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S-SEMIPRIME IDEALS AND WEAKLY S-SEMIPRIME IDEALS OF RINGS

Kamal Krishna Pathak and Jituparna Goswami

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Abstract Let R be a ring with identity and $S \subseteq R$ be a multiplicative closed subset. Hamed and Malek [10] introduced the concept of S-prime ideal of R which is a generalization of prime ideals. An ideal P of R disjoint with S is called S-prime ideal of R if there exists an $s \in S$ such that for all $a, b \in R$ if $ab \in P$, then $sa \in P$ or $sb \in P$. In this paper we introduce the concept of S-semiprime ideal and weakly S-semiprime ideal as generalizations of semiprime ideals. We show that S-semiprime ideals and weakly S-semiprime ideals enjoy analogs of many fundamental properties of semiprime ideals and we study their characterizations in the ring R.

1 Introduction

Throughout this paper R is considered to be a ring with unity $1(\neq 0)$. Let P be a proper ideal of ring R, then the ideal P is said prime ideal, if for $a, b \in R$ and $ab \in P$ implies either $a \in P$ or $b \in P$. The prime ideals play a very important role in the commutative ring theory. The notion of prime ideals is used to characterize certain classes of rings. For years, many researchers have shown immense interests on this issue and many generalizations of the same have been given by a lot of researchers. D. D. Anderson et al. [2] defined the weakly prime ideals as a generalization of prime ideals. A proper ideal P of ring R is called weakly prime if for all $a, b \in R$ with $0 \neq ab \in P$ implies either $a \in P$ or $b \in P$. Thus all prime ideal of a ring R are weakly prime ideal, the converse in general is not true. For example $< 0 >$ is always weakly prime ideal of R, and it is prime if and only if the given ring is an integral domain. Recently, various generalizations of prime ideals are studied in [3, 4, 7, 9]. Recall that a proper ideal P of R is a semiprime ideal if for $a \in R$, $a^2 \in P$ implies $a \in P$. A. Badawi [5] defined an ideal P as a weakly semiprime if for all $a \in R$ with $0 \neq a^2 \in P$ implies that $a \in P$. For example, all proper ideals of a quasilocal ring (R, M) with nilpotent module M with index 2 are weakly semiprime ideal. All the weakly prime ideals of a ring are weakly semiprime, but the converse in general is not true. Also all semiprime ideals of a ring are weakly semiprime, however the converse in general is not true. For example, the ideal $P = \{0, 8\}$ of $\mathbb{Z}_{16}$ is weakly semiprime which is not semiprime. Consider a nonempty subset S of R. We call S a multiplicative closed subset (briefly, m.c.s.) of R if $ss' \in S$ for all $s, s' \in S$. The concept of S-prime ideal was introduced by Hamed and Malek [10] which is also a generalization of prime ideals. A proper ideal P of R with a multiplicative closed subset S disjoint with P is called S-prime ideal if there exists an $s \in S$ such that for all $a, b \in R$ with $ab \in P$, we have $sa \in P$ or $sb \in P$. Every prime ideal of a ring disjoint with a multiplicative closed subset S is S-prime ideal, however the converse is not true in general [10, Example 1]. Later on many authors have extended the study of S-prime ideals of rings, for instance, see [1, 12, 14].

The main purpose of this paper is to introduce the notion of S-semiprime ideals and weakly S-semiprime ideals of rings which are generalizations of semiprime ideals of a ring and investigate their characterizations in the ring. A proper ideal P of R is said to be S-semiprime where S is multiplicative closed subset of R disjoint with P, if there exists an $s \in S$ such that for all $a \in R$ with $a^2 \in P$, we have $sa \in P$. All semiprime ideals disjoint with multiplicative closed set S are S-semiprime but the converse is not true in general, see Example 2.3. In the first section we study the basic properties of S-semiprime ideals. Among many results, it has been shown that $(P : s)$ is semiprime ideal of ring R implies that P is S-semiprime of ring R. However the converse is true only if P is 2-absorbing ideal of R. A nonzero proper ideal I of ring R is called 2-absorbing ideal of R if whenever a, b and $c \in R$ with $abc \in I$, then $ab \in I$ or $ac \in I$ or $bc \in I$ [6]. An element a of a commutative ring R is called a zero-divisor of R if $ab = 0$ for some non-zero element b of R. A nonzero element of a commutative ring R is called a regular element if it is not a zero divisor of R. It is proved that when S consists of regular elements, then 2-absorbing ideal P of R is S-semiprime ideal if and only if $(P : s)$ is semiprime ideal of ring R which implies $S^{-1}P$ is a semiprime ideal of $S^{-1}R$ and $(P : s) \subseteq S^{-1}P \cap R$. Also, we have the famous result of McCoy [13] known by the prime avoidance lemma which states that for a commutative ring R, an ideal I of R and for prime ideals $P_1, ..., P_n$ of R, if $I \subseteq \bigcup_{i=1}^{n} P_i$, then $I \subseteq P_i$ for some $i \in \{1, ..., n\}$. Using this prime avoidance lemma, we have established a characterization of S-semiprime ideals of ring R. Following the concepts of $m - system$ and $n - system$ [11], the concept of $S - m - system$ and $S - n - system$ has been introduced in this section to establish characterization of S-prime and S-semiprime ideals of ring R.

In the next section we study the weakly S-semiprime ideal of a ring R. Suppose the ring R is commutative, S a multiplicative closed subset of R and P an ideal of R disjoint with S. We say P is weakly S-semiprime ideal of R if there exists an $s \in S$ such that for all $a \in R$ with $0 \neq a^2 \in P$, we have $sa \in P$. All weakly semiprime ideal are weakly S-semiprime and all S-semiprime ideal are weakly S-semiprime but the converse are not true in general. In this section, it is shown that if P is a weakly S-semiprime ideal which is not S-semiprime, then $P^2 = 0$. Among many other results, it is also shown that the zero ideal is the only weakly S-semiprime ideal of ring R if and only if the ring R is reduced and $S^{-1}R$ is a field. We conclude this section with a study of the preservation of weakly S-semiprime ideals under ring homomorphism.

