Palestine Journal of Mathematics

Vol. 12(4)(2023) , 133–142

On $(n-1,n)$-φ-second submodules

S. Çeken, S. Koç and Ü. Tekir

Communicated by Ayman Badawi

MSC 2010 Classifications: Primary 13C05; Secondary 13C13.

Keywords and phrases: $(n-1,n)$-φ-second submodule, $(n-1,n)$-φ-prime submodule, n-absorbing ideal, $(n-1,n)$-m-almost second submodule, $(n-1,n)$-weak second submodule.

The authors would like to thank the reviewers and editor for their constructive comments and valuable suggestions that improved the quality of our paper.

Abstract *Let R be a commutative ring with identity, M be an R-module, $n \geq 2$ be a positive integer and $\varphi : S(M) \longrightarrow S(M)$ be a function where $S(M)$ is the set of all submodules of M. In this paper we introduce and study the concept of $(n-1,n)$-φ-second submodule. We call a non-zero submodule N of M as an $(n-1,n)$-φ-second submodule if $(a_1...a_{n-1})N \subseteq K$ and $(a_1...a_{n-1})\varphi(N) \nsubseteq K$, where $a_1,...,a_{n-1} \in R$ and K is a submodule of M, imply either $a_1...a_{n-1} \in ann_R(N)$ or $(a_1...a_{i-1}a_{i+1}...a_{n-1})N \subseteq K$ for some $i \in \{1,...,n-1\}$. We give a number of results concerning this submodule class. We characterize modules with the property that for some φ, every non-zero submodule is $(n-1,n)$-φ-second. We show that under some assumptions strongly $(n-1)$-absorbing second submodules and $(n-1,n)$-φ-second submodules coincide. We also focus on $(2,3)$-φ-second submodules and give some special results concerning them.*

1 Introduction

Prime ideals play a central role in commutative ring theory and algebraic geometry. In the literature, there are a number of generalizations of prime ideals in commutative rings (see for example [1], [2], [3], [14], [22], [24]). One of the generalization of prime ideals is the concept of n-absorbing ideal which was introduced in [2]. Let R be a commutative ring with identity and n be a positive integer. A proper ideal I of R is called an n-absorbing ideal if whenever $a_1...a_{n+1} \in I$ for $a_1,...,a_{n+1} \in R$, then $a_1...a_{i-1}a_{i+1}...a_{n+1} \in I$ for some $i \in \{1,...,n+1\}$ [2]. In [23], this concept was generalized to the concept of $(n-1,n)$-ϕ-prime ideal as in the following way. Let $\phi : S(R) \longrightarrow S(R) \cup \{\emptyset\}$ be a function where $S(R)$ is the set of all ideals of R. A proper ideal I of R is called an $(n-1,n)$-ϕ-prime ideal if $a_1...a_n \in I \backslash \phi(I)$, for $a_1,...,a_n \in R$, implies $a_1...a_{i-1}a_{i+1}...a_n \in I$ for some $i \in \{1,...,n\}$.

Prime submodules are module theoretic versions of prime ideals. The class of prime submodules has an important role in commutative ring theory as it gives characterizations of important ring classes such as Dedekind domains, Prüfer domains, arithmetical rings. The concept of prime submodule was first introduced in 1965 by E. H. Feller and E. W. Swokowski [26]. Let R be a commutative ring with non-zero identity. A proper submodule P of M is called a prime submodule if whenever $rm \in P$, where $r \in R$, $m \in M$, we have either $r \in (P :_R M)$ or $m \in P$. If P is a prime submodule of M, then $p = (P :_R M)$ is a prime ideal of R and in this case P is called a p–prime submodule [30]. If (0) is a prime submodule of M, then M is called a prime module.

Besides prime submodules, module theoretic versions of generalizations of prime ideals have been investigated since the begining of 2000s (see for example [12], [13], [21], [23], [27], [29], [31], [32], [33], [36]). Let R be a commutative ring with non-zero identity and M be an R-module. A proper submodule N of M is called an n-absorbing submodule if whenever $a_1...a_nm \in N$ for $a_1,...,a_n \in R$ and $m \in M$, then either $a_1...a_n \in (N :_R M)$ or there exists $i \in \{1,...,n\}$ such that $a_1...a_{i-1}a_{i+1}...a_nm \in N$ [21]. This notion was generalized in [23] as follows. Let $n \geq 2$ be a positive integer, $\phi : S(M) \longrightarrow S(M) \cup \{\emptyset\}$ be a function and P be a proper submodule of M. P is called an $(n-1,n)$-ϕ-prime submodule if $a_1...a_{n-1}x \in P\backslash\phi(P)$, for $a_1,...,a_{n-1} \in R$, $m \in M$, implies either $a_1...a_{n-1} \in (P :_R M)$ or $a_1...a_{i-1}a_{i+1}...a_{n-1}x \in P$ for some $i \in \{1,...,n-1\}$ [23]. Clearly, every $(n-1)$-absorbing submodule of M is $(n-1,n)$-ϕ-prime for any function ϕ on $S(M)$.

Second submodules of modules over commutative rings were introduced in [35] as the dual notion of prime submodules. According to this definition a non-zero submodule N of an R-module M is said to be a second submodule if for all $r \in R$, either $rN = 0$ or $rN = N$. If N is a second submodule of M, *then $p = ann_R(N)$ is a prime ideal of R. In this case, N is called a p-second submodule of M [35]. In recent years, second submodules have attracted attention of many researchers and it has been understood that this submodule class has an important role in determining characterizations of modules and rings (see for example [8], [9], [15], [16]). Along with the increased work on the second submodules, generalization of these submodules has begun to be investigated and it has been seen that these generalized second submodules also have interesting and important algebraic properties (see for example [6], [11], [17], [18], [19]). One of the generalization of second submodules is the concept of strongly n-absorbing second submodule which was introduced in [11]. Let R be a commutative ring with identity, M be an R-module and n be a positive integer. A non-zero submodule N of M is called a strongly n-absorbing second submodule if whenever $a_1...a_nN \subseteq K$ for $a_1, ..., a_n \in R$ and a submodule K of M, then either $a_1...a_n \in ann_R(N)$ or $a_1...a_{i-1}a_{i+1}...a_nN \subseteq K$ for some $i \in \{1, ..., n\}$ [11]. In this paper we extend this notion to $(n-1, n)$-φ-second submodules as follows. Let M be an R-module, $n \geq 2$ be a positive integer and N be a non-zero submodule of M. We call N as an $(n-1, n)$-φ-second submodule of M if $(a_1...a_{n-1})N \subseteq K$ and $(a_1...a_{n-1})\varphi(N) \nsubseteq K$, where $a_1, ..., a_{n-1} \in R$ and K is a submodule of M, imply either $a_1...a_{n-1} \in ann_R(N)$ or $(a_1...a_{i-1}a_{i+1}...a_{n-1})N \subseteq K$ for some $i \in \{1, ..., n-1\}$. Let $\varphi_M : S(M) \longrightarrow S(M)$ be the function defined by $\varphi_M(L) = M$ for every $L \in S(M)$. Then an $(n-1, n)$-φ_M-second submodule of M is said to be an $(n-1, n)$-weak second submodule of M. Let $m \geq 2$ be an integer and $\varphi_m : S(M) \longrightarrow S(M)$ be the function defined by $\varphi_m(L) = (L :_M ann_R(L)^{m-1})$ for every $L \in S(M)$. Then an $(n-1, n)$-φ_m-second submodule of M is said to be an $(n-1, n)$-m-almost second submodule of M. In particular, for $m = 2$, an $(n-1, n)$-2-almost second submodule of M is called an $(n-1, n)$-almost second submodule of M.*

Throughout this paper all rings will be commutative with non-zero identity and all modules will be unital left modules. Unless otherwise stated R will denote a ring. In the rest of the paper $S(M)$ will denote the set of all submodules of an R-module M and $\varphi : S(M) \to S(M)$ will be a function.

In Section 2, we give some characterizations of $(n-1, n)$-φ-second submodules and investigate their relationships with strongly $(n-1)$-absorbing second submodules. In Theorem 2.3, we give some equivalent conditions for a non-zero submodule Q of an R-module M to be $(n-1, n)$-φ-second. For an element $a \in R$ with $(0) \neq (0 :_M a) = aM$, we prove that $(0 :_M a)$ is an $(n-1, n)$-almost second submodule of M if and only if it is a strongly $(n-1)$-absorbing second submodule of M (see Theorem 2.4). Let $R = R_1 \times ... \times R_n$, $M = M_1 \times ... \times M_n$ where R_i is a ring, M_i is an R_i-module for $i = 1, ..., n$ and let $\varphi : S(M) \longrightarrow S(M)$ be a function. We investigate the structure of $(n-1, n)$-φ-second submodules of M (see Theorems 2.6, 2.9). We characterize R-modules M for which every non-zero submodule is $(n-1, n)$-weak second and $(n-1, n)$-n-almost second (see Theorems 2.10, 2.13). Let m, n be positive integers with $3 \leq m < n$, $R = R_1 \times ... \times R_m$ and $M = M_1 \times ... \times M_m$ where R_i is a ring and M_i is a non-zero Artinian R_i-module for each $i \in \{1, ..., m\}$. We investigate the structure of R-modules M in which every non-zero submodule is $(n-1, n)$-weak second (see Theorem 2.14). Let $m \geq 1$ and $n \geq 2$ be positive integers, $R = R_1 \times ... \times R_m$ and $M = M_1 \times ... \times M_m$ where (R_i, Q_i) is a local ring, M_i is an R_i-module. In Theorem 2.15, we give a condition for M to have the property that every non-zero submodule is $(n-1, n)$-weak second.

