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Hermite-Hadamard inequalities for s-convex functions via Caputo-Fabrizio fractional integrals

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Abstract: This paper introduces some new Hermite-Hadamard type integral inequalities for s-convex function via Caputo-Fabrizio fractional integral. At the end, applicability of obtained results to special means of real numbers has been discussed.

1 Introduction

For two decades, the field of fractional calculus got rapid attention due to its extensive application into the diversified fields such as biochemistry, physics, fluid mechanics, computer and modelling[13–18]. Thorough studies have been done on the existence and uniqueness of solutions for fractional differential equations. The existence and uniqueness of fractional differential equations demand certain inequalities. For this purpose, many mathematicians have developed such inequalities[6–10, 21]. In this paper, Caputo-Fabrizio fractional integral has been used. Caputo-Fabrizio operator has unique property in that it has a non-singular kernel.

Fractional calculus has very important applications in the field of inequality theory. Two types of inequalities arise in mathematics: integral inequalities and variational inequalities. Estimating the lower and upper bounds of the integration of the functions can be done with the help of integral inequalities. In comparison, the theory of variational inequalities has emerged as a fascinating branch of applied mathematics with wide-ranging applications in economics, industry and control theory. The Hermite-Hadamard inequality is essential for understanding convex functions and their generalisation. Hermite-Hadamard inequality has been generalised and refined across numerous classes of convex functions[7, 8, 10, 18, 21–25].

Theorem 1.1. *Let $\Psi : I \to \mathbb{R}$ be a convex function defined on the interval I of real numbers and $c_1, d_1 \in I$ with $c_1 < d_1$, then*

$$\Psi\left(\frac{c_1+d_1}{2}\right) \le \frac{1}{d_1-c_1}\int_{c_1}^{d_1} \Psi(u)\, du \le \frac{\Psi(c_1)+\Psi(d_1)}{2}. \tag{1.1}$$

2 Preliminaries

Before proving the main results, preliminary definitions and concepts have been recalled:

Definition 2.1. A function $\Psi : I \to \mathbb{R}$ is said to be convex if the following inequality holds:

$$\Psi\big(uc_1 + (1-u)d_1\big) \le u\Psi(c_1) + (1-u)\Psi(d_1), \tag{2.1}$$

for $c_1, d_1 \in I$ and $u \in [0, 1]$.

Definition 2.2. The function $\Psi : [0, \infty) \to \mathbb{R}$ is said to be s-convex if the following inequality holds:

$$\Psi(uc_1 + (1-u)d_1) \le u^s\Psi(c_1) + (1-u)^s\Psi(d_1), \tag{2.2}$$

for all $c_1, d_1 \in [0, \infty)$, $u \in [0, 1]$ and for some fixed $s \in (0, 1]$.

Definition 2.3. [3, 19, 20] Let $\Psi \in H^1(c_1, d_1)$, $c_1 < d_1$, $\delta \in [0,1]$, then left Caputo-Fabrizio fractional derivative is given by

$$\left({}_{c_1}^{\text{CFC}}D^{\delta}\Psi\right)(l) = \frac{B(\delta)}{1-\delta}\int_{c_1}^{l}\Psi'(u)e^{\frac{-\delta(l-u)^{\delta}}{1-\delta}}\,du, \tag{2.3}$$

and the associated left Caputo-Fabrizio integral operator is given by

$$\left({}_{c_1}^{\text{CF}}I^{\delta}\Psi\right)(l) = \frac{1-\delta}{B(\delta)}\Psi(l) + \frac{\delta}{B(\delta)}\int_{c_1}^{l}\Psi(u)\,du. \tag{2.4}$$

Where $B(\delta) > 0$ is a normalization function satisfying $B(0) = B(1) = 1$.

Definition 2.4. [3, 19, 20] Let $\Psi \in H^1(c_1, d_1)$, $c_1 < d_1$, $\delta \in [0,1]$, then right Caputo-Fabrizio fractional derivative is given by

$$\left({}^{\text{CFC}}D_{d_1}^{\delta}\Psi\right)(l) = \frac{-B(\delta)}{1-\delta}\int_{l}^{d_1}\Psi'(u)e^{\frac{-\delta(u-l)^{\delta}}{1-\delta}}\,du, \tag{2.5}$$

and the associated right Caputo-Fabrizio integral operator is given by

$$\left({}^{\text{CF}}I_{d_1}^{\delta}\Psi\right)(l) = \frac{1-\delta}{B(\delta)}\Psi(l) + \frac{\delta}{B(\delta)}\int_{l}^{d_1}\Psi(u)\,du. \tag{2.6}$$

Where $B(\delta) > 0$ is a normalization function satisfying $B(0) = B(1) = 1$.