For any undefined terminology of ring theory mentioned in this paper, we refer to [11, 15].

2 S-Semiprime Ideals

Let us consider R to be a commutative ring with unity and S be a multiplicative closed subset of R. In this section we introduce the notion of the S-semiprime ideals of R and study their characterization in R. We begin with the following definition.

Definition 2.1. Let R be a commutative ring, S be a multiplicative closed subset of R and P an ideal of R disjoint with S. We say P is S-semiprime ideal of R if there exists an $s \in S$ such that for all $a \in R$ with $a^2 \in P$, we have $sa \in P$.

Example 2.2. (1) Let R be a ring and let S be the set of all units of R, then S is a multiplicative closed subset of R. All the prime ideals of R are S-semiprime ideals of R.

(2) Let $R = \mathbb{Z}[X]$, $P = 4X\mathbb{Z}[X]$ and $S = \{2^n \mid n \in \mathbb{N}\}$. Clearly, P is an ideal of R and S is a multiplicative closed subset of R such that $P \cap S = \phi$. If $f \in R$ such that $f^2 \in P$ then $4X$ divides f which yields $4f \in P$. Hence, P is an S-semiprime ideal in R.

(3) Let us consider the ring $R = \mathbb{Z}$ and the multiplicative closed set $S = \{3^n \mid n \in \mathbb{N}\}$ of R. Then $P =< 6 >$ is S-semiprime ideal.

All semiprime ideal of R are S-semiprime. However, the converse is not true in genearal which can be illustrated by the following example.

Example 2.3. Let $R = \mathbb{Z}[X]$, $P = 4X\mathbb{Z}[X]$ and $S = \{2^n \mid n \in \mathbb{N}\}$. Then we have shown in the Example 2.2(2) that P is an S-semiprime ideal. Now we have $4X^2 = (2X)^2 \in P$ but $2X \notin P$, which implies that $4X\mathbb{Z}[X]$ is not semiprime ideal in $\mathbb{Z}[X]$.

Proposition 2.4. *Let R be a commutative ring, S be a multiplicative closed subset of R consisting of regular elements and P an 2-absorbing ideal of R disjoint with S. Then P is an S-semiprime ideal if and only if $(P : s)$ is semiprime ideal.*

Proof. Suppose P is S-semiprime ideal of R. Therefore there exists $s \in S$ such that, whenever $a^2 \in P$ implies $sa \in P$. We claim that $(P : s)$ is semiprime ideal i.e whenever $a^2 \in (P : s)$ implies $a \in (P : s)$. Let $a^2 \in (P : s)$, which implies $a^2 s = sa^2 \in P$. Since P is an ideal and S consists of regular elements, we can have $s.sa^2 = (sa)^2 \in P$. P is S-semiprime ideal, therefore $s.sa \in P$. As P is $2 - absorbing$, therefore we have $s.s \in P$ or $s.a \in P$ or $s.a \in P$. Since $P \cap S = \phi$, hence $s^2 \notin P$. Consequently $sa \in P$. Therefore $a \in (P : s)$ which implies $(P : s)$ is semiprime ideal. Conversely, suppose $(P : s)$ is semiprime ideal. We claim that P is S-semiprime. Let $a^2 \in P$ and since $s \in S \subset R$, implies $sa^2 \in P$ (since P is an ideal in R). Therefore $a^2 \in (P : s)$ implies $a \in (P : s)$ (since $(P : s)$ is semiprime). Consequently $sa \in P$. Hence P is S-semiprime in R. □

Proposition 2.5. *Let R be a commutative ring, S be a multiplicative closed subset of R consisting of regular elements and P be an 2-absorbing ideal of R disjoint with S. Consider the following conditions:*

(1) P is a S-semiprime ideal of R.

(2) $(P : s)$ is a semiprime ideal of R.

(3) $S^{-1}P$ is a semiprime ideal of $S^{-1}R$.

(4) $(P : s) \subseteq S^{-1}P \cap R$ for some $s \in S$.

Then we have the sequence of implications $(1) \Leftrightarrow (2) \Rightarrow (3) \Rightarrow (4)$

Proof. $(1) \Leftrightarrow (2)$ The proof is from the Proposition 2.4.

$(2) \Rightarrow (3)$ Suppose $(P : s)$ is a semiprime ideal of R. Which implies P is a S-semiprime ideal in R. There exists $s \in S$ such that, whenever $a^2 \in P$ implies $sa \in P$. We claim that $S^{-1}P$ is a semiprime ideal of $S^{-1}R$, that is, whenever $a^2 \in S^{-1}P$ implies $a \in S^{-1}P$. Let $\frac{a}{s} \in S^{-1}R$ such that $(\frac{a}{s})^2 \in S^{-1}P$. Then $\frac{a^2}{s^2} = \frac{p'}{s'}$ for some $p' \in P$ and $s' \in S$. Thus $a^2 s' = s^2 p' \in P$. Since P is an ideal, $a^2 s'^2 \in P$. Implies $(as')^2 \in P$. Since P is S-semiprime ideal, there exist $s \in S$ such that $sas' \in P$. This implies $\frac{a}{s} = \frac{ass'}{sss'} \in S^{-1}P$. Therefore $(\frac{a}{s})^2 \in S^{-1}P$. Implies $\frac{a}{s} \in S^{-1}P$. Hence $S^{-1}P$ is semiprime ideal in $S^{-1}R$.