In Section 3 we give our attention on $(2, 3)$-φ-second submodules. We present their various characterizations and investigate relationships with other concepts. In Theorem 3.2, we give some equivalent conditions for a non-zero submodule N of a module M over a um-ring to be $(2, 3)$-φ-second submodule. In Theorem 3.5, we prove that if N is a $(2, 3)$-φ-second submodule of M which is not strongly 2-absorbing second, then $ann_R(N)^2\varphi(N) \subseteq N$. This theorem has many consequences. In particular, by using this theorem, we show that if N is a $(2, 3)$-φ-second submodule of M that is not strongly 2-absorbing second, then $\sqrt{ann_R(N)} = \sqrt{ann_R(\varphi(N))}$. Additionally, if M is a finitely generated comultiplication R-module, then we show that $sec(N) = sec(\varphi(N))$ for such a submodule N where $\sec(N)$ *is the sum of all second submodules of N (see Corollary 3.8). Let N be a $(2, 3)$-φ-second submodule of M and suppose that $IJN \subseteq K$, $IJ\varphi(N) \nsubseteq K$ for some ideals I, J of R and a submodule K of M. Under a condition, we prove that $IJ \subseteq ann_R(N)$ or $IN \subseteq K$ or $JN \subseteq K$ (see Theorem 3.10). Let $R = F_1 \times F_2 \times F_3$ and $M = M_1 \times M_2 \times M_3$ where F_i is a field and M_i is a non-zero F_i-vector space for each $i \in \{1, 2, 3\}$.*

2 (*n-1*,*n*)-φ-Second Submodules

Definition 2.1. Let M be an R-module, $n \geq 2$ be a positive integer and N be a non-zero submodule of M. We call N as an $(n-1, n)$-φ-second submodule of M if $(a_1...a_{n-1})N \subseteq K$ and $(a_1...a_{n-1})\varphi(N) \nsubseteq K$, where $a_1, ..., a_{n-1} \in R$ and K is a submodule of M, imply either $a_1...a_{n-1} \in ann_R(N)$ or $(a_1...a_{i-1}a_{i+1}...a_{n-1})N \subseteq K$ for some $i \in \{1, ..., n-1\}$.

Let $\varphi_M : S(M) \longrightarrow S(M)$ be the function defined by $\varphi_M(L) = M$ for every $L \in S(M)$. Then an $(n-1, n)$-φ_M-second submodule of M is said to be an $(n-1, n)$-weak second submodule of M.

Let $m \geq 2$ be an integer and $\varphi_m : S(M) \longrightarrow S(M)$ be the function defined by $\varphi_m(L) = (L :_M ann_R(L)^{m-1})$ for every $L \in S(M)$. Then an $(n-1, n)$-φ_m-second submodule of M is said to be an $(n-1, n)$-

m-almost second submodule of M. In particular, for $m=2$, an $(n-1,n)$-2-almost second submodule of M is called an $(n-1,n)$-almost second submodule of M.

Throughout this paper we will assume that $N \subseteq \varphi(N)$ for every submodule N of an R-module M.

Example 2.2. (i) Let M be an R-module and $\varphi : S(M) \longrightarrow S(M)$ be any function. Then M is trivially an $(n-1,n)$-φ-second submodule.

(ii) Let $R=\mathbb{Z}$ and $M=\mathbb{Z}_{p_1p_2\dots p_{n-1}}$, where $p_1,p_2,\dots,p_{n-1}$ are distinct prime numbers. Suppose that $\varphi : S(M) \longrightarrow S(M)$ be any function. Now, we will show that every nonzero submodule N of M is an $(n-1,n)$-φ-second submodule. Let N be a nonzero submodule of M and K a submodule of M. Now, we may assume that N is proper. Then there exists $1<d<m=p_1p_2\dots p_{n-1}$ such that $d|m$ and $N=dM$. This implies that $d=p_1\dots\widehat{p_{i_1}}\widehat{p_{i_2}}\dots\widehat{p_{i_k}}\dots p_{n-1}$, where $p_1\dots\widehat{p_{i_1}}\widehat{p_{i_2}}\dots\widehat{p_{i_k}}\dots p_{n-1}$ denotes the product of all integers of the set $\{p_1,p_2,\dots,p_{n-1}\}-\{p_{i_1},p_{i_2},\dots,p_{i_k}\}$ and $1\le k\le n-2$. Let $a_1,a_2,\dots,a_{n-1}\in R$ such that $a_1a_2\dots a_{n-1}N\subseteq K$ but $a_1a_2\dots a_{n-1}\varphi(N)\nsubseteq K$. Now, we may assume that K is a nonzero proper submodule of M such that $N\nsubseteq K$. Then similarly, we have $1<t<m$ such that $t=p_1\dots\widehat{p_{j_1}}\widehat{p_{j_2}}\dots\widehat{p_{j_r}}\dots p_{n-1}$ and $K=tM$, where $1\le r\le n-2$. Now, it is clear that $(K:N)$ is an $(n-2)$-absorbing ideal of R. Since $a_1a_2\dots a_{n-1}N\subseteq K$, we get $a_1a_2\dots a_{n-1}\in(K:N)$ and so there exists $i\in\{1,2,\dots,n-1\}$ such that $a_1a_2\dots a_{i-1}a_{i+1}\dots a_{n-1}\in(K:N)$. Thus we conclude that $a_1a_2\dots a_{i-1}a_{i+1}\dots a_{n-1}N\subseteq K$. Therefore, N is an $(n-1,n)$-φ-second submodule.

Theorem 2.3. *Let Q be a non-zero submodule of an R-module M. Then the following are equivalent.*

(1) *Q is an $(n-1,n)$-φ-second submodule of M.*

(2) *If $a_1...a_{n-1}\notin ann_R(Q)$ for $a_1,...,a_{n-1}\in R$, then $a_1...a_{n-1}Q=a_1...a_{n-1}\varphi(Q)$ or $a_1...a_{n-1}Q=a_1...a_{i-1}a_{i+1}...a_{n-1}Q$ for some $i\in\{1,...,n-1\}$.*

(3) *For $a_1,..,a_{n-2}\in R$ and a submodule K of M with $a_1...a_{n-2}Q\nsubseteq K$;*
$(K:_R a_1...a_{n-2}Q)=\cup_{i=1}^{n-2}(K:_R a_1...a_{i-1}a_{i+1}...a_{n-2}Q)\cup ann_R(a_1...a_{n-2}Q)\cup(K:_R a_1...a_{n-2}\varphi(Q))$.

Proof. (1) $\Longrightarrow$ (2) This implication follows by using the inclusion $a_1...a_{n-1}Q\subseteq a_1...a_{n-1}Q$ for $a_1,...,a_{n-1}\in R$.

(2) $\Longrightarrow$ (1) Straightforward.

(1) $\Longrightarrow$ (3) Let $a_1,...a_{n-2}\in R$ and K be a submodule of M with $a_1...a_{n-2}Q\nsubseteq K$. Let $b\in(K:_R a_1...a_{n-2}Q)$. Then $ba_1...a_{n-2}Q\subseteq K$. If $ba_1...a_{n-2}\varphi(Q)\nsubseteq K$, then (1) implies that $ba_1...a_{n-2}\in ann_R(Q)$ or $ba_1...a_{i-1}a_{i+1}...a_{n-2}Q\subseteq K$ for some $i\in\{1,...,n-2\}$. If $ba_1...a_{n-2}\varphi(Q)\subseteq K$, then $b\in(K:_R a_1...a_{n-2}\varphi(Q))$. Thus $(K:_R a_1...a_{n-2}Q)\subseteq\cup_{i=1}^{n-2}(K:_R a_1...a_{i-1}a_{i+1}...a_{n-2}Q)\cup ann_R(a_1...a_{n-2}Q)\cup(K:_R a_1...a_{n-2}\varphi(Q))$. The other inclusion always holds since we assume $Q\subseteq\varphi(Q)$.

