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Sum Divisor Cordial Labeling For Path Union of Two Copies of Cycle Related Graphs

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Abstract A sum divisor cordial labeling (SDCL) of a graph G^+ with node set V is bijective. T^* from V to $\{1, 2, 3, \ldots, |V(G)|\}$ in such a way that if each edge xy is reserved the label 1 if $2|[T^*(x) + T^*(y)]$ and 0 otherwise, then the number of edges labeled with 0 and the number of edges labeled with 1 under the condition $|e_{T^*}(0) - e_{T^*}(1)| \leq 1$. A graph with a sum divisor cordial labeling (SDCL) is called a sum divisor cordial graph (SDCG). Thus, by using the above properties here we investigate that path union of two copies of cycle, cycle with triangle, wheel, helm, closed helm admit sum divisor cordial labeling.

1 Introduction

All the way through this paper, by a graph we meane a finite, undirected, simple graph $G = (V, E)$ with p vertices and q edges. We follow the central annotations and phraseology of graph theory as in [6] and number theory as in [3].
Labeling of a graph is a mapping that connects the graph elements to the numbers set, generally to the set of natural numbers. If the domain is a collection of nodes the labeling is called node labeling. If the domain is the collection of edges, then we allude to edge labeling. If the labels are given to both nodes and edges then the labeling is called total labeling. Graph Labeling is an energetic region that belongs to research in graph theory. The concept of graph labeling was introduced by Rosa. J. A. Gallian [5] developed graph labeling which performs as a boundary stuck flanked between number theory and the structure of graphs.
The present work is aimed to discuss one such labeling known as sum divisor cordial labeling.

2 Preliminaries

Definition 2.1. Cordial labeling was introduced by Cahit as a weaker version of graceful and harmonious labeling of graphs. Combining the concepts of divisibility and cordial labeling, Varatharajan et al. [8] introduced the concept of divisor cordial labeling of a graph.
A divisor cordial labeling (DCL) of a graph G^+ with node set V is bijective. T^* from V to $\{1, 2, 3, \ldots, |V(G)|\}$ in such a way that if each edge xy is reserved the label 1 if $T^*(x) \mid^* T(y)$ or $T^*(y) \mid T^*(x)$ and 0 otherwise, then the number of edges labeled with 0 and the number of edges labeled with 1 under the condition $|e_{T^*}(0) - e_{T^*}(1)| \leq 1$.
A graph with a divisor cordial labeling (DCL) is called a divisor cordial graph (DCG).

Varatharajan et al. [8] proved that some standard graphs and some special classes of graphs are DCG. Ghodasara and Adalja [4] determined DCL for the ringsum of some standard graphs with star graph.

Definition 2.2. Let $T^* : V(G) \to \{1, 2, 3, \ldots, |V(G)|\}$ be a bijection and let the induced function $T^{**} : E(G) \to \{0, 1\}$ be defined as

$$T^{**}(e = xy) = \begin{cases} 1; & \text{if } 2 \mid T^*(x) + T^*(y)] \\ 0; & \text{otherwise.} \end{cases}$$

Then T^* is called sum divisor cordial labeling of graph (SDCL) G if $|e_{T^*}(0) - e_{T^*}(1)| \leq 1$.
A graph with a sum divisor cordial labeling is called sum divisor cordial graph (SDCG).

Lourdusamy et al.[7] introduced the concept of sum divisor cordial labeling of graphs. In [7] the same authors have proved that shadow graph and splitting graph of $K_{1,n}$, shadow graph, subdivision graph, splitting graph and degree splitting graph of $B_{n,n}$, corona of a ladder and triangular ladder with K_1, closed helm are sum divisor cordial graphs. In [1], Adalja and Ghodasara derived sum divisor cordial labeling in the context of graph operations on bistar and in [2], they also derive that sum divisor cordial labeling in the context of corona products of some standard graphs with K_1.
In this paper, we have derived five new out-turns admitting SDCL in context of path union of two copies of various graphs are sum divisor cordial graph (SDCG).