$(3) \Rightarrow (4)$ Let $\alpha \in (P : s)$, which implies $\alpha s \in P$. Since $s \in S$, $\alpha = \frac{\alpha s}{s} \in S^{-1}P$. Therefore $(P : s) \subseteq S^{-1}P \cap R$ for some $s \in S$. □

Proposition 2.6. *Let R be a commutative ring with identity, S be a multiplicative closed subset and P an ideal of R disjoint with S. Then P is S-semiprime if and only if there exist $s \in S$, such that for all ideal I of R, if $I^2 \subseteq P$ then $sI \subseteq P$.*

Proof. Let there exists $s \in S$ such that for all ideals I of R, whenever $I^2 \subseteq P$ implies $sI \subseteq P$. We claim that P is S-semiprime ideal. Let $a \in R$ such that $a^2 \in P$, so $(aR)^2 \subseteq P$. Thus $s(aR) \subseteq P$, so $sa \in P$. Hence P is S-semiprime ideal. Conversely, suppose P is S-semiprime ideal. We suppose that for all $t \in S$, there exists ideal I_t of R with $(I_t)^2 \subseteq P$ but $tI_t \nsubseteq P$. Now since $s \in S$, there exist ideal I_s of R with $(I_s)^2 \subseteq P$ but $sI_s \nsubseteq P$. Therefore there exists $a_s \in I_s$ such that $sa_s \notin P$ with $a_s^2 \in P$. Which is a contradiction to the fact that P is S-semiprime. Therefore there exists $s \in S$ such that for all ideal I of R, if $I^2 \subseteq P$ then $sI \subseteq P$. □

Proposition 2.7. *Let R be a commutative ring with identity, S be a multiplicative closed subset of R and I an ideal of R disjoint with S. Let P be a proper ideal of R containing I such that $P/I \cap \bar{S} = \phi$. Then P is S-semiprime if and only if P/I is an $\bar{S}$-semiprime ideal of R/I.*

Proof. Suppose that P is S-semiprime ideal of R. By definition there exists $s \in S$ such that, whenever $a^2 \in P$ implies $sa \in P$. We claim that P/I is S-semiprime ideal of R/I. Let $\bar{a} \in R/I$ such that $(\bar{a})^2 \in P/I$. Then $\overline{a.a} \in P/I \implies \bar{a^2} \in P/I$. So $a^2 \in P$, thus $sa \in P$ (since P is S-semiprime ideal). Therefore $\overline{sa} \in P/I \implies \bar{s}\bar{a} \in P/I$ for $\bar{s} \in \bar{S}$. Conversely, Let us suppose that $P \cap S \neq \phi$, then there exists $s \in P \cap S$. Which implies $\bar{s} \in P/I \cap \bar{S}$. Which is a contradiction to the fact that $P/I \cap \bar{S} = \phi$. So $P \cap S = \phi$. Let $a \in R$ such that $a^2 \in P$. Now since P/I is $\bar{S}$-semiprime ideal of R/I, so there exists $s \in S$ such that for all $\bar{a} \in R/I$ with $(\bar{a})^2 \in P/I$ we have $\bar{s}\bar{a} \in P/I$. Thus $\overline{sa} \in P/I$ implies $sa \in P$. Hence for $a \in R$ with $a^2 \in P$ we have $sa \in P$. Which implies P is S-semiprime ideal. □

Proposition 2.8. *Let R be a commutative ring, S be a multiplicative closed subset of R and P an ideal of R disjoint with S.*

(1) Let J be an ideal of ring R such that J and S are not disjoint. If P is S-semiprime ideal then JP is S-semiprime ideal.

(2) Let $R \subseteq T$ be an extension of commutative rings. If Q is an S-semiprime ideal of T, then $Q \cap S$ is an S-semiprime ideal of R.

(3) Let $f : R \to T$ be a homomorphism of commutative rings such that $f(S)$ doesnot contain zero. If Q is $f(S)$-semiprime ideal of T, then $f^{-1}(Q)$ is S-semiprime ideal of R.

Proof. (1) Let P is S-semiprime ideal of R. So there exists $s \in S$ such that whenever $a \in R$ and $a^2 \in P$ implies $sa \in P$. J is an ideal of R such that $J \cap S \neq \phi$, suppose $t \in J \cap S$. Let $a \in R$ such that $a^2 \in JP \subseteq P$. Therefore $a^2 \in P$. So there exists $s \in S$ such that $sa \in P$ (since P is S-semiprime). Thus $tsa \in JP$ where $ts \in S$. Hence $a^2 \in JP$ implies $s'a \in JP$ ($ts = s' \in S$). Hence JP is S-semiprime ideal of R.

(2) Q is an S-semiprime ideal of T. We claim that $Q \cap R$ is an S-semiprime ideal of R. Let $a \in R$ such that $a^2 \in Q \cap R$. This implies $a^2 \in Q$. Since Q is S-semiprime ideal of R, therefore there exists $s \in S$ such that $sa \in Q$. And since $sa \in R$, hence $sa \in Q \cap R$. Thus $a^2 \in Q \cap R$ implies $sa \in Q \cap R$. Hence $Q \cap R$ is S-semiprime ideal of R.

(3) Let Q is an $f(S)$-semiprime ideal of T such that $f(S)$ does not contain zero. So $f(S)$ is multiplicative closed set of T. Let us suppose $P = f^{-1}(Q)$. Let us suppose $P \cap S \neq \phi$. Then $t \in P \cap S \implies t \in f^{-1}(Q) \cap S$. So $f(t) \in Q \cap f(S)$ which is a contradiction, since $Q \cap f(S) = \phi$ as Q is $f(S)$-semiprime ideal of T. Thus $P \cap S = \phi$. Let $a \in R$ such that $a^2 \in P = f^{-1}(Q)$. Then $f(a^2) \in Q$. Since f is a homomorphism, so $f(a)f(a) \in Q$. Which implies that $f(s)f(a) \in Q$ for some $s \in S$. So $f(sa) \in Q$, thus $sa \in f^{-1}(Q) = P$. Hence $f^{-1}(Q)$ is S-semiprime ideal of R. □

Definition 2.9. Let S be a multiplicative closed subset of a ring R not containing 0. Then R is said to be a S-prime ring if $< 0 >$ is a S-prime ideal of ring R.