(3) $\Longrightarrow$ (1) Let $a_1...a_{n-1}Q\subseteq K$ and $a_1...a_{n-1}\varphi(Q)\nsubseteq K$ for $a_1,...,a_{n-1}\in R$ and a submodule K of M. If $a_1...a_{n-2}Q\subseteq K$, then we are done. Let $a_1...a_{n-2}Q\nsubseteq K$. Since $a_{n-1}\in(K:_R a_1...a_{n-2}Q)$, (3) implies that
$a_1...a_{i-1}a_{i+1}...a_{n-1}Q\subseteq K$ for some $i\in\{1,..,n-1\}$ or $a_1...a_{n-2}a_{n-1}\in ann_R(Q)$. Thus Q is an $(n-1,n)$-φ-second submodule of M. □

Theorem 2.4. *Let M be an R-module, $a\in R$, $(0:_M a)\neq(0)$ and $(0:_M a)=aM$. Then $(0:_M a)$ is an $(n-1,n)$-almost second submodule of M if and only if it is a strongly $(n-1)$-absorbing second submodule of M.*

Proof. If $(0:_M a)$ is strongly $(n-1)$-absorbing second submodule, then clearly it is $(n-1,n)$-almost second submodule. Suppose that $(0:_M a)$ is an $(n-1,n)$-almost second submodule of M. Let $b_1...b_{n-1}(0:_M a)\subseteq K$ for $b_1,...,b_{n-1}\in R$ and a submodule K of M. If $b_1...b_{n-1}((0:_M a):_M ann_R(0:_M a))=b_1...b_{n-1}M\nsubseteq K$, then we are done. So we may assume that $b_1...b_{n-1}M\subseteq K$. Now, $(b_1+a)b_2...b_{n-1}(0:_M a)\subseteq K$. If $(b_1+a)b_2...b_{n-1}M\nsubseteq K$, then $(b_1+a)...b_{i-1}b_{i+1}...b_{n-1}(0:_M a)\subseteq K$ for some $i\in\{1,...,n-1\}$ or $(b_1+a)b_2...b_{n-1}\in ann_R(0:_M a)$. It follows that $b_1...b_{i-1}b_{i+1}...b_{n-1}(0:_M a)\subseteq K$ for some $i\in\{1,...,n-1\}$ or $b_1b_2...b_{n-1}\in ann_R(0:_M a)$ as desired. So we may assume that $(b_1+a)b_2...b_{n-1}M\subseteq K$. Then $b_2...b_{n-1}aM\subseteq K$ as $b_1...b_{n-1}M\subseteq K$. Since $(0:_M a)=aM$, we have $b_2...b_{n-1}(0:_M a)\subseteq K$. Thus $(0:_M a)$ is an $(n-1)$-absorbing second submodule of M. □

Proposition 2.5. *Let $R=R_1\times R_2$, $M=M_1\times M_2$ where R_i is a ring, M_i is an R_i-module for $i=1,2$ and let $\varphi : S(M)\longrightarrow S(M)$ be a function. Suppose that Q_1 is an $(n-1,n)$-weak-second submodule of M_1 such that $\varphi(Q_1\times\{0\})\subseteq M_1\times\{0\}$. Then $Q_1\times\{0\}$ is an $(n-1,n)$-φ-second submodule of M.*

Proof. Let $(a_1,b_1)...(a_{n-1},b_{n-1})(Q_1\times\{0\})\subseteq N_1\times N_2$ and $(a_1,b_1)...(a_{n-1},b_{n-1})\varphi(Q_1\times\{0\})\nsubseteq N_1\times N_2$ where $(a_1,b_1),...,(a_{n-1},b_{n-1})\in R$, $N_1\le M_1$, $N_2\le M_2$. Then $(a_1,b_1)...(a_{n-1},b_{n-1})(M_1\times\{0\})\nsubseteq N_1\times N_2$ by hypothesis. So $(a_1...a_{n-1})Q_1\subseteq N_1$ and $(a_1...a_{n-1})M_1\nsubseteq N_1$. Since Q_1 is $(n-1,n)$-weak second, we have $a_1...a_{n-1}\in ann_{R_1}(Q_1)$ or $(a_1...a_{i-1}a_{i+1}...a_{n-1})Q_1\subseteq N_1$ for some $i\in\{1,...,n-1\}$. These imply that

$(a_1,b_1)...(a_{n-1},b_{n-1}) \in ann_{R_1}(Q_1) \times R_2 = ann_R(Q_1 \times \{0\})$ or
$(a_1,b_1)...(a_{i-1},b_{i-1})(a_{i+1},b_{i+1})...(a_{n-1},b_{n-1})(Q_1 \times \{0\}) \subseteq N_1 \times N_2$. Thus $Q_1 \times \{0\}$ is an $(n-1,n)$-φ-second submodule of M. □

Theorem 2.6. *Let $R = R_1 \times ... \times R_n$, $M = M_1 \times ... \times M_n$ where R_i is a ring, M_i is an R_i-module for $i = 1,...,n$ and let $\varphi : S(M) \longrightarrow S(M)$ be a function. Suppose that $Q = Q_1 \times ... \times Q_n$ is an $(n-1,n)$-φ-second submodule of M where Q_i is a submodule of M_i for $i = 1,...,n$. Let $\psi_i : S(M_i) \longrightarrow S(M_i)$ be a function and $\varphi(Q) = \psi_1(Q_1) \times ... \times \psi_n(Q_n)$. Then Q_j is an $(n-1,n)$-ψ_j-second submodule of M_j for each j with $Q_j \neq (0)$.*

Proof. Let $Q_j \neq (0)$, $a_1...a_{n-1}Q_j \subseteq K$ and $a_1...a_{n-1}\psi_j(Q_j) \nsubseteq K$. Then
$(1,..,1,a_1,1,...,1)(1,..,1,a_2,1,...,1)...(1,..,1,a_{n-1},1,...,1)(Q_1 \times ... \times Q_j \times ... \times Q_n) \subseteq M_1 \times ... \times K \times ... \times M_n$
and
$(1,..,1,a_1,1,...,1)(1,..,1,a_2,1,...,1)...(1,..,1,a_{n-1},1,...,1)(\psi_1(Q_1) \times ... \times \psi_j(Q_j) \times ... \times \psi_n(Q_n))$
$= (1,..,1,a_1,1,...,1)(1,..,1,a_2,1,...,1)...(1,..,1,a_{n-1},1,...,1)\varphi(Q_1 \times ... \times Q_j \times ... \times Q_n) \nsubseteq M_1 \times ... \times K \times ... \times M_n$ where $a_1,...,a_{n-1}$ are in the j th components. Since Q is $(n-1,n)$-φ-second, we have $(1,...,a_1...a_{n-1},...,1) \in ann_R(Q)$ or $(1,...1,a_1...a_{i-1}a_{i+1}...a_n,1,...,1)Q_1 \times ... \times Q_n \subseteq M_1 \times ... \times K \times ... M_n$ for some $i \in \{1,...,n-1\}$. Thus we get that $a_1...a_{n-1} \in ann_{R_j}(Q_j)$ or $a_1...a_{i-1}a_{i+1}...a_nQ_j \subseteq K$ as needed. □

Corollary 2.7. *Let $R = R_1 \times ... \times R_n$, $M = M_1 \times ... \times M_n$ where R_i is a ring, M_i is an R_i-module for $i = 1,...,n$ and let $Q = Q_1 \times ... \times Q_n$ where Q_i is a submodule of M_i for $i = 1,...,n$. If Q is an $(n-1,n)$-m-almost second submodule of M, then Q_j is an $(n-1,n)$-m-almost second submodule of M_j for each j with $Q_j \neq (0)$.*

Proof. We have $\varphi_m(Q) = (Q :_M ann_R(Q)^{m-1}) = (Q_1 \times ... \times Q_n :_M ann_{R_1}(Q_1)^{m-1} \times ... \times ann_{R_n}(Q_n)^{m-1}) = (Q_1 :_{M_1} ann_{R_1}(Q_1)^{m-1}) \times ... \times (Q_n :_{M_n} ann_{R_n}(Q_n)^{m-1})$. So the result follows from Theorem 2.6. □

It is well know that annihilator $ann_R(N)$ of a second submodule N of an R-module M is a prime ideal. Now we present a new method for constructing $(n-1,n)$-ψ-prime ideal of a ring R, where $\psi : S(R) \longrightarrow S(R) \cup \{\emptyset\}$ is a function.

Proposition 2.8. *(i) Let M be an R-module $\varphi : S(M) \longrightarrow S(M)$ be a function. Suppose that N is an $(n-1,n)$-φ-second submodule and $\varphi^* : S(R) \longrightarrow S(R) \cup \{\emptyset\}$ is a function such that $\varphi^*(ann(N)) = ann(\varphi(N))$. Then $ann(N)$ is an $(n-1,n)$-φ^*-prime ideal of R.*

(ii) Suppose that M is a faithful R-module and N is an $(n-1,n)$-weak second submodule of M. Then $ann(N)$ is an $(n-1,n)$-weakly prime ideal of R.

Proof. (i) Let $a_1,a_2,\ldots,a_n \in R$ such that $a_1a_2\ldots a_n \in ann(N) - \varphi^*(ann(N))$. Then we have $a_1a_2\ldots a_nN = 0$ and $a_1a_2\ldots a_n\varphi(N) \neq 0$ since $\varphi^*(ann(N)) = ann(\varphi(N))$. This implies that $a_1a_2\ldots a_{n-1}N \subseteq (0 :_M a_n)$ and $a_1a_2\ldots a_{n-1}\varphi(N) \nsubseteq (0 :_M a_n)$. Since N is an $(n-1,n)$-φ-second submodule, we have either $a_1a_2\ldots a_{n-1} \in ann_R(N)$ or there exists $i \in \{1,2,\ldots,n-1\}$ such that $a_1a_2\ldots a_{i-1}a_{i+1}\ldots a_{n-1}N \subseteq (0 :_M a_n)$. Thus we conclude that $a_1a_2\ldots a_{n-1} \in ann_R(N)$ or
$a_1a_2\ldots a_{i-1}a_{i+1}\ldots a_{n-1}a_n \in ann_R(N)$. Hence, $ann(N)$ is an $(n-1,n)$-φ^*-prime ideal of R.