In [1] Dragomir and Agarwal gave important result regarding differntable mapping as:

Lemma 2.5. *Let $\Psi : I^{\circ} \subseteq \mathbb{R} \to \mathbb{R}$ be a differentiable mapping on I° and $c_1, d_1 \in I$. If $\Psi' \in L[c_1, d_1]$, then*

$$\frac{\Psi(c_1)+\Psi(d_1)}{2} - \frac{1}{d_1-c_1}\int_{c_1}^{d_1}\Psi(u)\,du = \frac{d_1-c_1}{2}\int_0^1(1-2u)\Psi'\big(uc_1+(1-u)d_1\big)\,du. \tag{2.7}$$

3 Main Results

Dragomir et al. [10] proved Hermite-Hadamard inequality for s-convex function.

Lemma 3.1. *Let $\Psi : [c_1, d_1] \subseteq \mathbb{R} \to \mathbb{R}$ be s-convex function on $[c_1, d_1]$ and $\Psi \in L_1[c_1, d_1]$. If $\delta \in [0,1]$ and $s \in (0,1]$, then*

$$2^{s-1}\Psi\left(\frac{c_1+d_1}{2}\right) \leq \frac{1}{(d_1-c_1)}\int_{c_1}^{d_1}\Psi(u)\,du \leq \frac{\Psi(c_1)+\Psi(d_1)}{s+1}. \tag{3.1}$$

In this section, Hermite-Hadamard inequality for s-convex function using Caputo-Fabrizio operator has been proved.

Theorem 3.2. *Let $\Psi : [c_1, d_1] \subseteq \mathbb{R} \to \mathbb{R}$ be s-convex function on $[c_1, d_1]$ and $\Psi \in L_1[c_1, d_1]$. If $\delta \in [0,1]$ and $l \in [c_1, d_1]$, then*

$$\Psi\left(\frac{c_1+d_1}{2}\right) \leq \frac{B(\delta)}{2^{s-1}\delta(d_1-c_1)}\left[\left({}_{c_1}^{\text{CF}}I^{\delta}\Psi\right)(l)+\left({}^{\text{CF}}I_{d_1}^{\delta}\Psi\right)(l)-\frac{2(1-\delta)}{B(\delta)}\Psi(l)\right] \leq \frac{\Psi(c_1)+\Psi(d_1)}{2^{s-1}(s+1)}. \tag{3.2}$$

Where $B(\delta) > 0$ is a normalization function satisfying $B(0) = B(1) = 1$.

Proof. First part of inequality (3.1) gives,

$$2^{s-1}\Psi\left(\frac{c_1+d_1}{2}\right) \leq \frac{1}{(d_1-c_1)}\int_{c_1}^{d_1}\Psi(u)\,du. \tag{3.3}$$

Multiplying both sides of (3.3) by $\frac{\delta(d_1-c_1)}{B(\delta)}$ and adding $\frac{2(1-\delta)}{B(\delta)}\Psi(l)$, we get

$$\begin{aligned}\frac{2(1-\delta)}{B(\delta)}\Psi(l)+\frac{2^{s-1}\delta(d_1-c_1)}{B(\delta)}\Psi\left(\frac{c_1+d_1}{2}\right)\\ \leq \frac{2(1-\delta)}{B(\delta)}\Psi(l)+\frac{\delta}{B(\delta)}\left(\int_{c_1}^{l}\Psi(u)\,du+\int_{l}^{d_1}\Psi(u)\,du\right)\\ =\left({}^{\mathrm{CF}}_{c_1}I^{\delta}\Psi\right)(l)+\left({}^{\mathrm{CF}}I^{\delta}_{d_1}\Psi\right)(l).\end{aligned}$$