Proposition 2.10. *Let R be a ring and S be a multiplicative closed subset of R. Then R is a S-prime ring if and only if for any $a, b \in R$ and $s \in S$, $ab = 0$ implies $sa = 0$ or $sb = 0$.*

Proof. Let us suppose that R is a S-prime ring. Then $< 0 >$ is a S-prime ideal. Let $a, b \in R$ such that $ab = 0 \in 0$. Then for $s \in S$, we have $sa \in < 0 >$ or $sb \in < 0 >$ as $< 0 >$ is S-prime ideal. Therefore, $sa = 0$ or $sb = 0$. Conversely, let $a, b \in R$ and $s \in S$ such that $ab = 0$ implies $sa = 0$ or $sb = 0$. Let $ab \in < 0 >$. Then $ab = 0$ which implies $sa = 0$ *or* $sb = 0$. Thus $sa \in < 0 >$ or $sb \in < 0 >$. Therefore, $< 0 >$ is S-prime ideal. Hence, R is a S-prime ring. □

Definition 2.11. Let S be a multiplicative closed subset of a ring R not containing 0. Then R is said to be a S-semiprime ring if $< 0 >$ is a S-semiprime ideal of ring R.

Proposition 2.12. *Let R be a ring and S be a multiplicative closed subset of R. Then R is a S-semiprime ring if and only if for any $a \in R$ and $s \in S$, $a^2 = 0$ implies $sa = 0$.*

Proof. Let us assume that R is a S-semiprime ring. Then $< 0 >$ is a S-semiprime ideal. Let $a \in R$ such that $a^2 = 0 \in < 0 >$. Then for $s \in S$, we have $sa \in < 0 >$ as $< 0 >$ is S-prime ideal. Therefore, $sa = 0$. Conversely, let $a \in R$ and $s \in S$ such that $a^2 = 0$ implies $sa = 0$. Let $a^2 \in < 0 >$. Then $a^2 = 0$ which implies $sa = 0$. Thus $sa \in < 0 >$. Therefore $< 0 >$ is S-semiprime ideal. Hence, R is a S-semiprime ring. □

Proposition 2.13. *Let R be a commutative ring and $S \subseteq R$ is a multiplicative closed set. Let I be an ideal of R and let $P_1, P_2, P_3, ..., P_n$ be S-semiprime ideals of R. If $I \subseteq \bigcup_{i=1}^n P_i$, then there exist $s \in S$ and $j = \{1, 2, 3, ..., n\}$ such that $sI \subseteq P_j$.*

Proof. Let $P_1, P_2, P_3, ..., P_n$ be S-semiprime ideals of R. And suppose that $I \subseteq \bigcup_{i=1}^n P_i$. By Proposition 2.4, for all $i = \{1, 2, 3, ..., n\}$ there exists $s \in S$ such that $(P_i : s_i)$ is a semiprime ideal of R. And we have $I \subseteq \bigcup_{i=1}^n P_i \subseteq \bigcup_{i=1}^n (P_i : s_i)$. Using the prime avoidance lemma, there exists $j \in \{1, 2, 3, ..., n\}$ such that $I \subseteq (P_j : s_j)$; which implies $s_j I \subseteq P_j$. □

Definition 2.14. [11] Let R be a ring. A nonempty set $S \subseteq R$ is said to be an $m - system$ if, for any $a, b \in S$, there exists $r \in R$ such that $arb \in S$.

Definition 2.15. [11] Let R be a ring. A nonempty set $S \subseteq R$ is called an $n - system$ if, for any $a \in S$, there exists $r \in R$ such that $ara \in S$.

Now following the definitions of $m - system$ and $n - system$, next we introduce a characterization for S-prime ideals and S-semiprime ideals. For that we establish the notion of $S - m - system$ and $S - n - system$.

Definition 2.16. Let R be a ring. A nonempty subset M of R containing a multiplicative closed subset S is called a $S - m - system$ if for any $x, y \in R$, there exists $s \in S$ and $r \in R$ such that $sx, sy \in M$ implies that $xry \in M$.

Proposition 2.17. *Let R be a ring and S be multiplicative closed subset of R. A proper ideal P of R is an S-prime ideal of R if and only if P^c is a $S - m - system$.*

Proof. Let P be an S-prime ideal of ring R if and only if for any $x, y \in R$ with $xRy \subseteq P$ there exists $s \in S$ such that $sx \in P$ or $sy \in P$ if and only if there exists $s \in S$ with $sx \in P^c$ and $sy \in P^c$ then there exists $x \in R$ such that $xry \notin P$ and so $xry \in P^c$ if and only if P^c is a $S - m - system$. □

Definition 2.18. Let R be a ring. A nonempty subset N of R containing a multiplicative closed subset S is called a $S - n - system$ if for any $x \in R$, there exists $s \in S$ and $r \in R$ such that $sx \in N$ implies that $xrx \in N$.