(ii) Suppose that M is a faithful R-module and N is an $(n-1,n)$-weak second submodule of M. Thus N is an $(n-1,n)$-φ_M-second submodule of M. Now, consider the function $\varphi^* : S(R) \to S(R)$ defined by $\varphi(I) = 0$ for each $I \in S(R)$. Note that $\varphi^*(ann(N)) = 0 = ann(\varphi_M(N)) = ann(M)$ since M is a faithful module. The rest follows from (i). □

Theorem 2.9. *Let $R = R_1 \times ... \times R_n$ and $M = M_1 \times ... \times M_n$ be a faithful R-module where R_i is a ring and M_i is a non-zero R_i-module for all $i = 1,...,n$. If Q is a proper $(n-1,n)$-weak second submodule of M such that $ann_R(Q) \neq (0)$, then $Q = Q_1 \times ... \times Q_{i-1} \times (0) \times Q_{i+1} \times ... \times Q_n$ for some $i \in \{1,...,n\}$ and if $Q_j \neq (0)$ for $j \neq i$, then Q_j is a strongly $(n-1)$-absorbing second submodule of M_j.*

Proof. Let $Q = Q_1 \times ... \times Q_n$ where Q_i is a submodule of M_i for $i \in \{1,...,n\}$. Then $(0) \neq ann_R(Q) = ann_{R_1}(Q_1) \times ... \times ann_{R_n}(Q_n)$ is a non-zero proper ideal of R. By Proposition 2.8-(ii), $ann_R(Q)$ is an $(n-1,n)$-weakly prime ideal of R. By [23, Lemma 3.6], $ann_{R_i}(Q_i) = R_i$ for some $i \in \{1,...,n\}$ and so $Q_i = (0)$. Thus $Q = Q_1 \times ... \times Q_{i-1} \times (0) \times Q_{i+1} \times ... \times Q_n$. Let $Q_j \neq (0)$ for $j \neq i$. We claim that Q_j is a strongly $(n-1)$-absorbing second submodule of M_j. Assume that $i < j$. Let $a_1,...,a_{n-1} \in R$ and X be a submodule of M_j such that $(a_1...a_{n-1})Q_j \subseteq X$. Then we have

$(0,...,1,...0,a_1...a_{n-1},0,...,0)(Q_1 \times ... \times Q_{i-1} \times (0) \times Q_{i+1} \times ... \times Q_n) \subseteq (0) \times ... \times (0) \times ... \times X \times ... \times (0)$
and

$(0,...,1,...0,a_1,0,...,0)...(0,...,1,...0,a_{n-1},0,...,0)(M_1 \times ... \times M_i \times ... \times M_j \times ... \times M_n) \nsubseteq (0) \times ... \times (0) \times ... \times X \times ...(0)$ as $M_i \neq (0)$. Since Q is an $(n-1,n)$-weak second submodule of M, we have $a_1...a_{n-1} \in ann_{R_j}(Q_j)$ or $(a_1...a_{k-1}a_{k+1}...a_{n-1})Q_j \subseteq X$ for some $k \in \{1,...,n-1\}$. Thus Q_j is a strongly $(n-1)$-absorbing second submodule of M_j. The proof for $j < i$ can be seen as a similar way. □

Note that the previous theorem is still valid under the condition that "every proper ideal of R is $(n-1,n)$-weakly prime of R".

In the following theorem, we give a characterization of simple modules in terms of $(n-1,n)$-weak second submodules.

Theorem 2.10. *Let $n \geq 2$, $R = R_1 \times ... \times R_n$ and $M = M_1 \times ... \times M_n$ where R_i is a ring and M_i is a non-zero R_i-module for all $i = 1, ..., n$. The following statements are equivalent:*

(i) Every non-zero submodule of M is $(n-1,n)$-weak second submodule.

(ii) M_i is a simple R_i-module for each $i \in \{1, ..., n\}$.

Proof. $(i) \Longrightarrow (ii)$: Assume that M_1 is not a simple R_1-module. So there exists a non-zero proper submodule Q_1 of M_1. By hypothesis, the submodule $Q = Q_1 \times M_2 \times ... \times M_n$ is an $(n-1,n)$-weak second submodule of M. We have

$(1,0,...,0)(Q_1 \times M_2 \times ... \times M_n) = (1,0,1,...,1)(1,1,0,1,...,1)...(1,1,...,0)(Q_1 \times M_2 \times ... \times M_n) \subseteq Q_1 \times (0) \times ... \times (0)$ and $(1,0,...,0)(M_1 \times M_2 \times ... \times M_n) = (1,0,1,...,1)(1,1,0,1,...,1)...(1,1,...,0)(M_1 \times M_2 \times ... \times M_n) \not\subseteq Q_1 \times (0) \times ... \times (0)$. Since Q is $(n-1,n)$-weak second, we have two cases:

Case 1: $(1,0,...,0) \in ann_R(Q)$ which gives the contradiction that $Q_1 = (0)$

Case 2: $M_j = (0)$ for some $j \in \{2, ..., n\}$ which is again a contradiction.

Thus M_1 is a simple R_1-module. By a similar argument, we can prove that M_j is a simple R_j-module for all $j \in \{2, ..., n\}$.

$(ii) \Longrightarrow (i)$: Suppose that M_i is a simple R_i-module for each $i \in \{1, ..., n\}$. Let $N = N_1 \times N_2 \times \ldots \times N_n$ be a nonzero submodule of M and $K = K_1 \times K_2 \times \ldots \times K_n$ be a submodule of M. Take $x_i = (a_{i1}, a_{i2}, \ldots, a_{in}) \in R$ for $i = 1, 2, \ldots, n-1$ such that $x_1x_2 \ldots x_{n-1}N \subseteq K$ and $x_1x_2 \ldots x_{n-1}M \not\subseteq K$. We may assume that $N \not\subseteq K$. Since M_i is a simple R_i-module for each $i \in \{1, ..., n\}$ and $N \not\subseteq K$, we get $N_t = M_t$ and $K_t = 0$ for some $t \in \{1, ..., n-1\}$. Thus we have $(K : N) = (K_1 : N_1) \times (K_2 : N_2) \times \ldots \times ann_{R_t}(M_t) \times \ldots \times (K_n : N_n)$. Since M_i is simple, it is clear that $ann_{R_i}(M_i)$ is a prime ideal of R_i. Also, note that $(K_i : N_i)$ is either R_i or $ann_{R_i}(M_i)$. If $(K_i : N_i) = R_i$ for all $i \neq t$, then $(K : N) = R_1 \times R_2 \times \ldots \times ann_{R_t}(M_t) \times \ldots \times R_n$ is a prime ideal so is $(n-2)$-absorbing. This implies that there exists $i \in \{1, 2, \ldots, n-1\}$ such that $x_1x_2 \ldots x_{i-1}x_{i+1} \ldots x_{n-1} \in (K : N)$, namely $x_1x_2 \ldots x_{i-1}x_{i+1} \ldots x_{n-1}N \subseteq K$. If $(K_i : N_i) \neq R_i$ for all $i \neq t$, then $N_i \not\subseteq K_i$ for all i $(1 \leq i \leq n\}$. Since M_i is a simple R_i-module, $K_i = (0)$ for all i $(1 \leq i \leq n)$. Thus $K = (0)$ and so $x_1...x_{n-1} \in ann_R(N)$. If at least two of $(K_i : N_i)$'s equal R_i, the $(K :_R N)$ is an $(n-2)$-absorbing ideal of R by [2, Corollary 4.8 and Theorem 2.1]. Since $x_1x_2 \ldots x_{n-1} \in (K : N)$, there exists $i \in \{1, 2, \ldots, n-1\}$ such that $x_1x_2 \ldots x_{i-1}x_{i+1} \ldots x_{n-1} \in (K : N)$, namely $x_1x_2 \ldots x_{i-1}x_{i+1} \ldots x_{n-1}N \subseteq K$. Suppose only one of $(K_j : N_j)$ equals R_j. Then, by using the simplicity of each M_i, one can show that $K = 0 \times ...K_j \times 0 \times ... \times 0$ and $N = M_1 \times ... \times N_j \times ... \times M_n$. Since $x_1x_2 \ldots x_{n-1}N \subseteq K$ and $x_1x_2 \ldots x_{n-1}M \not\subseteq K$, we can see that $K_j \neq M_j$. Thus $K_j = (0)$ and so $K = (0)$. This shows that $x_1...x_{n-1} \in ann_R(N)$. Hence, N is an $(n-1,n)$-weak second submodule of M. □

Theorem 2.11. *Let $R = F_1 \times ... \times F_n$ and $M = M_1 \times ... \times M_n$ where $n \geq 2$, F_i is a field and M_i is a non-zero F_i-vector space for each $i \in \{1, ..., n\}$. Every non-zero submodule of M is $(2,3)$-weak second if and only if* $\dim(M_i) = 1$ *for all $i \in \{1, 2, 3\}$.*

Proof. Note that a vector space M_i over a field F_i is a simple module if and only if $\dim(M_i) = 1$. The rest follows from Theorem 2.10. □

Let M be an R-module and N, K be submodules of M. The coproduct of N and K is defined by $(0 :_M ann_R(N)ann_R(K))$ and it is denoted by $C(NK)$ [5].

Recall from [7] that an R-module M is said to be fully coidempotent if $N = C(N^2)$ for every submodule N of M.