Definition 2.3. The path union of a graph G is the graph obtained by adding an edge between corresponding vertices of G_j to $G_{j+1}, 1 \leq j \leq n-1$, where $G_1, , G_2, G_3, \ldots, G_n (n \geq 2)$ are n copies of G. It is denoted by $P(n.G)$.

3 Main Results

Definition 3.1. A closed path is called a cycle. A cycle with n vertices is denoted as C_n. In a cycle graph, all the vertices are of degree 2.

Theorem 3.2. *The path union of two copies of cycle C_n ($n \in \mathbb{N}, n \geq 3$) is a SDCG.*

Proof. Let $G = P(2.C_n)$ be the path union of two copies of cycle C_n. Let $v_1, v_2, \ldots, v_n$ be the successive vertices of C_n and let $u_1 = v_{n+1}, u_2 = v_{n+2}, \ldots, u_n = v_{2n}$ be the successive vertices of another C_n. The vertex v_1 is connected to u_1 by an edge P_1, which collectively makes the path union of two copies of the cycle.
We characterize labeling function $T^* : V(G) \to \{1, 2, 3, \ldots, 2n\}$ as follows .
Case 1: For $n \equiv 0, 1, 3 (mod\, 4)$
In this case $|V(G)| = 2n$ and $|E(G)| = 2n + 1$.
Here we characterize labeling T^* as follows.

$$T^*(v_j) = \begin{cases} j & ; j \equiv 1, 0 (mod\, 4) \\ j+1 & ; j \equiv 2 (mod\, 4) \\ j-1 & ; j \equiv 3 (mod\, 4); \quad 1 \leq j \leq n. \end{cases}$$

Subcase 1: For $n \equiv 0 (mod\, 4)$:

$$T^*(u_j) = \begin{cases} n+j & ; j \equiv 1, 0 (mod\, 4) \\ n+j+1 & ; j \equiv 2 (mod\, 4) \\ n+j-1 & ; j \equiv 3 (mod\, 4); \quad 1 \leq j \leq n. \end{cases}$$

Subcase 2: For $n \equiv 1 (mod\, 4)$:

$$T^*(u_j) = \begin{cases} n+j+1 & ; j \equiv 1 (mod\, 4) \\ n+j-1 & ; j \equiv 2 (mod\, 4) \\ n+j & ; j \equiv 3, 0 (mod\, 4); \quad 1 \leq j \leq n-1. \end{cases}$$
$$T^*(u_n) = 2n.$$

Subcase 3: For $n \equiv 3 (mod\, 4)$:

$$T^*(u_j) = \begin{cases} n+j+1 & ; j \equiv 1 (mod\, 4) \\ n+j-1 & ; j \equiv 2 (mod\, 4) \\ n+j & ; j \equiv 3, 0 (mod\, 4); \quad 1 \leq j \leq n. \end{cases}$$

Case 2: For $n \equiv 2(mod\,4)$:

$$
\begin{aligned}
T^*(v_j) &= \begin{cases} j & ;\ j \equiv 1,0(mod\,4) \\ j+1 & ;\ j \equiv 2(mod\,4) \\ j-1 & ;\ j \equiv 3(mod\,4); \quad 1 \leq j \leq n-1. \end{cases} \\
T^*(v_n) &= n. \\
T^*(u_j) &= \begin{cases} n+j & ;\ j \equiv 1,0(mod\,4) \\ n+j+1 & ;\ j \equiv 2(mod\,4) \\ n+j-1 & ;\ j \equiv 3(mod\,4); \quad 1 \leq j \leq n-2. \end{cases} \\
T^*(u_{n-1}) &= 2n, \\
T^*(u_n) &= 2n-1.
\end{aligned}
$$