Proposition 2.19. *Let S be multiplicative closed subset of ring R. A proper ideal P of R is an S-semiprime ideal of R if and only if P^c is a $S - n - system$.*

Proof. Let us suppose P be an S-semiprime ideal of ring R if and only if for any $x \in R$ with $xRx \subseteq P$ there exists $s \in S$ such that $sx \in P$ if and only if there exists $s \in S$ with $sx \in P^c$ then there exists $x \in R$ such that $xrx \notin P$ and so $xrx \in P^c$ if and only if P^c is a $S - n - system$. □

Example 2.20. Consider the ring $R = \mathbb{Z}$ and the multiplicative closed set $S = \{3n \mid n \in \mathbb{N}\}$ of R. $P =< 6 >$ is S-semiprime ideal. Then P^c is an $S - n - system$ in $R = \mathbb{Z}$

Proposition 2.21. *Let R be a ring and S be a multiplicative closed subset of R. Let N be an $S - n - system$ of R and P be a maximal ideal of R with respect to the condition that N is disjoint with P. Then P is an S-semiprime ideal of R.*

Proof. Let us suppose that P is not S-semiprime. Then $sx \notin P$ for all $s \in S$ but $x^2 \in P$ for $x \in R$. Since P is maximal with respect to $N \cap P = \phi$, therefore we can write there exists $n \in N$ such that $n \in P+ < x^2 >$. There exists $s' \in S$ and $r \in R$ such that $s'n \in N$ implies that $nrn \in N$ since N is an $S - n - system$. Again we have $nrn \in (P+ < x^2 >)R(P+ < x^2 >) \subseteq P+ < x^2 > \subseteq P$. Which is a contradiction to the fact that $N \cap P = \phi$. Hence P is an S-semiprime ideal in R. □

Definition 2.22. Let S be any multiplicative closed subset of a ring R. For any ideal I of R, we define $\beta(I)$ = $\{r \in R : N \cap I \neq \phi$ for any $S - n - system$ N containing $r\}$.

Proposition 2.23. *Let R be a ring and S be a multiplicative closed subset of R. For any ideal I of R, $\beta(I) = \bigcap_{I \subseteq P,\ P\ is\ an\ S-semiprime\ ideal} P$.*

Proof. Suppose $x \in \beta(I)$. Let P ne an S-semiprime ideal of R such that $I \subseteq P$. Let us consider that $x \notin P$ then $x \in P^c$. And by Proposition 2.19 we have P^c is an $S - n - system$. So $P^c \cap I \neq \phi$. Which is a contradiction as $I \subseteq P$. Hence $x \in P$ for all S-semiprime ideals P such that $I \subseteq P$. Hence $x \in \bigcap_{I \subseteq P,\ P\ is\ an\ S-semiprime\ ideal} P$. Conversely, let $x \in \bigcap_{I \subseteq P,\ P\ is\ an\ S-semiprime\ ideal} P$. Let us suppose that $x \notin \beta(I)$. So by definition of $\beta(I)$, there exists an $S - n - system$ N such that $x \in N$ and $N \cap I = \phi$. By Zorn's lemma, there exists a maximal ideal J of R such that $N \cap J = \phi$. Now by Proposition 2.21, J is an S-semiprime ideal of ring R. Since $x \in N$ so $x \notin J$ and therefore $x \notin I$. Hence $x \notin \bigcap_{I \subseteq P,\ P\ is\ an\ S-semiprime\ ideal} P$, which is a contradiction. Therefore $x \in \beta(I)$. □

Lemma 2.24. *Let R be a commutative ring. A nonempty subset S of R is a multiplicative closed set if and only if $M_n^d(S)$ is a multiplicative closed subset of $M_n(R)$.*

Proof. Let S be multiplicative closed subset of ring R. Then $1 \in S$ and for $x, y \in S$ implies that $xy \in S$. It follows that $I \in M_n^d(S)$ and let $A, B \in M_n^d(S)$. Then $A = diag(a_1, a_2, \ldots, a_n)$ and $B = diag(b_1, b_2, \ldots, b_n)$ where $a_i, b_i \in S$. So, $AB = diag(a_1b_1, a_2b_2, \ldots, a_nb_n)$. Which shows that $AB \in M_n^d(S)$. Thus $M_n^d(S)$ is multiplicative closed set.
Conversely, let $M_n^d(S)$ is a multiplicative closed sibset of $M_n(R)$. Then for any $A, B \in M_n^d(S)$ we have $AB \in M_n^d(S)$. We have to show that for any $x, y \in S$ implies that $xy \in S$. We construct $A = diag(x, x, \ldots, x)$ and $B = diag(y, y, \ldots, y)$. This implies that $diag(xy, xy, \ldots, xy) \in M_n^d(S)$ and thus $xy \in S$. Hence S is a multiplicative closed subset of R. □

In the following we establish a relationship between S-semiprime ideal of a ring and S-semiprime ideal of its corresponding matrix ring.

Lemma 2.25. [11] *If A and B are two ideal of a ring R then (i) $M_n(AB) = M_n(A)M_n(B)$ and (ii) $A \subseteq B$ if and only if $M_n(A) \subseteq M_n(B)$.*

Proposition 2.26. *Let R be a ring with identity and S be a multiplicative closed subset of R. A proper ideal J of R is an S-semiprime ideal of R if and only if $M_n(J)$ is an $M_n^d(S)$-semiprime ideal of $M_n(R)$.*

Proof. Let J be an S-semiprime ideal of R. We know that the ideals of $M_n(R)$ are of the form $M_n(J)$ for every ideal J of R. Suppose $M_n(A)$ be ideal of $M_n(R)$ such that $(M_n(A))^2 \subseteq M_n(J)$. By the above Lemma 2.25. we have $(M_n(A))^2 = M_n(A^2) \subseteq M_n(J)$. Which implies that $A^2 \subseteq J$. Since J is an S-semiprime ideal of R so there exists $s \in S$ such that $sA \subseteq J$. It follows that $M_n(sA) \subseteq M_n(J)$. Thus, there exists scalar matrix $sI \in M_n^d(S)$ such that $sIM_n(A) \subseteq M_n(J)$. Hence $M_n(J)$ is an $M_n^d(S)$-semiprime ideal of $M_n(R)$. Conversely, let $M_n(J)$ is an $M_n^d(S)$-semiprime ideal of $M_n(R)$. Suppose A be ideal of R such that $A^2 \subseteq J$. Which implies that $M_n(A^2)$ is an ideal of $M_n(R)$ and by above Lemma 2.25. we have $M_n(A^2) \subseteq M_n(J)$. It follows that $(M_n(A))^2 \subseteq M_n(J)$. Since $M_n(J)$ is an $M_n^d(S)$-semiprime ideal of $M_n(R)$ so there exists $sI \in M_n^d(S)$ such that $sIM_n(A) = M_n(sA) \subseteq M_n(J)$ and hence $sA \subseteq J$. Thus J is an S-semiprime ideal of R. □

3 Weakly S-Semiprime Ideals

In this section we introduce the notion of the weakly S-semiprime ideals of R and discuss their characterization in R. We begin with the following definition.