Lemma 2.12. *An R-module M is fully coidempotent if and only if $N = (N :_M ann_R(N)^m)$ for every submodule N of M and positive integer m.*

Proof. Suppose that M is a fully coidempotent R-module. Let N be a submodule of M and m be a positive integer. It is sufficient to show that $N = (N :_M ann_R(N))$. We have $N = C(N^2) = (0 :_M ann_R(N)^2)$. Also, $N \subseteq (0 :_M ann_R(N))$ implies that $(N :_M ann_R(N)) \subseteq ((0 :_M ann_R(N)) :_M ann_R(N)) = (0 :_M ann_R(N)^2) = N$ and so $(N :_M ann_R(N)) \subseteq N$. Since the other inclusion always holds we have $(N :_M ann_R(N)) = N$ and hence $N = (N :_M ann_R(N)^m)$ for all $m \geq 1$.

Conversely, suppose that $N = (N :_M ann_R(N)^m)$ for every submodule N of M and positive integer m. Then $N = (N :_M ann_R(N))$. We have

$C(N^2) = (0 :_M ann_R(N)^2) \subseteq (N :_M ann_R(N)^2) = ((N :_M ann_R(N)) :_M ann_R(N)) = (N :_M ann_R(N)) = N$. Thus we get that $C(N^2) \subseteq N$ and so $N = C(N^2)$. □

Theorem 2.13. *Let $R = R_1 \times ... \times R_n$ and $M = M_1 \times ... \times M_n$ where R_i is a ring, $0 \neq M_i$ is an R_i-module for all $i \in \{1, ..., n\}$ and $n \geq 2$. Then every non-zero submodule of M is $(n-1,n)$-n-almost second if and only if M is a fully coidempotent R-module.*

Proof. $(\Longleftarrow)$ Clear.

$(\Longrightarrow)$ Suppose that every non-zero submodule of M is $(n-1,n)$-n-almost second. It is sufficient to show that M_i is a fully coidempotent R_i-module for each $i \in \{1, ..., n\}$. Suppose on the contrary that M_1 is not fully coidempotent. So there exists a submodule N_1 of M_1 such that $(N_1 :_{M_1} ann_{R_1}(N_1)^{n-1}) \not\subseteq N_1$. By hypothesis, $N := N_1 \times M_2 \times ... \times M_n$ is $(n-1,n)$-n-almost second submodule of M. We have

$$(1,0,...,0)N = (1,0,1,...,1)(1,1,0,1,...,1)...(1,1,...,1,0)N \subseteq N_1 \times (0) \times ... \times (0)$$

and $(1,0,1,...,1)(1,1,0,1,...,1)...(1,1,...,1,0)(N :_M ann_R(N)^{n-1}) \not\subseteq N_1 \times (0) \times ... \times (0)$. Since N is $(n-1,n)$-n-almost second, we have $1 \in ann_{R_1}(N_1)$ or $M_i = (0)$ for some $i \in \{2, ..., n\}$ which are both contradictions. Similarly, M_i is a fully coidempotent R_i-module for each $i \in \{2, ..., n\}$. This implies that M is a fully coidempotent R-module. □

Theorem 2.14. *Let m, n be positive integers with $3 \leq m < n$, $R = R_1 \times ... \times R_m$ and $M = M_1 \times ... \times M_m$ where R_i is a ring and M_i is a non-zero Artinian R_i-module for each $i \in \{1, ..., m\}$. Let J_i denote the Jacobson radical of R_i for each $i \in \{1, ..., m\}$. If every non-zero submodule of M is $(n-1,n)$-weak second, then $J_i^{n-m} M_i = (0)$ for each $i \in \{1, ..., m\}$.*

Proof. Assume that $J_1^{n-m} M_1 \neq (0)$. Then there exist $a_1, ..., a_{n-m} \in J_1$ such that $a_1 ... a_{n-m} M_1 \neq (0)$. By hypothesis, $Q = (0 :_{M_1} a_1 ... a_{n-m} R_1) \times M_2 \times ... \times M_m$ is an $(n-1,n)$-weak second submodule of M. We have
$(a_1 ... a_{n-m}, 0, ..., 0, 1)Q = (a_{11}, ..., a_{1m})...(a_{(n-1)1}, ... a_{(n-1)m})(0 :_{M_1} a_1 ... a_{n-m} R) \times M_2 \times ... \times M_m \subseteq (0) \times (0) \times ... \times M_m$ and $(a_{11}, ..., a_{1m})...(a_{(n-1)1}, ... a_{(n-1)m}) M_1 \times ... \times M_m \not\subseteq (0) \times (0) \times ... \times M_m$ where $a_{k1} = a_k$ for $1 \leq k \leq n-m$, $a_{(n-m+t)(t+1)} = 0$ for $1 \leq t \leq m-2$, in other places $a_{ij} = 1$. Since Q is $(n-1,n)$-weak second, we get the following three cases:

Case 1: $(a_1 ... a_{n-m}, 0, ..., 0, 1) \in ann_R(Q)$ which gives the contradiction that $M_m = (0)$.

Case 2: $M_j = (0)$ for some $2 \leq j \leq m-1$ which is a contradiction.

Case 3: $a_1 ... a_{j-1} a_{j+1} ... a_{n-m}(0 :_{M_1} a_1 ... a_{n-m}) = (0)$ which implies that $(0 :_{M_1} a_1 ... a_{j-1} a_{j+1} ... a_{n-m} R_1) = (0 :_{M_1} a_1 ... a_{j-1} a_j a_{j+1} ... a_{n-m} R_1) = ((0 :_{M_1} a_1 ... a_{j-1} a_{j+1} ... a_{n-m}) :_{M_1} a_j R)$. Since M_1 is an Artinian R_1-module and $a_j R \subseteq J_1$, [28, Proposition 3.5] implies that $M_1 = (0 :_{M_1} a_1 ... a_{j-1} a_{j+1} ... a_{n-m} R_1)$, i.e., $a_1 ... a_{j-1} a_{j+1} ... a_{n-m} M_1 = (0)$, a contradiction. Thus $J_1^{n-m} M_1 = (0)$. By a similar argument, we can prove that $J_i^{n-m} M_i = (0)$ for each $i \in \{2, ..., m\}$. □

Let (R, Q) be a local ring and M be an R-module. If t is the smallest positive integer such that $Q^t M = (0)$, then t is called the associated degree of M. If $Q^t M \neq (0)$, for all $t \geq 1$, then the associated degree of M is defined as ∞ [25].

Theorem 2.15. *Let $m \geq 1$ be a positive integer, $R = R_1 \times ... \times R_m$ and $M = M_1 \times ... M_m$ where (R_i, Q_i) is a local ring, M_i is an R_i-module and the associated degree of M_i is t_i for all $i \in \{1, ..., m\}$. If $\sum_{i=1}^m t_i \leq n-1$, then every non-zero submodule of M is $(n-1,n)$-weak second where $n \geq 2$.*

Proof. Let $N = N_1 \times ... \times N_m$ be a non-zero submodule of M where N_i is a submodule of M_i for $1 \leq i \leq m$. Let $(a_{11}, ..., a_{1m})...(a_{(n-1)1}, ..., a_{(n-1)m})N \subseteq K_1 \times ... \times K_m$ and $(a_{11}, ..., a_{1m})...(a_{(n-1)1}, ..., a_{(n-1)m})M \not\subseteq K_1 \times ... \times K_m$ where $a_{ij} \in R_j$, $K_j \leq M_j$ for $1 \leq i \leq n-1$ and $1 \leq j \leq m$. Then there exists $j \in \{1, ..., m\}$ such that $(\prod_{k=1}^{n-1} a_{kj}) M_j \not\subseteq K_j$. Since $Q_j^{t_j} M_j = (0)$, there exist at most $t_j - 1$ elements of $\{a_{1j}, ..., a_{(n-1)j}\}$ that are nonunits in R_j. So we need at most $t_j - 1$ parentheses such that the product of their j th components with N_j is in K_j. Let $i \neq j$. We have $Q_i^{t_i} M_i = (0)$. If there exist t_i elements of $\{a_{1i}, ..., a_{(n-1)i}\}$ that are nonunits in R_i, then the product of these elements is zero and we need t_i parentheses such that the product of their i th components with N_i is in K_i. If there exist less than t_i elements that are nonunits in R_i, then we need less than t_i parentheses such that the product of their i th components with N_i is in K_i. Thus we need at most $(t_j - 1) + \sum_{i \neq j, i=1}^m t_i = \sum_{i=1}^m t_i - 1$ parentheses such that their product with N is in $K_1 \times ... \times K_m$. Since $\sum_{i=1}^m t_i \leq n-1$, we conclude that N is $(n-1,n)$-weak second. □

Corollary 2.16. *Let $m < n$ be two positive integers, $R = F_1 \times ... \times F_m$ and $M = M_1 \times ... M_m$ where F_i is a field and M_i is an F_i-vector space for all $i \in \{1, ..., m\}$. Then every non-zero submodule of M is $(n-1,n)$-weak second where $n \geq 2$.*

Proof. The associated degree of M_i is $t_i := 1$ for all $i \in \{1, ..., m\}$. Thus $\sum_{i=1}^m t_i = m \leq n-1$. By Theorem 2.15, every non-zero submodule of M is $(n-1,n)$-weak second. □

3 (2,3)-φ-second Submodules

In this section we focus on $(2,3)$-φ-second submodules and investigate their various properties and relatonships with other concepts.

Recall from [34] that R is called a u-ring provided R has the property that an ideal contained in a finite union of ideals must be contained in one of those ideals. A um-ring is a ring R with the property that an R-module which is equal to a finite union of submodules must be equal to one of them.