In view of the above labeling pattern, we observe that,

Cases of n	Edge conditions
$n \equiv 1,3(mod\,4)$	$e^*_T(1) = \lfloor \frac{2n+1}{2} \rfloor, e^*_T(0) = \lceil \frac{2n+1}{2} \rceil$
$n \equiv 2,0(mod\,4)$	$e^*_T(0) = \lfloor \frac{2n+1}{2} \rfloor, e^*_T(1) = \lceil \frac{2n+1}{2} \rceil$

Clearly, it satisfies the condition $|e_{T^*}(0) - e_{T^*}(1)| \leq 1$.
Hence, the path union of two copies of cycle C_n is a SDCG. □

Example 3.3. SDCL of the path union of two copies of C_5 is shown in Figure 1.

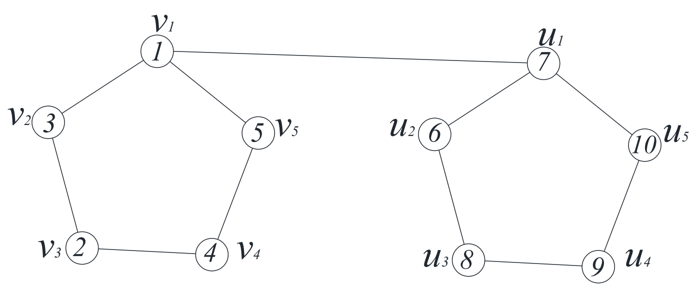

Figure 1.

Definition 3.4. A cycle with a triangle is a cycle with three chords which by themselves form a triangle.
For positive integers p, q, r and $n \geq 6$ with $p+q+r+3 = n$, $C_n(p,q,r)$ denotes a cycle with triangle whose edges form the edges of cycles C_{p+2}, C_{q+2}, C_{r+2} without chords.

Theorem 3.5. *The path union of two copies of cycle $C_{1,1,n-2}$ ($n \in \mathbb{N}, n \geq 5$) with three chords is a SDCG, where three chords which by themselves form a triangle.*

Proof. Let $G = P(2.C_{1,1,n-2})$ be the path union of two copies of cycle with three chords $C_{1,1,n-2}$. Let $v_1, v_2, \ldots, v_n$ be the successive vertices of C_n and let $u_1 = v_{n+1}, u_2 = v_{n+2}, \ldots, u_n = v_{2n}$ be the successive vertices of another C_n. let $e_1 = v_1v_3$ and $e_2 = v_3v_{n-1}$ and $e_2 = v_1v_{n-1}$ be the chords of C_n and let $e'_1 = u_1u_3$ and $e'_2 = u_3u_{n-1}$ and $e'_2 = u_1u_{n-1}$ be the chords of another C_n. The vertex v_1 is connected to u_1 by an edge P_1 which collectively makes the path union of 2 copies of cycle with three chords.
We characterize labeling function $T^* : V(G) \to \{1, 2, 3, \ldots, 2n\}$ as follows.
Case 1: For $n \equiv 0, 2(mod\,4)$
In this case $|V(G)| = 2n$ and $|E(G)| = 2n + 7$.

Here we characterize labeling T^* as follows.

$$T^*(v_j) = \begin{cases} j & ; j \equiv 1,0(mod\,4) \\ j+1 & ; j \equiv 2(mod\,4) \\ j-1 & ; j \equiv 3(mod\,4); \quad 1 \le j \le n. \end{cases}$$

Subcase 1: For $n \equiv 0(mod\,4)$:

$$T^*(u_j) = \begin{cases} n+j & ; j \equiv 1,0(mod\,4) \\ n+j+1 & ; j \equiv 2(mod\,4) \\ n+j-1 & ; j \equiv 3(mod\,4); \quad 1 \le j \le n. \end{cases}$$