Definition 3.1. Let R be a commutative ring, S be a multiplicative closed subset of R and P an ideal of R disjoint with S. We say P is weakly S-semiprime ideal of R if there exists an $s \in S$ such that for all $a \in R$ with $0 \neq a^2 \in P$, we have $sa \in P$.

It is obvious that all weakly semiprime ideal of R are weakly S-semiprime. However, the converse is not true in genearal which can be illustrated by the following example.

Example 3.2. Let $R = \mathbb{Z}[X]$, $P = 4X\mathbb{Z}[X]$ and $S = \{2^n | n \in \mathbb{N}\}$. Then we have shown in the Example 2.2(2) that P is an S-semiprime ideal, which implies that P is weakly S-semiprime ideal. We have $0 \neq 4X^2 = (2X)^2 \in P$ but $2X \notin P$. Thus, P is not weakly semiprime.

Also, an S-semiprime ideal of R is weakly S-semiprime. However, the converse is not true in general. The following example shows that the converse implication is not true.

Example 3.3. Let $R = \mathbb{Z}_{12}$ and $S = \{3, 9\}$. Clearly, S is a multiplicative closed subset of R. The ideal $< 0 >$ is weakly S-semiprime ideal of $\mathbb{Z}_{12}$. However, $< 0 >$ ideal is not S-semiprime in $\mathbb{Z}_{12}$. Here $6 \in \mathbb{Z}_{12}$, and $6^2 \in < 0 >$. However $6.3 = 6.9 = 6 \notin < 0 >$. Hence $< 0 >$ is not S-semiprime in $\mathbb{Z}_{12}$.

Remark 3.4. $< 0 >$ is always weakly S-semiprime ideal of R, and it is S-semiprime if and only if R is an integral domain.

Let S be a multiplicative closed subet of a ring R and P an ideal of R disjoint with S. It is obvious that if $(P : s)$ is a weakly semiprime ideal of R for some $s \in S$, then P is a weakly S-semiprime ideal. However, the converse implication is not true in general. That is P is a weakly S-semiprime ideal of R doesnot imply in general that $(P : s)$ is weakly semiprime ideal.

Example 3.5. Let $R = \mathbb{Z}_{12}$ and $S = \{3, 9\}$. The ideal $< 0 >$ is weakly S-semiprime, but the ideal $(0 : 3) = (0 : 9) = \{0, 4\}$ is not weakly semiprime. Since $0 \neq 2^2 \in \{0, 4\}$ but $2 \notin \{0, 4\}$.

Proposition 3.6. *Let R be a commutative ring, S be a multiplicative closed subset of R consisting of regular elements and P be an 2-absorbing ideal of R disjoint with S. Then P is a weakly S-semiprime ideal of R if and only if $(P : s)$ is a weakly semiprime ideal of R for some $s \in S$.*

Proof. Suppose P is weakly S-semiprime ideal in R, then there exists $s \in S$ such that for all $a \in R$ with $0 \neq a^2 \in P$, we have $sa \in P$. We claim that $(P : s)$ is a weakly semiprime ideal of R. Let $0 \neq a^2 \in (P : s)$. Then, $0 \neq sa^2 \in P$. Since P is an ideal, we can have $(sa)^2 \in P$. Hence $s.sa \in P$. Since P is 2-absorbing and $P \cap S = \phi$. Hence $sa \in P$. Which implies $a \in (P : s)$. This implies $(P : s)$ is weakly semiprime ideal. Conversely, suppose $(P : s)$ is weakly semiprime ideal in R. Let $0 \neq a^2 \in P$. For $s \in S \subseteq R$, we have $0 \neq sa^2 \in P$. Therefore, $a^2 \in (P : s)$. As $(P : s)$ is weakly semiprime ideal, hence $0 \neq a^2 \in (P : s)$ implies $a \in (P : s)$. Hence $sa \in P$. This implies that P is weakly S-semiprime ideal in R. □

Definition 3.7. Let P be a weakly S-semiprime ideal of a ring R and S be multiplicative closed subset of R disjoint with P. We say a is an *unbreakable zero element of* P if $a^2 = 0$ and $sa \notin P$ for all $s \in S$.

Proposition 3.8. *Let P be a weakly S-semiprime ideal and S be multiplicative closed subset of a ring R such that $P \cap S = \phi$. Suppose that a is unbreakable-zero element of P. Then $(a + i)^2 = (a - i)^2 = 0$.*

Proof. It is given that P is an S-semiprime ideal and S is a multiplicative closed subset of R such that $P \cap S = \phi$. Also a is an unbreakable zero element of P. Therefore $a^2 = 0$ and $sa \notin P$. Let $i \in P$. Since $(a + i)^2 = a^2 + 2ai + i^2 = 0 + 2ai + i^2 \in P$ and $sa \notin P$, we have $(sa + si) = s(a + i) \notin P$. Since $P \cap S = \phi$, hence $(a + i) \notin P$. Thus $(a + i)^2 = 0$.
Similarly $(a - i)^2 = a^2 - 2ai + i^2 \in P$ and $sa \notin P$ which implies $s(a - i) \notin P$, we have $(a - i) \notin P$. Thus $(a - i)^2 = 0$. □

Proposition 3.9. *Let P be a weakly S-semiprime ideal of a ring R that is not S-semiprime and S is multiplicative closed subset of R. Then $P \subseteq Nil(R)$.*