Lemma 3.1. *[33, Lemma 2.40]A ring R is a um-ring if and only if $M \subseteq \cup_{i=1}^{n} M_i$ where M_i's are some R-modules, implies that $M \subseteq M_i$ for some $1 \leq i \leq n$.*

Theorem 3.2. *Let R be a um-ring, M be an R-module and N be a non-zero submodule of M. Then the following are equivalent.*

(1) N is a $(2,3)$-φ-second submodule of M.

(2) If $ab \notin ann_R(N)$ for $a, b \in R$, then $abN = aN$ or $abN = bN$ or $abN = ab\varphi(N)$.

(3) If $aN \not\subseteq K$ for $a \in R$ and a submodule K of M, then $(K :_R aN) = (K :_R N)$ or $(K :_R aN) = ann_R(aN)$ or $(K :_R aN) = (K :_R a\varphi(N))$.

(4) If $aIN \subseteq K$ and $aI\varphi(N) \not\subseteq K$ for $a \in R$, any ideal I of R and any submodule K of M, then $aN \subseteq K$ or $IN \subseteq K$ or $aI \subseteq ann_R(N)$.

(5) If $IN \not\subseteq K$ for any ideal I of R and any submodule K of M, then $(K :_R IN) = (K :_R N)$ or $(K :_R IN) = ann_R(IN)$ or $(K :_R IN) = (K :_R I\varphi(N))$.

(6) If $IJN \subseteq K$ and $IJ\varphi(N) \not\subseteq K$ for any ideals I, J of R and any submodule K of M, then $IN \subseteq K$ or $JN \subseteq K$ or $IJ \subseteq ann_R(N)$.

Proof. (1) $\Longleftrightarrow$ (2) By Theorem 2.3.

(1) $\Longrightarrow$ (3) Let $aN \not\subseteq K$ for $a \in R$ and any submodule K of M. Let $b \in (K :_R aN)$. Then $abN \subseteq K$. If $ab\varphi(N) \subseteq K$, then $b \in (K :_R a\varphi(N))$. If $ab\varphi(N) \not\subseteq K$, then $bN \subseteq K$ or $ab \in ann_R(N)$. It follows that $b \in (K :_R N)$ or $b \in ann_R(aN)$. Thus $(K :_R aN) = (K :_R a\varphi(N)) \cup (K :_R N) \cup ann_R(aN)$. Since R is a um-ring, we have $(K :_R aN) = (K :_R N)$ or $(K :_R aN) = ann_R(aN)$ or $(K :_R aN) = (K :_R a\varphi(N))$.

(3) $\Longrightarrow$ (4) Let $aIN \subseteq K$ and $aI\varphi(N) \not\subseteq K$ for $a \in R$, any ideal I of R and any submodule K of M. If $aN \subseteq K$, then we are done. Let $aN \not\subseteq K$. By (3), $(K :_R aN) = (K :_R N)$ or $(K :_R aN) = ann_R(aN)$ or $(K :_R aN) = (K :_R a\varphi(N))$. In the first case, we have $IN \subseteq K$. In the second case, $Ia \subseteq ann_R(N)$. The third case cannot hold since $aI\varphi(N) \not\subseteq K$.

(4) $\Longrightarrow$ (5) Let $IN \not\subseteq K$ where I is an ideal of R and K is a submodule of M. Let $a \in (K :_R IN)$. Then $aIN \subseteq K$. If $aI\varphi(N) \subseteq K$, then $a \in (K :_R I\varphi(N))$. If $aI\varphi(N) \not\subseteq K$, then $aN \subseteq K$ or $aI \subseteq ann_R(N)$. In the first case $a \in (K :_R N)$. In the second case $a \in ann_R(IN)$. Thus $(K :_R IN) = (K :_R N) \cup ann_R(IN) \cup (K :_R I\varphi(N))$. Since R is a um-ring, $(K :_R IN) = (K :_R N)$ or $(K :_R IN) = ann_R(IN)$ or $(K :_R IN) = (K :_R I\varphi(N))$.

(5) $\Longrightarrow$ (6) Let $IJN \subseteq K$ and $IJ\varphi(N) \not\subseteq K$ where I, J are ideals of R and K is a submodule of M. If $IN \subseteq K$, then we are done. Let $IN \not\subseteq K$. Then, by (5), $(K :_R IN) = (K :_R N)$ or $(K :_R IN) = ann_R(IN)$ or $(K :_R IN) = (K :_R I\varphi(N))$. In the first case, we have $JN \subseteq K$. In the second case we have $IJ \subseteq ann_R(N)$.

(6) $\Longrightarrow$ (1) Clear. □

Definition 3.3. Let M be an R-module, N be a $(2,3)$-φ-second submodule of M, K be a submodule of M and $a, b \in R$. If $ab\varphi(N) \subseteq K$, $ab \notin ann_R(N)$, $aN \not\subseteq K$ and $bN \not\subseteq K$, then (a, b, K) is called a φ-triple of N.

Theorem 3.4. *Let N be a $(2,3)$-φ-second submodule of an R-module M and (a, b, K) be a φ-triple of N for some $a, b \in R$ and a submodule K of M. Then*

(1) $ab\varphi(N) \subseteq N$.

(2) $a(ann_R(N))\varphi(N) \subseteq K$.

(3) $b(ann_R(N))\varphi(N) \subseteq K$.

(4) $(ann_R(N))^2\varphi(N) \subseteq K$.

(5) $a(ann_R(N))\varphi(N) \subseteq N$.

(6) $b(ann_R(N))\varphi(N) \subseteq N$.

Proof. (1) Suppose that $ab\varphi(N) \not\subseteq N$. Then $ab\varphi(N) \not\subseteq N \cap K$ and $abN \subseteq N \cap K$. Since N is $(2,3)$-φ-second submodule, we have $ab \in ann_R(N)$ or $aN \subseteq N \cap K \subseteq K$ or $bN \subseteq N \cap K \subseteq K$ which contradicts the assumption that (a, b, K) is a φ-triple of N.

(2) Suppose that $a(ann_R(N))\varphi(N) \not\subseteq K$. Then $ax\varphi(N) \not\subseteq K$ for some $x \in ann_R(N)$. Then $a(b+x)\varphi(N) \not\subseteq K$ because $ab\varphi(N) \subseteq K$. Also, $a(b+x)N = abN \subseteq K$. Since N is $(2,3)$-φ-second submodule, we have $a(b+x) \in ann_R(N)$ or $aN \subseteq K$ or $(b+x)N = bN \subseteq K$. The first case implies that $ab \in ann_R(N)$ which is a contradiction. Clearly, the other two cases contradict with the hypothesis.

(3) The proof is similar to part (2).

(4) Suppose that $x_1x_2\varphi(N) \nsubseteq K$ for some $x_1, x_2 \in ann_R(N)$. Then parts (2) and (3) imply that $(a+x_1)(b+x_2)\varphi(N) \nsubseteq K$. Clearly, $(a+x_1)(b+x_2)N = abN \subseteq K$. Since N is $(2,3)$-φ-second submodule, we have $(a+x_1)(b+x_2) \in ann_R(N)$ or $(a+x_1)N = aN \subseteq K$ or $(b+x_2)N = bN \subseteq K$ which are contradictions.

(5) Suppose that $a(ann_R(N))\varphi(N) \nsubseteq N$. Then there exists $x \in ann_R(N)$ such that $ax\varphi(N) \nsubseteq N$. By part (1), $a(b+x)\varphi(N) \nsubseteq N \cap K$ and $a(b+x)N \subseteq N \cap K$. Since N is $(2,3)$-φ-second submodule, we have $a(b+x) \in ann_R(N)$ or $aN \subseteq K$ or $(b+x)N = bN \subseteq K$ which are contradictions. Thus $a(ann_R(N))\varphi(N) \subseteq N$.

(6) The proof is similar to part (5). □

Theorem 3.5. *Let M be an R-module and N be a $(2,3)$-φ-second submodule of M which is not strongly 2-absorbing second submodule. Then $ann_R(N)^2\varphi(N) \subseteq N$.*

Proof. Let N be a $(2,3)$-φ-second submodule of M which is not strongly 2-absorbing second submodule. Then there exists a φ-triple (a,b,K) of N for some $a, b \in R$ and a submodule K of M. Suppose that $(ann_R(N))^2\varphi(N) \nsubseteq N$. Hence there exist $x_1, x_2 \in ann_R(N)$ such that $x_1x_2\varphi(N) \nsubseteq N$. Then $(a+x_1)(b+x_2)\varphi(N) \nsubseteq K \cap N$ by Theorem 3.4. Also, clearly, $(a+x_1)(b+x_2)N = abN \subseteq K \cap N$. Since N is $(2,3)$-φ-second submodule of M, we have $(a+x_1)(b+x_2) \in ann_R(N)$ or $(a+x_1)N = aN \subseteq K \cap N \subseteq K$ or $(b+x_2)N = bN \subseteq K \cap N \subseteq K$ which are contradictions. □

Let M be an R-module. We define the function $\varphi_\omega : S(M) \longrightarrow S(M)$ as $\varphi_\omega(L) = \sum_{i\in\mathbb{Z}^+}(L :_M ann_R(L)^i)$ for every $L \in S(M)$.