Subcase 2: For $n \equiv 2(mod\,4)$:

$$T^*(u_j) = \begin{cases} n+j-1 & ; j \equiv 1(mod\,4) \\ n+j & ; j \equiv 2,3(mod\,4) \\ n+j+1 & ; j \equiv 0(mod\,4); \quad 1 \le j \le n-1. \end{cases}$$

$$T^*(u_n) = 2n.$$

Case 2: For $n \equiv 1(mod\,4)$:

$$T^*(v_j) = \begin{cases} j & ; j \equiv 1,0(mod\,4) \\ j+1 & ; j \equiv 2(mod\,4) \\ j-1 & ; j \equiv 3(mod\,4); \quad 1 \le j \le n-1. \end{cases}$$

$$T^*(v_n) = n+1.$$

$$T^*(u_j) = \begin{cases} n+j-1 & ; j \equiv 1(mod\,4) \\ n+j & ; j \equiv 2,3(mod\,4) \\ n+j+1 & ; j \equiv 0(mod\,4); \quad 1 \le j \le n. \end{cases}$$

Case 3: For $n \equiv 3(mod\,4)$:

$$T^*(v_j) = \begin{cases} j & ; j \equiv 1,0(mod\,4) \\ j+1 & ; j \equiv 2(mod\,4) \\ j-1 & ; j \equiv 3(mod\,4); \quad 1 \le j \le n. \end{cases}$$

$$f(u_1) = n+1,$$
$$f(u_2) = n+2,$$
$$f(u_3) = n+3,$$

$$T^*(u_j) = \begin{cases} n+j & ; j \equiv 0,3(mod\,4) \\ n+j+1 & ; j \equiv 1(mod\,4) \\ n+j-1 & ; j \equiv 2(mod\,4); \quad 4 \le j \le n. \end{cases}$$

In view of the above labeling pattern, we observe that,

Cases of n	Edge conditions
$n \equiv 0,3(mod\,4)$	$e^*_T(1) = \lfloor \frac{2n+7}{2} \rfloor, e^*_T(0) = \lceil \frac{2n+7}{2} \rceil$
$n \equiv 1,2(mod\,4)$	$e^*_T(0) = \lfloor \frac{2n+7}{2} \rfloor, e^*_T(1) = \lceil \frac{2n+7}{2} \rceil$

Clearly, it satisfies the condition $|e_{T^*}(0) - e_{T^*}(1)| \le 1$.
Hence, the path union of two copies of cycle with three chords $C_{1,1,n-2}$ is a SDCG. □

Example 3.6. SDCL of the path union of two copies of $C_{1,1,n-2}$ is shown in Figure 2.

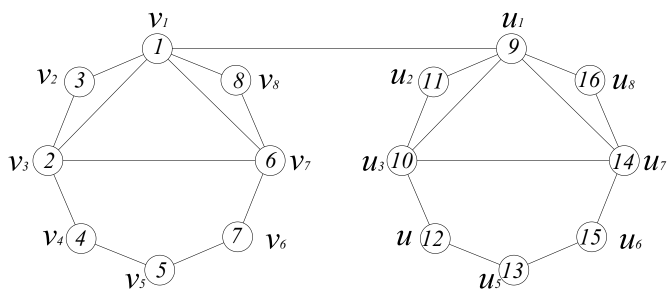

Figure 2.

Definition 3.7. The wheel graph W_n is the join of C_n and K_1, i.e. $W_n = C_n + K_1$. Here the edges(vertices) of C_n are called rim edges(rim vertices) and the vertex corresponding to K_1 is called apex.

Theorem 3.8. *The path union of two copies of wheel W_n ($n \in \mathbb{N}, n \geq 3$) is a SDCG.*

Proof. Let $G = P(2.W_n)$ be the path union of two copies of wheel W_n. Let $v_0, v_1, v_2, \ldots, v_n$ be the successive vertices of W_n and let $u_0, u_1 = v_{n+1}, u_2 = v_{n+2}, \ldots, u_n = v_{2n}$ be the successive vertices of another W_n. The vertex v_1 is connected to u_1 by an edge P_1 which collectively makes the path union of 2 copies of wheel.
We characterize labeling function $T^* : V(G) \to \{1, 2, 3, \ldots, 2n+2\}$ as follows.
In this case $|V(G)| = 2n+2$ and $|E(G)| = 4n+1$.
Here we characterize labeling T^* as follows.