Reorganizing terms we get,

$$\frac{2^{s-1}\delta(d_1-c_1)}{B(\delta)}\Psi\left(\frac{c_1+d_1}{2}\right)\leq\left({}^{\mathrm{CF}}_{c_1}I^{\delta}\Psi\right)(l)+\left({}^{\mathrm{CF}}I^{\delta}_{d_1}\Psi\right)(l)-\frac{2(1-\delta)}{B(\delta)}\Psi(l). \tag{3.4}$$

Multiplying $\frac{B(\delta)}{2^{s-1}\delta(d_1-c_1)}$ both sides of (3.4) proves first part of desired inequality. For second part, consider,

$$\frac{1}{2^{s-1}(d_1-c_1)}\int_{c_1}^{d_1}\Psi(u)\,du\leq\frac{\Psi(c_1)+\Psi(d_1)}{2^{s-1}(s+1)}. \tag{3.5}$$

Similarly, second inequality follows the exact operation of (3.3) in (3.5).

□

Theorem 3.3. *Let $\Psi,\Phi: I\subseteq\mathbb{R}\to\mathbb{R}$ be a s-convex function. If $\Psi\Phi\in L[c_1,d_1]$, then*

$$\begin{aligned}\frac{B(\delta)}{\delta(d_1-c_1)}\left[\left({}^{\mathrm{CF}}_{c_1}I^{\delta}\Psi\Phi\right)(l)+\left({}^{\mathrm{CF}}I^{\delta}_{d_1}\Psi\Phi\right)(l)-\frac{2(1-\delta)}{B(\delta)}\Psi(l)\Phi(l)\right]\\ \leq\frac{1}{2s+1}P(c_1,d_1)+\beta(s,s)Q(c_1,d_1).\end{aligned} \tag{3.6}$$

Where

$P(c_1,d_1)=\Psi(c_1)\Phi(c_1)+\Psi(d_1)\Phi(d_1)$,
$Q(c_1,d_1)=\Psi(c_1)\Phi(d_1)+\Psi(d_1)\Phi(c_1)$,
and $l\in[c_1,d_1]$, where $B(\delta)>0$ is a normalization function satisfying $B(0)=B(1)=1$.

Proof. Given that Ψ and Φ are s-convex function,

$$\Psi\big(c_1v+(1-v)d_1\big)\leq v^s\Psi(c_1)+(1-v)^s\Psi(d_1).\quad\forall\, v\in[0,1]\ \ and\ \ c_1,d_1\in I,$$

and

$$\Phi\big(c_1v+(1-v)d_1\big)\leq v^s\Phi(c_1)+(1-v)^s\Phi(d_1).\quad\forall\, v\in[0,1]\ \ and\ \ c_1,d_1\in I.$$

Multiplying above inequalities both sides gives,

$$\begin{aligned}&\Psi\big(c_1v+(1-v)d_1\big)\Phi\big(c_1v+(1-v)d_1\big)\\ &\leq v^{2s}\Psi(c_1)\Phi(c_1)+(1-v)^{2s}\Psi(d_1)\Phi(d_1)+v^s(1-v)^s\big[\Psi(c_1)\Phi(d_1)+\Psi(d_1)\Phi(c_1)\big].\end{aligned} \tag{3.7}$$

Integrating (3.7) with respect to v over [0,1] and making substitution $c_1v+(1-v)d_1=u$, we have

$$\begin{aligned}\frac{1}{d_1-c_1}\int_{c_1}^{d_1}\Psi(u)\Phi(u)\,du\leq\frac{1}{2s+1}\big[\Psi(c_1)\Phi(c_1)+\Psi(d_1)\Phi(d_1)\big]\\ +\beta(s,s)\big[\Psi(c_1)\Phi(d_1)+\Psi(d_1)\Phi(c_1)\big].\end{aligned}$$