Proof. Suppose that P be a weakly S-semiprime ideal of ring R, such that P is not S-semiprime. Therefore we can conclude that there is $a \in R$ which is *unbreakable zero element of* P. That is there is $a \in R$ such that $a^2 = 0$ and $sa \notin P$. Let $i \in P$. Then by the previous theorem, $(a + i)^2 = 0$ for *unbreakable zero element* a of P. This implies $(a + i) \in Nil(R)$. Also $a^2 = 0$ implies $a \in Nil(R)$. Thus we have $i \in Nil(R)$. Therefore we can conclude that $P \subseteq Nil(R)$. □

Proposition 3.10. *Let A be an ideal of R and let S be a multiplicative closed subset in R such that $A \cap S$ is empty. Then there is an ideal P of R which is maximal with respect to the property that $A \subseteq P$ and $P \cap S$ are empty. Furthermore, P is a semiprime ideal.*

Proof. Let Γ be the set of all ideals M of R containg A and $M \cap S$ is empty. Γ is not empty since $A \in \Gamma$. By Zorn's lemma Γ has a maximal element P. To show that P is semiprime, let us suppose that P is not semiprime. Let $a^2 \in P$ such that $a \notin P$. Then $P \subset P + < a >$ and so there is an element $s \in S$ such that $s \in P + < a >$. Hence $s = p + ra$ where $p \in P$ and $r \in R$. Then $s^2 = (p + ra)^2 = p^2 + 2pra + r^2a^2 \in P \cap S$, a contradiction to the assumption that $P \cap S = \phi$. Hence P is semiprime ideal of R. □

Proposition 3.11. *Let R be a commutative ring and S be a multiplicative closed subset of R consisting of regular elements. Then the following assertions are equivalent:*

(1) $< 0 >$ is the only weakly S-semiprime ideal of R.
(2) $< 0 >$ is the only S-semiprime ideal of R.
(3) R is a reduced and $S^{-1}R$ is a field.

Proof. $(1) \Rightarrow (2)$ Let P be an S-semiprime ideal of R. Then P is weakly S-semiprime ideal of R, and then $P = < 0 >$. So $< 0 >$ is the only S-semiprime ideal of R.

$(2) \Rightarrow (3)$ By Proposition 3.10. there is a semiprime ideal P with $P \cap S = \phi$. Hence P is an S-semiprime ideal of R. Hence, $P =< 0 >$ and so the ring R is a reduced. Let $a \in R - \{0\}$ and $s \in S$. We claim that $\frac{a}{s}$ is invertible in $S^{-1}R$ to prove that $S^{-1}R$ is field. If $a \in S$ then we have the desired result. Hence, we may assume that $a \notin S$. Suppose that $< a > \cap S = \phi$. Then, by Proposition 3.10. there is a semiprime ideal P of R such that $< a >\subseteq P$. Hence, $< a >\subseteq P = 0$. Which is a contradiction to the fact that $a \in R - \{0\}$. Hence, $< a > \cap S \neq \phi$. Let $s' \in< a > \cap S$. Set $s' = at$. We have, $\frac{st}{s'} \in S^{-1}R$ and $\frac{a}{s}.\frac{st}{s'} = \frac{ast}{ss'} = \frac{ss'}{ss'} = 1$. Therefore, $\frac{st}{s'}$ is the inverse of $\frac{a}{s}$ and hence $\frac{a}{s}$ is invertible in $S^{-1}R$ as desired. So, $S^{-1}R$ is a field.

$(3) \Rightarrow (1)$ Let P be a weakly S-semiprime ideal of R. Let us suppose $p \in P - \{0\}$. Now since $S^{-1}R$ is field, hence there exists $a \in R - \{0\}$ and $s \in S$ such that $\frac{p}{1}.\frac{a}{s} = 1$. This implies $pa = s \in P \cap S$. Which is contradiction to the fact that $P \cap S = \phi$. Thus $P =< 0 >$ is the only weakly S-semiprime ideal of R. □

Proposition 3.12. *Let R be a commutative ring and S be a multiplicative closed subset of R consisting of regular elements. Then the following assertions are equivalent:*

(1) Every weakly S-semiprime ideal of R is semiprime.

(2) R is a reduced and every S-semiprime ideal of R is semiprime.

Proof. $(1) \Rightarrow (2)$ We have that $< 0 >$ is weakly S-semiprime ideal of R. Since every weakly S-semiprime ideal of R is semiprime, hence $< 0 >$ is semiprime ideal of R. Thus the ring R is reduced. Again since all S-semiprime ideal are weakly S-semiprime ideal and hence every S-semiprime ideal of R is semiprime ideal.

$(2) \Rightarrow (1)$ Since the ring R is reduced, therefore every weakly S-semiprime ideal of R is S-semiprime. And since every S-semiprime ideal of R is semiprime ideal, thus every weakly S-semiprime ideal of R is semiprime. □

Proposition 3.13. *Let S be a multiplicative closed subset of a ring R and P be a weakly S-semiprime ideal of R such that P is not S-semiprime; then $P^2 = 0$.*

Proof. It is given that P is weakly S-semiprime ideal. Hence there exists $s \in S$ such that whenever $a \in R$ and $0 \neq a^2 \in P$ implies $sa \in P$. Suppose that $P^2 \neq 0$. We claim that P is S-semiprime. Let $a \in R$ such that $a^2 \in P$. If $0 \neq a^2 \in P$, then $sa \in P$ (since P is weakly S-semiprime). Now suppose $a^2 = 0$. If $aP \neq 0$, there exists $p \in P$ such that $ap \neq 0$. Therefore $0 \neq ap \in P \implies 0 \neq ap = a(p + a) \in P$ (since $a^2 = 0$). Thus $(a + p)(a + p) \in P$ implies $(a + p)^2 \in P$ implies $s(a + p) \in P$. So $sa \in P$ implying P is S-semiprime. Now suppose $aP = 0$. Since $P^2 \neq 0$, there exists $p \in P$ such that $p^2 \neq 0$ and $p^2 \in P$. Thus, $0 \neq p^2 \in P \implies 0 \neq (a + p)(a + p) \in P$ ($a^2 = 0,\ ap = 0$ *and* $p^2 \in P$). So $s(a + p) \in P$ implies $sa \in P$. □