$$
\begin{aligned}
T^*(v_0) &= 1. \\
T^*(v_j) &= 2j+2 \quad 1 \leq j \leq n. \\
T^*(u_0) &= 2. \\
T^*(u_j) &= 2j+1 \quad 1 \leq j \leq n.
\end{aligned}
$$

In view of the above labeling pattern, we observe that,

Cases of n	Edge conditions
$n \equiv 0, 2, 1, 3 (mod\, 4)$	$e^*_T(1) = \lfloor \frac{4n+1}{2} \rfloor, e^*_T(0) = \lceil \frac{4n+1}{2} \rceil$

Clearly, it satisfies the condition $|e_{T^*}(0) - e_{T^*}(1)| \leq 1$.
Hence, the path union of two copies of wheel W_n is a SDCG. □

Example 3.9. SDCL of the path union of two copies of W_6 is shown in Figure 3.

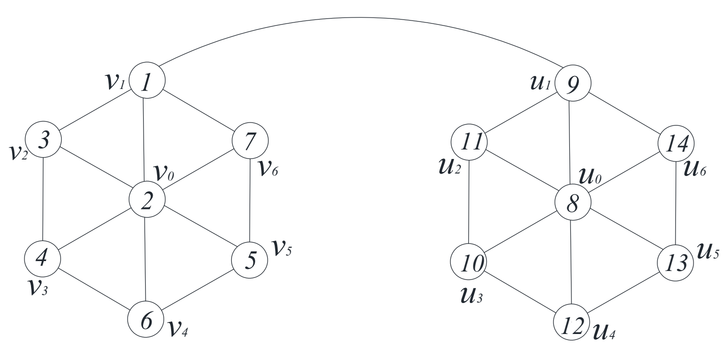

Figure 3.

Definition 3.10. The helm H_n is the graph obtained from a wheel by attaching a pendent edge at each vertex of the n - cycle.

Theorem 3.11. *The path union of two copies of helm H_n ($n \in \mathbb{N}, n \geq 3$) is a SDCG.*

Proof. Let $G = P(2.H_n)$ be the path union of two copies of helm H_n. Let $v_1, v_2, \dots, v_n$ be the vertices of degree 4 and $w_1, w_2, \dots, w_n$ be the pendant vertices, v_0 be apex vertex of H_n. Let $u_1 = v_{n+1}, u_2 = v_{n+2}, \dots, u_n = v_{2n}$ be the vertices of degree 4 and $w'_1, w'_2, \dots, w'_n$ be the pendant vertices, u_0 be apex vertex of another helm H_n. The vertex v_1 is connected to u_1 by an edge P_1 which collectively makes the path union of 2 copies of helm.
We characterize labeling function $T^* : V(G) \to \{1, 2, 3, \dots, 4n + 2\}$ as follows.
Case 1: For $n \equiv 0, 2(mod\, 4)$
In this case $|V(G)| = 4n + 2$ and $|E(G)| = 6n + 1$.
Here we characterize labeling $T*$ as follows.

$$T^*(v_0) = 1.$$

$$T^*(v_j) = \begin{cases} 2j+1 & ; j \equiv 1, 3mod\, 4) \\ 2j-1 & ; j \equiv 0, 2(mod\, 4); \quad 1 \leq j \leq n. \end{cases}$$

$$T^*(w_j) = \begin{cases} 2j+3 & ; j \equiv 1, 3(mod\, 4) \\ 2j & ; j \equiv 0, 2(mod\, 4); \quad 1 \leq j \leq n. \end{cases}$$

$$T^*(u_0) = 2n + 2.$$

$$T^*(u_j) = \begin{cases} 2n+2j+2 & ; j \equiv 1, 3mod\, 4) \\ 2n+2j-1 & ; j \equiv 0, 2(mod\, 4); \quad 1 \leq j \leq n. \end{cases}$$

$$T^*(w'_j) = \begin{cases} 2n+2j+4 & ; j \equiv 1, 3(mod\, 4) \\ 2n+2j+1 & ; j \equiv 0, 2(mod\, 4); \quad 1 \leq j \leq n. \end{cases}$$