Multiply both sides by $\frac{\delta(d_1-c_1)}{B(\delta)}$ and adding $\frac{2(1-\delta)}{B(\delta)}\Psi(l)\Phi(l)$ gives,

$$\frac{\delta}{B(\delta)}\left[\int_{c_1}^{l}\Psi(u)\Phi(u)\,du+\int_{l}^{d_1}\Psi(u)\Phi(u)\,du\right]+\frac{2(1-\delta)}{B(\delta)}\Psi(l)\Phi(l)$$
$$\leq\frac{\delta(d_1-c_1)}{B(\delta)}\left[\frac{1}{2s+1}P(c_1,d_1)+\beta(s,s)Q(c_1,d_1)\right]+\frac{2(1-\delta)}{B(\delta)}\Psi(l)\Phi(l).$$

$$\left({}^{\mathrm{CF}}_{c_1}I^{\delta}(\Psi\Phi)\right)(l)+\left({}^{\mathrm{CF}}I^{\delta}_{d_1}(\Psi\Phi)\right)(l)\leq\frac{\delta(d_1-c_1)}{B(\delta)}\left[\frac{1}{2s+1}P(c_1,d_1)+\beta(s,s)Q(c_1,d_1)\right]+\frac{2(1-\delta)}{B(\delta)}\Psi(l)\Phi(l).$$

By reorganizing terms, proof is completed. □

Theorem 3.4. *Let $\Psi,\Phi:I\subseteq\mathbb{R}\to\mathbb{R}$ be s-convex function. If $\Psi\Phi\in L[c_1,d_1]$, then*

$$2^{s-1}\Psi\left(\frac{c_1+d_1}{2}\right)\Phi\left(\frac{c_1+d_1}{2}\right)-\frac{B(\delta)}{\delta(d_1-c_1)}\left[\left({}^{\mathrm{CF}}_{c_1}I^{\delta}\Psi\Phi\right)(l)+\left({}^{\mathrm{CF}}I^{\delta}_{d_1}\Psi\Phi\right)(l)\right]+\frac{2(1-\delta)}{\delta(d_1-c_1)}\Psi(l)\Phi(l)\leq\ \beta(s,s)M(c_1,d_1)+\frac{1}{2s+1}N(c_1,d_1).\tag{3.8}$$

Where $P(c_1,d_1)$ and $Q(c_1,d_1)$ are given in Theorem 3.3, $l\in[c_1,d_1]$ and $B(\delta)>0$ is a normalization function satisfying $B(0)=B(1)=1$.

Proof. Given that the Ψ and Φ are s-convex function on $[c_1,d_1]$ and for $u=\frac{1}{2}$, we have

$$\Psi\left(\frac{c_1+d_1}{2}\right)\leq\left(\frac{1}{2}\right)^{s}\Big[\Psi((1-u)c_1+ud_1)+\Psi(uc_1+(1-u)d_1)\Big].\quad\forall\,c_1,d_1\in I,u\in[0,1].$$

and

$$\Phi\left(\frac{c_1+d_1}{2}\right)\leq\left(\frac{1}{2}\right)^{s}\Big[\Phi((1-u)c_1+ud_1)+\Phi(uc_1+(1-u)d_1)\Big].\quad\forall\,c_1,d_1\in I,u\in[0,1].$$

Multiplying above two inequalities,

$$\begin{aligned}&\Psi\left(\frac{c_1+d_1}{2}\right)\Phi\left(\frac{c_1+d_1}{2}\right)\\&\leq\left(\frac{1}{2}\right)^{2s}\big[\Psi\big((1-u)c_1+ud_1\big)\Phi\big((1-u)c_1+ud_1\big)+\Psi\big(uc_1+(1-u)d_1\big)\Phi\big(uc_1+(1-u)d_1\big)\\&\quad+\Psi\big((1-u)c_1+ud_1\big)\Phi\big(uc_1+(1-u)d_1\big)+\Psi\big(uc_1+(1-u)d_1\big)\Phi\big((1-u)c_1+ud_1\big)\big]\\&\leq\left(\frac{1}{2}\right)^{2s}\big[\Psi\big((1-u)c_1+ud_1\big)\Phi\big((1-u)c_1+ud_1\big)+\Psi\big(uc_1+(1-u)d_1\big)\Phi\big(uc_1+(1-u)d_1\big)\\&\quad+2\big\{u^{s}(1-u)^{2s}\big[\Psi(c_1)\Phi(c_1)+\Psi(d_1)\Phi(d_1)\big]+(1-u)^{2s}\Psi(c_1)\Phi(d_1)+u^{2s}\Psi(d_1)\Phi(c_1)\big\}\big].\end{aligned}\tag{3.9}$$