Corollary 3.6. *Let M be an R-module and N be a $(2,3)$-φ-second submodule of M such that $(N :_M ann_R(N)^3) \subseteq \varphi(N)$. Then N is $(2,3)$-φ_ω-second submodule of M.*

Proof. If N is a strongly 2-absorbing second submodule of M, then the result is clear. So we may assume that N is not a strongly 2-absorbing second submodule of M. Therefore, by Theorem 3.5, we have $(N :_M ann_R(N)^3) \subseteq \varphi(N) \subseteq (N :_M ann_R(N)^2) \subseteq (N :_M ann_R(N)^3)$, that is, $\varphi(N) = (N :_M ann_R(N)^2) = (N :_M ann_R(N)^3)$. It follows that $\varphi(N) = (N :_M ann_R(N)^j)$ for all $j \geq 2$ and the result follows. □

Recall from [4] that an R-module M is said to be a comultiplication module if for every submodule N of M there exists an ideal I of R such that $N = (0 :_M I)$. It also follows that M is a comultiplication module if and only if $N = (0 :_M ann_R(N))$ for every submodule N of M [4].

Corollary 3.7. *Let M be a comultiplication R-module and N be a submodule of M. Then the following hold.*

(1) *If N is a $(2,3)$-φ-second submodule of M that is not strongly 2-absorbing second, then $\varphi(N) \subseteq C(N^3)$.*

(2) *If $n \geq 3$ is an integer and N is a $(2,3)$-n-almost second submodule of M that is not strongly 2-absorbing second, then $C(N^3) = C(N^n)$.*

Proof. (1) Since M is comultiplication, $N = (0 :_M ann_R(N))$. By Theorem 3.5, $\varphi(N) \subseteq (N :_M ann_R(N)^2) = ((0 :_M ann_R(N)) :_M ann_R(N)^2) = (0 :_M ann_R(N)^3) = C(N^3)$ and hence $\varphi(N) \subseteq C(N^3)$.

(2) Notice that $\varphi_n(N) = (N :_M ann_R(N)^{n-1}) = (0 :_M ann_R(N)^n) = C(N^n)$. By part (1), $C(N^n) \subseteq C(N^3)$. Since the reverse inclusion always holds we have $C(N^3) = C(N^n)$. □

Let M be an R-module and N be a submodule of M. The sum of all second submodules of N is called the second radical of N and denoted by $sec(N)$. If there is no second submodule of N, then we define $sec(N) = 0$ [16].

Corollary 3.8. *Let M be an R-module and N be a $(2,3)$-φ-second submodule of M that is not strongly 2-absorbing second. Then the following hold.*

(1) $\sqrt{ann_R(N)} = \sqrt{ann_R(\varphi(N))}$.

(2) *If M is a finitely generated R-module, then $sec(N) = sec(\varphi(N))$.*

Proof. (1) By Theorem 3.5, $ann_R(N)^2\varphi(N) \subseteq N$. Then $ann_R(N)^3 \subseteq ann_R(\varphi(N))$ and so $\sqrt{ann_R(N)} \subseteq \sqrt{ann_R(\varphi(N))}$. Since the reverse inclusion always holds we have the equality.

(2) By [10, Theorem 2.12], $sec(N) = (0 :_M \sqrt{ann_R(N)})$ and by part (1), $sec(N) = (0 :_M \sqrt{ann_R(\varphi(N))}) = sec(\varphi(N))$. □

Definition 3.9. Let N be a $(2,3)$-φ-second submodule of an R-module M and suppose that $IJN \subseteq K$ for some ideals I, J of R and a submodule K of M. We call N as a free φ-triple with respect to I, J, K if (a,b,K) is not a φ-triple of N for each $a \in I, b \in J$.

Theorem 3.10. *Let N be a $(2,3)$-φ-second submodule of M and suppose that $IJN \subseteq K$, $IJ\varphi(N) \not\subseteq K$ for some ideals I, J of R and a submodule K of M such that N is a free φ-triple with respect to I, J, K. Then $IJ \subseteq ann_R(N)$ or $IN \subseteq K$ or $JN \subseteq K$.*

Proof. Suppose that $IJ \not\subseteq ann_R(N)$. We show that $IN \subseteq K$ or $JN \subseteq K$. Assume on the contrary that $IN \not\subseteq K$ and $JN \not\subseteq K$. Then there exist $a_1 \in I$ and $b_1 \in J$ such that $a_1N \not\subseteq K$ and $b_1N \not\subseteq K$. If $a_1b_1\varphi(N) \not\subseteq K$, then $a_1b_1 \in ann_R(N)$ as N is $(2,3)$-φ-second submodule. If $a_1b_1\varphi(N) \subseteq K$, then again $a_1b_1 \in ann_R(N)$ as (a_1, b_1, K) is not a φ-triple of N. Since $IJ \not\subseteq ann_R(N)$, there are $a \in I$ and $b \in J$ such that $ab \notin ann_R(N)$. Since (a, b, K) is not a φ-triple of N and N is a $(2,3)$-φ-second submodule, we have $aN \subseteq K$ or $bN \subseteq K$. There are three cases.

Case 1: Suppose that $aN \subseteq K$ but $bN \not\subseteq K$. We have $a_1bN \subseteq K$. Since (a_1, b, K) is not a φ-triple of N and N is a $(2,3)$-φ-second submodule, we have $a_1b \in ann_R(N)$. Also, we have $(a + a_1)bN \subseteq K$. Since $(a + a_1, b, K)$ is not a φ-triple of N and N is a $(2,3)$-φ-second submodule, we have $(a + a_1)b \in ann_R(N)$ which gives the contradiction that $ab \in ann_R(N)$ or $(a + a_1)N \subseteq K$ which gives the contradiction that $a_1N \subseteq K$.

Case 2: Suppose that $bN \subseteq K$ but $aN \not\subseteq K$. We have $ab_1N \subseteq K$. Since (a, b_1, K) is not a φ-triple of N and N is a $(2,3)$-φ-second submodule, we have $ab_1 \in ann_R(N)$. Also, we have $a(b + b_1)N \subseteq K$. Since $(a, b + b_1, K)$ is not a φ-triple of N and N is a $(2,3)$-φ-second submodule, $a(b + b_1) \in ann_R(N)$ which gives the contradiction that $ab \in ann_R(N)$ or $(b + b_1)N \subseteq K$ which gives the contradiction that $b_1N \subseteq K$.

Case 3: Suppose that $aN \subseteq K$ and $bN \subseteq K$. Then $(b+b_1)N \not\subseteq K$ as $b_1N \not\subseteq K$. We have $a_1(b+b_1)N \subseteq K$. Since $(a_1, b + b_1, K)$ is not a φ-triple of N and N is a $(2,3)$-φ-second submodule, we have $a_1(b + b_1) \in ann_R(N)$. Since $a_1b_1 \in ann_R(N)$, we have $a_1b \in ann_R(N)$. Also, $(a + a_1)N \not\subseteq K$ since $a_1N \not\subseteq K$. We have $(a + a_1)b_1N \subseteq K$. Since $(a + a_1, b_1, K)$ is not a φ-triple of N and N is a $(2,3)$-φ-second submodule, $(a + a_1)b_1 \in ann_R(N)$ and so $ab_1 \in ann_R(N)$. On the other hand, we have $(a + a_1)(b + b_1)N \subseteq K$. Since $(a + a_1, b + b_1, K)$ is not a φ-triple of N and N is a $(2,3)$-φ-second submodule, $(a + a_1)(b + b_1) \in ann_R(N)$ which gives the contradiction that $ab \in ann_R(N)$. □

Proposition 3.11. *Let $R = R_1 \times R_2$ and $M = M_1 \times M_2$ where R_i is a ring, M_i is an R_i-module for $i = 1, 2$. Let $\psi_i : S(M_i) \longrightarrow S(M_i)$ be a function for each $i = 1, 2$ and $\varphi = \psi_1 \times \psi_2$. Suppose that $N = N_1 \times (0)$ where N_1 is a non-zero submodule of M_1.*

(1) *If $\psi_2((0)) = (0)$, then N is a $(2,3)$-φ-second submodule of M if and only if N_1 is a $(2,3)$-ψ_1-second submodule of M_1.*

(2) *If $\psi_2((0)) \neq (0)$, then N is a $(2,3)$-φ-second submodule of M if and only if N_1 is a strongly 2-absorbing second submodule of M_1.*

Proof. (1) Suppose that N is a $(2,3)$-φ-second submodule of M. Let $a_1b_1N_1 \subseteq K_1$ and $a_1b_1\psi_1(N_1) \not\subseteq K_1$ for $a_1, b_1 \in R_1$ and $K_1 \leq M_1$. Then $(a_1, 1)(b_1, 1)(N_1 \times (0)) \subseteq K_1 \times (0)$ and $(a_1, 1)(b_1, 1)\varphi(N_1 \times (0)) \not\subseteq K_1 \times (0)$. Since N is a $(2,3)$-φ-second submodule of M, we get that $(a_1, 1)(N_1 \times (0)) \subseteq K_1 \times (0)$ or $(b_1, 1)(N_1 \times (0)) \subseteq K_1 \times (0)$ or $(a_1, 1)(b_1, 1) \in ann_R(N_1 \times (0)) = ann_{R_1}(N_1) \times R_2$. Thus $a_1N_1 \subseteq K_1$ or $b_1N_1 \subseteq K_1$ or $a_1b_1 \in ann_{R_1}(N_1)$ and so N_1 is a $(2,3)$-ψ_1-second submodule of M_1.