Case 2: For $n \equiv 1, 3(mod\, 4)$:

$$T^*(v_0) = 1.$$

$$T^*(v_j) = \begin{cases} 2j+1 & ; j \equiv 1, 3mod\, 4) \\ 2j-1 & ; j \equiv 0, 2(mod\, 4); \quad 1 \leq j \leq n-1. \end{cases}$$

$$T^*(v_n) = 2n + 1.$$

$$T^*(w_j) = \begin{cases} 2j+3 & ; j \equiv 1, 3(mod\, 4) \\ 2j & ; j \equiv 0, 2(mod\, 4); \quad 1 \leq j \leq n-1. \end{cases}$$

$$T^*(w_n) = 2n.$$

$$T^*(u_0) = 2n + 2.$$

$$T^*(u_j) = \begin{cases} 2n+2j+2 & ; j \equiv 1, 3mod\, 4) \\ 2n+2j-1 & ; j \equiv 0, 2(mod\, 4); \quad 1 \leq j \leq n-1. \end{cases}$$

$$T^*(u_n) = 4n + 2.$$

$$T^*(w'_j) = \begin{cases} 2n+2j+4 & ; j \equiv 1, 3(mod\, 4) \\ 2n+2j+1 & ; j \equiv 0, 2(mod\, 4); \quad 1 \leq j \leq n-1. \end{cases}$$

$$T^*(w'_n) = 4n + 1.$$

In view of the above labeling pattern, we observe that,

Cases of n	Edge conditions
$n \equiv 0, 2, 1, 3(mod\, 4)$	$e^*_T(0) = \lfloor \frac{6n+1}{2} \rfloor, e^*_T(1) = \lceil \frac{6n+1}{2} \rceil$

Clearly, it satisfies the condition $|e_{T^*}(0) - e_{T^*}(1)| \leq 1$.
Hence, the path union of two copies of helm H_n is a SDCG. □

Example 3.12. SDCL of the path union of two copies of helm H_3 is shown in Figure 4.

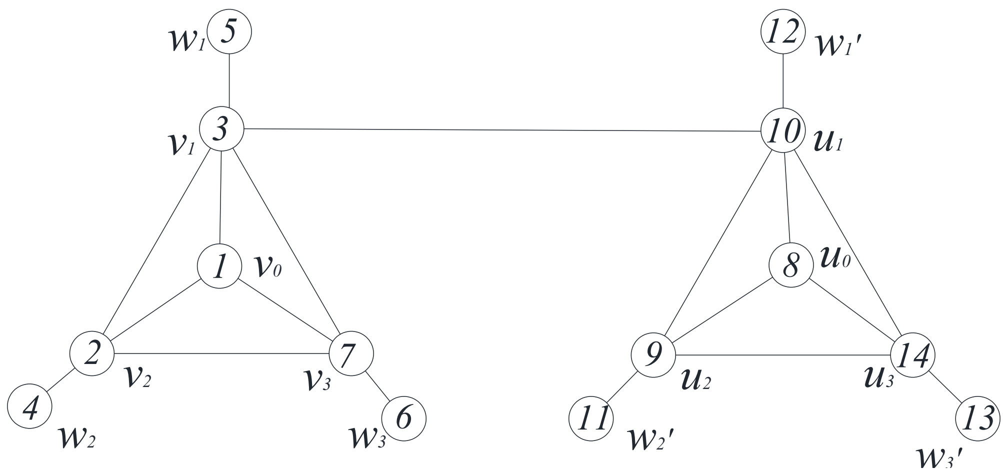

Figure 4.