Integrating (3.9) with respect to u over [0,1] and making substitution, we have

$$\Psi\Big(\frac{c_1+d_1}{2}\Big)\Phi\Big(\frac{c_1+d_1}{2}\Big)$$
$$\le \Big(\frac{1}{2}\Big)^s\Big[\frac{2}{(d_1-c_1)}\int_{c_1}^{d_1}\Psi(u)\Phi(u)\,du+2\beta(s,s)\big[\Psi(c_1)\Phi(c_1)+\Psi(d_1)\Phi(d_1)\big]$$
$$+\frac{2}{2s+1}\big[\Psi(c_1)\Phi(d_1)+\Psi(d_1)\Phi(c_1)\big]\Big].$$

Thus

$$2^{s-1}\Psi\Big(\frac{c_1+d_1}{2}\Big)\Phi\Big(\frac{c_1+d_1}{2}\Big)\le\frac{1}{(d_1-c_1)}\int_{c_1}^{d_1}\Psi(u)\Phi(u)\,du$$
$$+\beta(s,s)P(c_1,d_1)+\frac{1}{2s+1}Q(c_1,d_1). \tag{3.10}$$

Multiplying both sides with $\frac{\delta(d_1-c_1)}{B(\delta)}$ and subtracting $\frac{2(1-\delta)}{B(\delta)}\Psi(l)\Phi(l)$

$$\frac{2^{s-1}\delta(d_1-c_1)}{B(\delta)}\Psi\Big(\frac{c_1+d_1}{2}\Big)\Phi\Big(\frac{c_1+d_1}{2}\Big)-\frac{\delta}{B(\delta)}\Big[\int_{c_1}^{l}\Psi(u)\Phi(u)\,du+\int_{l}^{d_1}\Psi(u)\Phi(u)\,du\Big]$$
$$-\frac{2(1-\delta)}{B(\delta)}\Psi(l)\Phi(l)\le\frac{\delta(d_1-c_1)}{B(\delta)}\Big[\beta(s,s)P(c_1,d_1)+\frac{1}{2s+1}Q(c_1,d_1)\Big]-\frac{2(1-\delta)}{B(\delta)}\Psi(l)\Phi(l). \tag{3.11}$$

which leads to

$$\frac{2^{s-1}\delta(d_1-c_1)}{B(\delta)}\Psi\Big(\frac{c_1+d_1}{2}\Big)\Phi\Big(\frac{c_1+d_1}{2}\Big)-\Big(\big({}^{\mathrm{CF}}_{c_1}I^{\delta}(\Psi\Phi)\big)(l)+\big({}^{\mathrm{CF}}I^{\delta}_{d_1}(\Psi\Phi)\big)(l)\Big)$$
$$\le\frac{\delta(d_1-c_1)}{B(\delta)}\Big[\beta(s,s)P(c_1,d_1)+\frac{1}{2s+1}Q(c_1,d_1)\Big]-\frac{2(1-\delta)}{B(\delta)}\Psi(l)\Phi(l). \tag{3.12}$$

Multipying both sides by $\frac{B(\delta)}{\delta(d_1-c_1)}$ and rearranging terms, proof is completed.

□

Gürbüz et al. [5] proved the following lemma in context of Caputo-Fabrizio fractional operator.

Lemma 3.5. *Let a function $\Psi:[c_1,d_1]\subseteq\mathbb{R}\to\mathbb{R}$ be a differntiable function on I°, $c_1,d_1\in I$ with $c_1<d_1$. If $\Psi'\in L_1[c_1,d_1]$, $\delta\in[0,1]$ and $l\in[c_1,d_1]$, then*

$$\frac{d_1-c_1}{2}\int_0^1(1-2u)\Psi'\big(uc_1+(1-u)d_1\big)\,du-\frac{2(1-\delta)}{\delta(d_1-c_1)}\Psi(l)$$
$$=\frac{\Psi(c_1)+\Psi(d_1)}{2}-\frac{B(\delta)}{\delta(d_1-c_1)}\big[\big({}^{\mathrm{CF}}_{c_1}I^{\delta}\Psi\big)(l)+\big({}^{\mathrm{CF}}I^{\delta}_{d_1}\Psi\big)(l)\big]. \tag{3.13}$$

Where $B(\delta)>0$ is a normalization function satisfying $B(0)=B(1)=1$.

