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} \quad \text{for all} \quad a \in \mathbb{R}.$$

If n and m are both greater than 1, then it is found that the commutator of x^n and y^m is equal to 0 for any values of x and y in $\mathcal{X}$. It is important to note that although this is true, $\mathcal{X}$ is not a commutative set.

It is not redundant to demonstrate that both hypotheses, $\mathcal{H}_1$ and $\mathcal{H}_2$, are open, as shown in the following example.

Example 2.7. Consider the field of real numbers $\mathbb{R}$ and let $\mathcal{X}$ be $\mathcal{M}_2(\mathbb{R})$ equipped with regular matrix addition and multiplication. Also, consider the norm defined by $\|A\|_1 = \max\limits_{1 \le j \le 2} \sum\limits_{1 \le i \le 2} |a_{i,j}|$ for all $A = (a_{i,j})_{1 \leqslant i,j \leqslant 2} \in \mathcal{X}$. This makes $\mathcal{X}$ a prime unital Banach algebra.

Let $\mathcal{F}_1 = \left\{ \begin{pmatrix} t & 0 \\ 0 & t \end{pmatrix} \mid t \in \mathbb{R} \right\}$ and $\mathcal{F}_2 = \left\{ \begin{pmatrix} t^2 & 0 \\ 0 & t^2 \end{pmatrix} \mid t \in \mathbb{R} \right\}$. It is worth noting that $\mathcal{F}_1$ is not open in $\mathcal{X}$. To prove this, we need to show that the complement of $\mathcal{F}_1^c$ is not closed. For this, consider the sequence $\left(\begin{pmatrix} 1 + \frac{1}{n} & \frac{-1}{n} \\ \frac{1}{n} & 1 + \frac{1}{n} \end{pmatrix} \right)_{n \in \mathbb{N}^*}$, which belongs to $\mathcal{F}_1^c$ and converges to $\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \notin \mathcal{F}_1^c$. Therefore, $\mathcal{F}_1^c$ is not closed, implying that $\mathcal{F}_1$ is not open in $\mathcal{X}$.

Furthermore, we have

$$A^n \circ B^m = \begin{pmatrix} 2a^n b^{2m} & 0 \\ 0 & 2a^n b^{2m} \end{pmatrix} \in Z(\mathcal{X})$$

for all $A = \begin{pmatrix} a & 0 \\ 0 & a \end{pmatrix} \in \mathcal{F}_1$, $B = \begin{pmatrix} b^2 & 0 \\ 0 & b^2 \end{pmatrix} \in \mathcal{F}_2$ and for all $(m, n) \in \mathbb{N}^2$. However, it should be noted that $\mathcal{X}$ is not commutative.

This example demonstrates that $\mathbb{Z}/3\mathbb{Z}$ cannot be used in place of $\mathbb{K} = \mathbb{R}$ or $\mathbb{C}$.

Example 2.8. Consider the Banach algebra $(\mathcal{M}_2(\mathbb{Z}/3\mathbb{Z}), +, \times, .)$ consisting of 2×2 matrices with coefficients in $\mathbb{Z}/3\mathbb{Z}$ and usual matrix addition and multiplication. The norm is defined by $\|A\|_1 = \sum\limits_{1 \le i,j \le 2} |a_{i,j}|$ for any $A = (a_{i,j})_{1 \leqslant i,j \leqslant 2} \in \mathcal{M}_2(\mathbb{Z}/3\mathbb{Z})$, where $|.|$ is the norm defined on $\mathbb{Z}/3\mathbb{Z}$ by $|\bar{0}| = 0$, $|\bar{1}| = 1$, and $|\bar{2}| = 2$.

Note that the subset $\mathcal{H} = \left\{ \begin{pmatrix} a & 0 \\ 0 & a \end{pmatrix} \mid a \in \mathbb{Z}/3\mathbb{Z} \right\}$ is open in $\mathcal{M}_2(\mathbb{Z}/3\mathbb{Z})$. In fact, for any $A \in \mathcal{H}$, the open ball $B(A, 1) = \{X \in \mathcal{M}_2(\mathbb{Z}/3\mathbb{Z})$ such that $\|A - X\|_1 < 1\}$ is contained in $\mathcal{H}$, showing that $\mathcal{H}$ is a non-empty open subset of $\mathcal{M}_2(\mathbb{Z}/3\mathbb{Z}, +, \times, .)$.