Definition 3.13. The closed helm CH_n is the graph obtained from a helm H_n by joining each pair of consecutive pendant vertices (by an edge) to form a cycle.

Theorem 3.14. *The path union of two copies of closed helm CH_n ($n \in \mathbb{N}, n \geq 3$) is a SDCG.*

Proof. Let $G = P(2.CH_n)$ be the path union of two copies of closed helm CH_n. Let $v_1, v_2, \ldots, v_n$ be the internal vertices, $w_1, w_2, \ldots, w_n$ be the external vertices of CH_n and let $u_1 = v_{n+1}, u_2 = v_{n+2}, \ldots, u_n = v_{2n}$ be the internal vertices, $w'_1, w'_2, \ldots, w'_n$ be the external vertices of another CH_n. The vertex w_1 is connected to w'_1 by an edge P_1 which collectively makes the path union of 2 copies of closed helm.
We characterize labeling function $T^* : V(G) \to \{1, 2, 3, \ldots, 4n + 2\}$ as follows.
Case 1: For $n \equiv 0, 1, 2, 3 (mod\, 4)$
In this case $|V(G)| = 4n + 2$ and $|E(G)| = 8n + 1$.
Here we characterize labeling T^* as follows.

$$\begin{aligned}
T^*(v_0) &= 1. \\
T^*(v_j) &= 2j \quad 1 \leq j \leq n. \\
T^*(w_j) &= 2j + 1 \quad 1 \leq j \leq n. \\
T^*(u_0) &= 2n + 2. \\
T^*(u_j) &= 2n + 2j + 1 \quad 1 \leq j \leq n. \\
T^*(w'_j) &= 2n + 2j + 2 \quad 1 \leq j \leq n.
\end{aligned}$$

In view of the above labeling pattern, we observe that,

Cases of n	Edge conditions
$n \equiv 0, 2, 1, 3 (mod\, 4)$	$e^*_T(1) = \lfloor \frac{8n+1}{2} \rfloor, e^*_T(0) = \lceil \frac{8n+1}{2} \rceil$

Clearly, it satisfies the condition $|e_{T^*}(0) - e_{T^*}(1)| \leq 1$.
Hence, the path union of two copies of closed helm CH_n is a SDCG. □

Example 3.15. SDCL of the path union of two copies of CH_7 is shown in Figure 5.

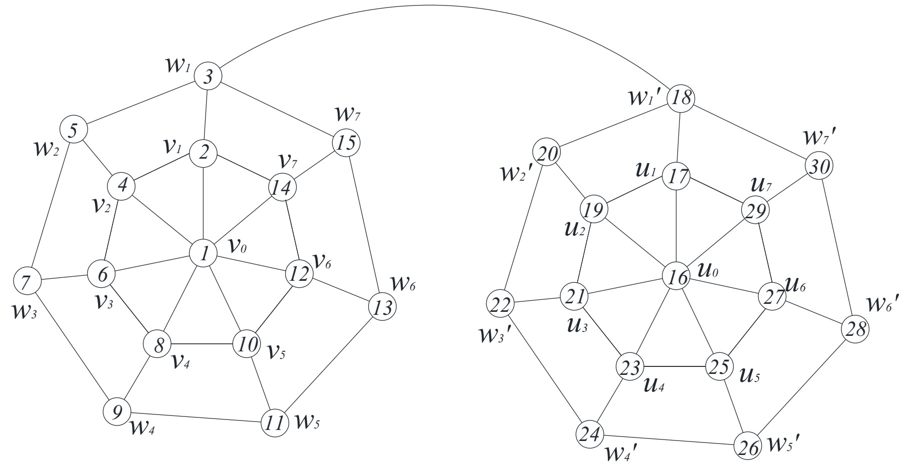

Figure 5.

In closing, we have investigated path union of two copies of cycle, cycle with triangle, wheel, helm and closed helm are SDCG. A similar investigation can be carried out for some other graph families.

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