Theorem 3.6. *Let a function $\Psi:I\subseteq\mathbb{R}\to\mathbb{R}$ be a differntiable positive function on I° and $|\Psi'|$ be s-convex function on $[c_1,d_1]$, where $c_1,d_1\in I$ with $c_1<d_1$. If $\Psi'\in L_1[c_1,d_1]$, $\delta\in[0,1]$ and $l\in[c_1,d_1]$, then*

$$\Big|\frac{\Psi(c_1)+\Psi(d_1)}{2}+\frac{2(1-\delta)}{\delta(d_1-c_1)}\Psi(l)-\frac{B(\delta)}{\delta(d_1-c_1)}\big[\big({}^{\mathrm{CF}}_{c_1}I^{\delta}\Psi\big)(l)+\big({}^{\mathrm{CF}}I^{\delta}_{d_1}\Psi\big)(l)\big]\Big|$$
$$\le\frac{(d_1-c_1)(|\Psi'(c_1)|+|\Psi'(d_1)|)(2^s.s+1)}{2^{s+1}(s+1)(s+2)}. \tag{3.14}$$

Where $B(\delta) > 0$ *is a normalization function satisfying* $B(0) = B(1) = 1$.

Proof. Application of Lemma 3.5, the s-convexity of $\left|\Psi'\right|$ and the properties of the absolute value gives,

$$\begin{aligned}
&\left|\frac{\Psi(c_1)+\Psi(d_1)}{2}+\frac{2(1-\delta)}{\delta(d_1-c_1)}\Psi(l)-\frac{B(\delta)}{\delta(d_1-c_1)}\left[\left({}^{\mathrm{CF}}_{c_1}I^{\delta}\Psi\right)(l)+\left({}^{\mathrm{CF}}I^{\delta}_{d_1}\Psi\right)(l)\right]\right|\\
&\le \frac{d_1-c_1}{2}\int_0^1 |1-2u|\left|\Psi'\left(uc_1+(1-u)d_1\right)\right| du\\
&\le \frac{d_1-c_1}{2}\int_0^1 |1-2u|\left(u^s\left|\Psi'(c_1)\right|+(1-u)^s\left|\Psi'(d_1)\right|\right) du\\
&= \frac{d_1-c_1}{2}\left(\int_0^1 u^s|(1-2u)|\left|\Psi'(c_1)\right| du+\int_0^1 (1-u)^s|1-2u|\left|\Psi'(d_1)\right|\right) du\Bigg)\\
&= \frac{(d_1-c_1)(|\Psi'(c_1)|+|\Psi'(d_1)|)(2^s.s+1)}{2^{s+1}(s+1)(s+2)}.
\end{aligned}$$

□

Theorem 3.7. *Let a function* $\Psi : [c_1, d_1] \subseteq \mathbb{R} \to \mathbb{R}$ *be a differntiable on* I° *and* $\left|\Psi'\right|^q$ *be s-convex function on* $[c_1, d_1]$*, where* $c_1, d_1 \in I$ *with* $c_1 < d_1$, $p > 1$ *and* $\frac{1}{p}+\frac{1}{q}=1$. *If* $\Psi' \in L_1[c_1, d_1]$, $\delta \in [0, 1]$ *and* $l \in [c_1, d_1]$*, then*

$$\begin{aligned}
&\left|\frac{\Psi(c_1)+\Psi(d_1)}{2}+\frac{2(1-\delta)}{\delta(d_1-c_1)}\Psi(l)-\frac{B(\delta)}{\delta(d_1-c_1)}\left[\left({}^{\mathrm{CF}}_{c_1}I^{\delta}\Psi\right)(l)+\left({}^{\mathrm{CF}}I^{\delta}_{d_1}\Psi\right)(l)\right]\right|\\
&\le \frac{d_1-c_1}{2}\left(\frac{1}{p+1}\right)^{\frac{1}{p}}\left(\frac{|\Psi'(c_1)|^q+|\Psi'(d_1)|^q}{s+1}\right)^{\frac{1}{q}}. \qquad (3.15)
\end{aligned}$$

Where $B(\delta) > 0$ *is a normalization function satisfying* $B(0) = B(1) = 1$.