Conversely, suppose that N_1 is a $(2,3)$-ψ_1-second submodule of M_1. Let $a = (a_1, a_2)$, $b = (b_1, b_2) \in R_1 \times R_2$ and $K = K_1 \times K_2$ be a submodule of $M_1 \times M_2$ where K_i is a submodule of M_i for each $i = 1, 2$. Suppose that $abN \subseteq K$ and $ab\varphi(N) \not\subseteq K$. Since $\psi_2((0)) = (0)$, we have $a_1b_1N_1 \subseteq K_1$ and $a_1b_1\psi_1(N_1) \not\subseteq K_1$. Since N_1 is a $(2,3)$-ψ_1-second submodule of M_1, we have $a_1b_1 \in ann_{R_1}(N_1)$ or $a_1N_1 \subseteq K_1$ or $b_1N_1 \subseteq K_1$. Then we get that $ab = (a_1b_1, a_2b_2) \in ann_{R_1}(N_1) \times R_2 = ann_R(N)$ or $aN = (a_1, a_2)(N_1 \times (0)) \subseteq K_1 \times K_2$ or $bN = (b_1, b_2)(N_1 \times (0)) \subseteq K_1 \times K_2$. Thus N is a $(2,3)$-φ-second submodule of M.

(2) Suppose that N is a $(2,3)$-φ-second submodule of M. Let $a_1b_1N_1 \subseteq K_1$ for $a_1, b_1 \in R_1$ and $K_1 \leq M_1$. Then $(a_1, 1)(b_1, 1)(N_1 \times (0)) \subseteq K_1 \times (0)$ and $(a_1, 1)(b_1, 1)(\psi_1(N_1) \times \psi_2((0))) = (a_1, 1)(b_1, 1)\varphi(N) \not\subseteq K_1 \times (0)$. Since N is a $(2,3)$-φ-second submodule of M, we get that $(a_1, 1)(N_1 \times (0)) \subseteq K_1 \times (0)$ or $(b_1, 1)(N_1 \times (0)) \subseteq K_1 \times (0)$ or $(a_1, 1)(b_1, 1) \in ann_R(N_1 \times (0)) = ann_{R_1}(N_1) \times R_2$. Thus $a_1N_1 \subseteq K_1$ or $b_1N_1 \subseteq K_1$ or $a_1b_1 \in ann_{R_1}(N_1)$ and so N_1 is a strongly 2-absorbing second submodule of M_1.

Conversely, assume that N_1 is a strongly 2-absorbing second submodule of M_1. Then $N_1 \times (0)$ is a strongly 2-absorbing second submodule of M. Hence N is a $(2,3)$-φ-second submodule of M. □

References

[1] *D. D. Anderson, M. Bataineh,* Generalizations of prime ideals, *Comm. Algebra,* **36(2)**, *686–696, (2008).*

[2] *D. F. Anderson, A. Badawi,* On n-absorbing ideals of commutative rings, *Comm. Algebra,* **39(5)**, *1646–1672, (2011).*

[3] *D. D. Anderson, E. Smith,* Weakly prime ideals, *Houston J. of Math.,* **29**, *831–840, (2003).*

[4] *H.Ansari-Toroghy, F. Farshadifar,* The dual notion of multiplication modules, *Taiwanese J. Math.,* **11(4)**, *1189–1201, (2007).*

[5] *H. Ansari-Toroghy, F. Farshadifar,* Product and dual product of submodules, *Far East J. Math. Sci.,* **25(3)**, *447–455, (2007).*

[6] *H. Ansari-Toroghy, F. Farshadifar,* The dual notions of some generalizations of prime submodules, *Comm. Algebra,* **39(7)**, *2396-2416, (2011).*

[7] *H. Ansari-Toroghy, F. Farshadifar,* Fully idempotent and coidempotent modules, *Bull. Iranian Math. Soc.,* **38(4)**, *987–1005, (2012).*

[8] *H. Ansari-Toroghy, F. Farshadifar,* On the dual notion of prime submodules, *Algebra Colloq.,* **19(1)**, *1109–1116, (2012).*

[9] *H. Ansari-Toroghy, F. Farshadifar,* On the dual notion of prime submodules II, *Mediterr. J. Math.,* **9(2)**, *327-336, (2012).*

[10] *H. Ansari-Toroghy, F. Farshadifar,* On the dual notion of prime radicals of submodules, *Asian-Eur. J. Math.,* **6(2)**, *1350024, (2013).*

[11] *H. Ansari-Toroghy, F. Farshadifar, S. Maleki-Roudposhti,* n-absorbing and Strongly n-absorbing Second Submodules, *Bol. Soc. Paran. Mat.,* **39(1)**, *9–22, (2021).*

[12] *M. Behboodi,* On weakly prime radical of modules and semi-compatible modules, *Acta Math Hung.,* **113**, *243–254, (2006).*

[13] *P. K. Beiranvand , R. Beyranvand,* Almost prime and weakly prime submodules, *Journal of Algbera and Its Applications,* **18(7)**, *1950129, (2019).*

[14] *S. M. Bhatwadekar, P. K. Sharma,* Uniqe factorization and birth of almost primes, *Comm. Algebra,* **33**, *43–49, (2005).*

[15] *S. Çeken, M. Alkan, P. F. Smith,* Second modules over noncommutative rings, *Communications in Algebra,* **41(1)**, *83–98, (2013).*

[16] *S. Çeken, M. Alkan and P. F. Smith,* The dual notion of the prime radical of a module, *Journal of Algebra,* **392**, *265–275, (2013).*

[17] *S. Çeken, M. Alkan,* On the Weakly Second Spectrum of a Module, *Filomat,* **30(4)**, *1013–1020, (2016).*

[18] *S. Çeken, M. Alkan,* On Strongly 2-Absorbing Second Submodules, *AIP Conference Proceedings,* **1991(1)**, *020003, (2018).*

[19] *S. Çeken,* On The Interior of a Submodule With Respect to a Set of Ideal, *Quaestiones Mathematicae,* **42(4)**, *431–446, (2019).*

[20] *S. Çeken, Ü. Tekir, S. Koç, A. Kelleqi,* Generalizations of second submodules, *Filomat,* **34(12)**, *3995–4004, (2020).*

[21] *A. Y. Darani, F. Soheilnia,* On n-absorbing submodules, *Math. Commun.,* **17**, *547–557, (2012).*

[22] *M. Ebrahimpour, R. Nekooei,* On generalizations of prime ideals, *Comm. Algebra,* **40(4)**, *1268–1279, (2012).*

[23] *M. Ebrahimpour, R. Nekooei,* On generalizations of prime submodules, *Bull. Iranian Math. Soc.,* **39(5)**, *919–939, (2013).*

[24] *M. Ebrahimpour,* On generalizations of prime ideals (II), *Comm. Algebra,* **42(9)**, *3861–3875, (2014).*

[25] *M. Ebrahimpour,* (n-1,n)-Weakly Prime Submodules In Direct Product Of Modules, *Journal of Hyperstructures,* **6(1)**, *10–16, (2017).*

[26] *E. H. Feller, E. W. Swokowski,* Prime modules, *Canad. J. Math.,* **17**, *1041–1052, (1965).*

[27] *A. Haghany, M. Vedadi,* Endoprime modules, *Acta Math. Hung.,* **106**, *89–99, (2005).*

[28] *N. S. Karamzadeh, O. A. S. Karamzadeh,* On Artinian Modules Over Duo Rings, *Communications in Algebra,* **38**, *3521–3531, (2010).*

[29] *H. A. Khashan,* On almost prime submodules, *Acta Math. Sci. B,* **32(2)**, *645–651, (2012).*

[30] *C. P. Lu,* Prime submodules of modules, *Comment. Math. Univ. St. Paul,* **33(1)**, *61–69, (1984).*

[31] *S. Moradi, A. Azizi,* On n-almost prime submodules, *Indian J. Pure and Appl. Math.,* **44(5)**, *605–619, (2013).*

[32] *S. Moradi, A. Azizi,* Generalizations of prime submodules, *Hacettepe Journal of Mathematics and Statistics,* **44(3)**, *587–595, (2015).*

[33] *H. Mostafanasab, Ü. Tekir, E. Yetkin Çelikel, E. A. Uğurlu, G. Ulucak, A. Y. Darani,* Generalizations of 2-absorbing and 2-absorbing primary submodules, *Hacettepe Journal of Mathematics and Statistics,* **48(4)**, *1001–1016, (2019).*

[34] *P. Quartararo and H.S. Butts,* Finite unions of ideals and modules, *Proc. Amer. Math. Soc.,* **52**, *91–96, (1975).*

[35] *S. Yassemi,* The dual notion of prime submodules, *Arch. Math (Brno),* **37**, *273–278, (2001).*

[36] *N. Zamani,* Φ-prime submodules, *Glasg. Math. J.,* **52(2)**, *253–259, (2010).*

Author information

S. Çeken, Department of Mathematics, Trakya University, Turkey.
E-mail: `cekensecil@gmail.com`

S. Koç, Department of Mathematics, Istanbul Medeniyet University, Turkey.
E-mail: `suat.koc@medeniyet.edu.tr`

Ü. Tekir, Department of Mathematics, Marmara University, Turkey.
E-mail: `utekir@marmara.edu.tr`

Received: October 26, 2022.
Accepted: August 7, 2023.