For positive integers m and n, we have the following properties:
1. $A^n B^m \in Z(\mathcal{X})$ for all $A, B \in \mathcal{H}$
2. $A^n \circ B^m \in Z(\mathcal{X})$ for all $A, B \in \mathcal{H}$
3. $[A^n, B^m] \in Z(\mathcal{X})$ for all $A, B \in \mathcal{H}$
4. $A^n \in Z(\mathcal{X})$ for all $A \in \mathcal{H}$

However, note that while $\mathcal{M}_2(\mathbb{Z}/3\mathbb{Z})$ is not commutative.

3 Applications

In this section, we will discuss some applications of Theorem 2.2.

Application 1

Consider the set $\mathcal{X}$, which is comprised of all $n \times n$ strictly upper triangular matrices with either real or complex values, where n is greater than or equal to 2. The norm $\|.\|_1$ of $\mathcal{X}$ is defined as the sum of the absolute values of all elements in the matrix. Using the usual matrix operations and this norm, it can be easily verified that $\mathcal{X}$ is a non-commutative real Banach Algebra.

It should be noted that for any $(x, y) \in \mathcal{X}^2$, $[x^n, y^n]$ is a member of the center of $\mathcal{X}$. This implies, according to Theorem 2.2, that $\mathcal{X}$ is not prime.

Application 2

Let's consider the field of complex numbers, denoted by $\mathbb{C}$. We have a set of 2×2 matrices with matrix addition and multiplication, denoted by $\mathcal{X} = \mathcal{M}_2(\mathbb{C})$. For any $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathcal{X}$, we define $\|A\|_2 = (|a|^2 + |b|^2 + |c|^2 + |d|^2)^{\frac{1}{2}}$. This makes $(\mathcal{X}, \|.\|_2)$ a normed linear space.

We can observe that $\mathcal{H} = \left\{ \begin{pmatrix} e^{it} & 0 \\ 0 & e^{-it} \end{pmatrix} \mid t \in \mathbb{R} \right\}$ is an open subset of $\mathcal{B}$ (refer to Application 3.1 in [11]). It's worth noting that $[A^n, B^m] = 0$ for all $A, B \in \mathcal{H}$ and for all $n, m \in \mathbb{N}^*$. From Theorem 2.2, we can conclude that $\mathcal{X}$ is not a Banach algebra under the defined norm.

4 Conclusion

In this article, we studied the effects of topology and endomorphism on the Banach algebra.

5 Acknowledgements

The authors would like to express their heartfelt gratitude to the referees for providing numerous helpful comments.

References

[1] E. C. Posner: *Derivations in Prime rings*. Proc. Amer. Math. Soc. 8, 1093-1100 (1957).

[2] F. F. Bonsall and J. Duncan, *Complete normed algebras*, Springer-Verlag, New York, (1973).

[3] H. E. Bell: *On some commutativity theorems of Herstein*. Arch. Math. (Basel) 24(1), 34-38 (1973).

[4] Herstein, I. N.: the Lie structure of an assosiative ring, Journal of Algebra, 14, 561-571, (1970)

[5] J. H. Mayne. mappings of prime rings. Canad. Math. Bull., 2(1)(1984), 122-126.

[6] M. Moumen and L. Taoufiq. *Characterization of the center of a prime Banach algebra by its homoderivations, Boletim da Sociedade Paranaense de Matemática (to appear).*

[7] M. Moumen and L. Taoufiq, Commutativity Theorems and projection on the center of a Banach algebra , Tatra Mt. Math. Publ. 83 (2023), 119–130, DOI: 10.2478/tmmp-2023-0009.

[8] M. Moumen and L. Taoufiq, Derivations Mapping into The Jacobson radical of a Banach algebra, Journal of Algebra and Related Topics (2022) (to appear).

[9] M. Moumen, L. Taoufiq and A. Boua, *On prime Banach algebras with continuous derivations*, Mathematica, 65(88), n○1, 2023, pp. 122-132.

[10] M. Moumen, L. Taoufiq and L. Oukhtite, *Some differential identities on prime Banach algebras*, Journal of Algebra and Its Applications (2022), doi:10.1142/S0219498823502584.

[11] Shakir Ali and Abdul Nadim Khan, *On Commutativity Of Banach Algebras With Derivations*, Bull Aust.Math.Soc.91(2015), 419-425 doi:10.1017/S0004972715000118.

[12] Yood, B: *Commutativity theorems for Banach algebras*. Mich. Math. J. 37(2), 203-210 (1990).

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Received: 2022-06-22
Accepted: 2023-07-30