Proof. Application of Lemma 3.5, s-convexity of $|\Psi'|^q$ and the Hölder inequality gives,

$$\begin{aligned}
&\left|\frac{\Psi(c_1)+\Psi(d_1)}{2}+\frac{2(1-\delta)}{\delta(d_1-c_1)}\Psi(l)-\frac{B(\delta)}{\delta(d_1-c_1)}\left[\left({}^{\mathrm{CF}}_{c_1}I^{\delta}\Psi\right)(l)+\left({}^{\mathrm{CF}}I^{\delta}_{d_1}\Psi\right)(l)\right]\right|\\
&\le \frac{d_1-c_1}{2}\int_0^1 |1-2u|\left|\Psi'\left(uc_1+(1-u)d_1\right)\right| du\\
&\le \frac{d_1-c_1}{2}\left(\int_0^1 |1-2u|^p du\right)^{\frac{1}{p}}\left(\int_0^1 \left|\Psi'\left(uc_1+(1-u)d_1\right)\right|^q du\right)^{\frac{1}{q}}\\
&= \frac{d_1-c_1}{2}\left(\int_0^1 |1-2u|^p du\right)^{\frac{1}{p}}\left(\int_0^1 \left(u^s\left|\Psi'(c_1)\right|^q+(1-u)^s\left|\Psi'(d_1)\right|^q\right) du\right)^{\frac{1}{q}}\\
&= \frac{d_1-c_1}{2}\left(\frac{1}{p+1}\right)^{\frac{1}{p}}\left(\frac{\left|\Psi'(c_1)\right|^q+\left|\Psi'(d_1)\right|^q}{s+1}\right)^{\frac{1}{q}}.
\end{aligned}$$

Which completes the proof. □

Theorem 3.8. *Let* $\Psi : [c_1, d_1] \subseteq \mathbb{R} \to \mathbb{R}$ *be a differntiable positive function on* I° *and* $\left|\Psi'\right|^q$ *be s-convex function on* $[c_1, d_1]$*, where* $c_1, d_1 \in I$ *with* $c_1 < d_1$. *If* $\Psi' \in L_1[c_1, d_1]$, $\delta \in [0, 1]$ *and* $l \in [c_1, d_1]$*, then*

$$\left|\frac{\Psi(c_1)+\Psi(d_1)}{2}+\frac{2(1-\delta)}{\delta(d_1-c_1)}\Psi(l)-\frac{B(\delta)}{\delta(d_1-c_1)}\left[\left({}^{\mathrm{CF}}_{c_1}I^{\delta}\Psi\right)(l)+\left({}^{\mathrm{CF}}I^{\delta}_{d_1}\Psi\right)(l)\right]\right|$$
$$\leq \frac{d_1-c_1}{2^{\frac{p+1}{p}}}\left(\frac{(2^s.s+1)}{2^{s+1}(s+1)(s+2)}\right)^{\frac{1}{q}}\left(|\Psi'(c_1)|^q+|\Psi'(d_1)|^q\right)^{\frac{1}{q}}. \tag{3.16}$$

Where $B(\delta)>0$ is a normalization function satisfying $B(0)=B(1)=1$.

Proof. Application of Lemma 3.5, s-convexity of $\left|\Psi'\right|^q$ and Hölder's inequality gives,