Proposition 3.14. *Let R be a ring and $S_1 \subset S_2$ be multiplicative closed subsets of R such that for any $s \in S_2$, there is an element $t \in S_2$ satisfying $st \in S_1$. If P is a weakly S_2-semiprime ideal of R, then P is a weakly S_1-semiprime ideal of R.*

Proof. Let $a \in R$ such that $0 \neq a^2 \in P$. Since P is S_2-semiprime ideal, there exists $s \in S_2$ such that $sa \in P$. By the assumption, there exists $t \in S_2$ such that $s' = st \in S_1$. And then $s'a = sta \in P$. Consequently, P is weakly S_1-semiprime ideal of R. □

Let S be multiplicative closed subset of a ring R and S^*=$\{r \in R : \frac{r}{1}$ is unit in $S^{-1}R\}$ denotes the saturation of S. S^* is a multiplicative closed subset of R containing S. S^* is always a saturated multiplicative subset of R [8].

Proposition 3.15. *Let R be a ring, S be multiplicative closed subset of R and P an ideal of R disjoint with S. Then P is a weakly S-semiprime ideal of R if and only if P is a weakly S^*-semiprime ideal.*

Proof. S^* is saturation of multiplicative closed set S, so $S \subseteq S^*$. $P \cap S = \phi$. We claim that $S^* \cap P = \phi$. Let us suppose $S^* \cap P \neq \phi$. So, there exists $t \in S^* \cap P$. That is $t \in P$ and $t \in S^*$. By definition of S^*, $\frac{t}{1}$ is unit in $S^{-1}R$. Hence there exists $\frac{m}{n} \in S^{-1}R$ such that $\frac{t}{1}.\frac{m}{n} = 1$ for some $m \in R$ and $n \in S$. This implies $tm = n \in S$. But $t \in P$ implies $tm \in P$, which is a contradiction to the fact that $P \cap S = \phi$. Hence $S^* \cap P = \phi$. We have $S \subseteq S^*$ and

$P \cap S^* = \phi$. Clearly, if P is a weakly S-semiprime ideal of R, then P is weakly S^*-semiprime ideal of ring R. Now suppose P is weakly S^*-semiprime ideal. To prove the converse part we will use Proposition 3.14. We will show that for any $r \in S^*$, there exists $r' \in S^*$ such that $rr' \in S$. For $r \in S^*$, then $\frac{r}{1}.\frac{a}{s} = 1$ for some $a \in R$ and $s \in S$. Which implies that $tar = ts \in S$ for some $t \in S$. Now by taking $r' = ta$ we have $r' \in S^*$ with $rr' \in S$. Therefore, by replacing $S = S_1$ and $S^* = S_2$ we can conclude that P is weakly S-semiprime ideal. □

Proposition 3.16. *Let $f : R \to R'$ be a ring homomorphism and S be a multiplicative closed subset of the ring R. The following conditions hold:*

(1) If f is an epimorphism and P is a weakly S-semiprime ideal of R containing $Ker(f)$ then $f(P)$ is a weakly $f(S)$-semiprime ideal of R'.

(2) If f is a monomorphism and P' is a weakly $f(S)$-semiprime ideal of R', then $f^{-1}(P')$ is a weakly S-semiprime ideal of R.

Proof. (1) Let f is an epimorphism and P is weakly S-semiprime ideal of R containing $Ker(f)$. To prove $f(P)$ to be weakly $f(S)$-semiprime ideal, first we claim that $f(P) \cap f(S) = \phi$. Let $r \in f(P) \cap f(S)$. Then there exists some $p \in P$ and $s \in S$ such that $r = f(p) = f(s)$. Therefore $s - p \in Ker(f) \subseteq P$, which implies $s \in P$. A contradiction to the fact that $P \cap S = \phi$. Therefore $f(P) \cap f(S) = \phi$. Now, let us suppose $0 \neq a'^2 \in f(P)$. Since f is epimorphism, there is $a \in R$ such that $f(a) = a'$ and $0 \neq f(a^2) = a'^2 \in f(P)$. Since $Ker(f) \subseteq P$, we get $0 \neq a^2 \in P$, and since P is weakly S-semiprime we have $sa \in P$ for some $s \in S$. This implies $f(s)a' \in f(P)$. Thus $0 \neq a'^2 \in f(P)$ implies that $sa' \in f(P)$. Hence $f(P)$ is weakly $f(S)$-semiprime ideal of R'.

(2) Let f is a monomorphism and P' be an $f(S)$-semiprime ideal of R'. So there exists $s \in S$ such that for all $a' \in R'$, $0 \neq a'^2 \in P'$ implies $f(s)a' \in P'$.The condition $f^{-1}(P') \cap S = \phi$ is trivial. Let $a \in R$ such that $0 \neq a^2 \in f^{-1}(P')$. f is monomorphism implies $Ker(f) = \{0\}$, so we get $= 0 \neq f(a^2) = (f(a))^2 \in P'$. Then $f(s)f(a) = f(sa) \in P'$ since P' is weakly $f(S)$-semiprime ideal. Thus $sa \in f^{-1}(P')$. So we can conclude that $f^{-1}(P')$ is a weakly S-semiprime ideal of R. □

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Author information

Kamal Krishna Pathak, Department of Mathematics, Gauhati University, Guwahati-14, Assam, India.
E-mail: kamalkrishna2k18@gmail.com

Jituparna Goswami, Department of Mathematics, Gauhati University, Guwahati-14, Assam, India (Corresponding Author).
E-mail: jituparnagoswami18@gmail.com

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