$$\left|\frac{\Psi(c_1)+\Psi(d_1)}{2}+\frac{2(1-\delta)}{\delta(d_1-c_1)}\Psi(l)-\frac{B(\delta)}{\delta(d_1-c_1)}\left[\left({}^{\mathrm{CF}}_{c_1}I^{\delta}\Psi\right)(l)+\left({}^{\mathrm{CF}}I^{\delta}_{d_1}\Psi\right)(l)\right]\right|$$
$$\leq \frac{d_1-c_1}{2}\int_0^1|1-2u|\left|\Psi'\big(uc_1+(1-u)d_1\big)\right|du$$
$$\leq \frac{d_1-c_1}{2}\left(\int_0^1|1-2u|\,du\right)^{\frac{1}{p}}\left(\int_0^1|1-2u|\left|\Psi'\big(uc_1+(1-u)d_1\big)\right|^q du\right)^{\frac{1}{q}}$$
$$\leq \frac{d_1-c_1}{2}\left(\int_0^1|1-2u|\,du\right)^{\frac{1}{p}}\left(\int_0^1\left(u^s|1-2u|\left|\Psi(c_1)'\right|^q+(1-u)^s|1-2u|\left|\Psi(d_1)'\right|^q\right)du\right)^{\frac{1}{q}}$$
$$=\frac{d_1-c_1}{2}\left(\int_0^1|1-2u|\,du\right)^{\frac{1}{p}}\left(\left|\Psi(c_1)'\right|^q\int_0^1u^s|1-2u|\,du+\left|\Psi(d_1)'\right|^q\int_0^1(1-u)^s|1-2u|\,du\right)^{\frac{1}{q}}$$
$$=\frac{d_1-c_1}{2^{\frac{p+1}{p}}}\left(\frac{(2^s.s+1)}{2^{s+1}(s+1)(s+2)}\right)^{\frac{1}{q}}\left(|\Psi'(c_1)|^q+|\Psi'(d_1)|^q\right)^{\frac{1}{q}}.$$

□

4 Applications to Special Means

(i) Arithmetic mean :

$$\mathcal{A}=\mathcal{A}(c_1,d_1)=\frac{c_1+d_1}{2}. \qquad Where \quad c_1,d_1\in\mathbb{R}.$$

(ii) The generalized logarithmic mean :

$$\mathcal{L}=\mathcal{L}_l^l(c_1,d_1)=\frac{d_1^{l+1}-c_1^{l+1}}{(l+1)(d_1-c_1)}. \qquad Where \quad l\in\mathbb{R}\setminus\{-1,0\},\ c_1,d_1\in\mathbb{R},\ c_1\neq d_1.$$

Proposition 4.1. *Let* $c_1,d_1\in\mathbb{R}^+$ $c_1<d_1$, *then*

$$\left|\mathcal{A}\big(c_1^2,d_1^2\big)-\mathcal{L}_2^2(c_1,d_1)\right|\leq\frac{(d_1-c_1)(2^s.s+1)}{2^s(s+1)(s+2)}\big[|c_1|+|d_1|\big].$$

Proof. In Theorem 3.6, set $\Psi(u)=u^2$ with $\delta=1$ and $B(\delta)=B(1)=1$, then proof is obvious. □

Proposition 4.2. *Let* $c_1,d_1\in\mathbb{R}^+$, $c_1<d_1$, *then*

$$\left|\mathcal{A}\big(e^{c_1},e^{d_1}\big)-\mathcal{L}\big(e^{c_1},e^{d_1}\big)\right|\leq\frac{(d_1-c_1)(2^s.s+1)}{2^{s+1}(s+1)(s+2)}\big(e^{c_1}+e^{d_1}\big).$$

Proof. In Theorem 3.6, set $\Psi(u)=e^u$ with $\delta=1$ and $B(\delta)=B(1)=1$, then proof is obvious. □

Proposition 4.3. *Let* $c_1, d_1 \in \mathbb{R}^+,\ c_1 < d_1$, *then*

$$\left|\mathcal{A}\left(c_1^n, d_1^n\right) - \mathcal{L}_n^n(c_1, d_1)\right| \leq \frac{n(d_1 - c_1)(2^s.s + 1)}{2^{s+1}(s+1)(s+2)}\left(\left|c_1^{n-1}\right| + \left|d_1^{n-1}\right|\right).$$

Proof. In Theorem 3.6, set $\Psi(u) = u^n$ with $\delta = 1$ and $B(\delta) = B(1) = 1$, then proof is obvious. □

5 Conclusion

In this paper, we have derived some Hermite-Hadamard inequality for s-convex function using Caputo-Fabrizio fractional integral. Also, the inequalities involving the product of two s-convex functions have been derived. By using the result derived by Gürbüz et al.[5], we have derived several inequalities involving Caputo-Fabrizio fractional integral using s-convex function. Lastly, the comparison of arithmetic mean and the generalized logarithmic mean has been derived using the results obtained in this paper. If we replace $s = 1$ in Theorem 3.2, we obtain the Hermite-Hadamard inequality for convex function using Caputo-Fabrizio fractional integral. So results obtained in this manuscript are more generalized than some previously published results.

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