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ON FUSION FRAMES IN QUATERNIONIC HILBERT SPACES

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Abstract In [15] authors have studied frames of operators in Quaternionic Hilbert spaces and silently discussed that fusion frames are particular case of frames of operators. In this paper, we have studied and extended their properties and present some of their characterizations in quaternionic Hilbert spaces. Further, we have examined the existence of synthesis, analysis, and frame operators and investigated their properties for fusion frames in Quaternionic Hilbert spaces. Furthermore, we have given some perturbation results for fusion frames in Quaternionic Hilbert spaces similar to the results in Hilbert spaces. Finally, woven fusion frames in Quaternionic Hilbert spaces are studied.

1 Introduction

Frames [11] in Hilbert spaces were introduced in 1952 while studying the non-harmonic Fourier series. But their potential was realized by the researchers after the work done by Daubechies, Grossman, and Meyer [3], due to its vast applications in various fields like signal and image processing, sigma-delta quantization, filter bank theory, and wireless communication. For more details, one may refer to [6]. In recent years, many generalizations of frames were introduced and studied. One of a generalization which is much appreciated by the researchers is fusion frames in Hilbert spaces introduced by P. Casazza and G. Kutyniok [8]. Fusion frames are used in filter bank theory, time-frequency analysis, and signal and image processing. For further details, regarding the applications and properties of fusion frames and its extension see [8, 9, 1]. In [19] Bemrose et al. have defined and studied the properties of weaving frames in Hilbert spaces which are used in distributed signal processing.

Hamilton discovered the field of quaternion which is a generalization of complex numbers, it is a four-dimensional non-commutative real algebra. Quaternions are used to study rotation in the higher dimension, theory of relativity, Newtonian and quantum mechanics, and general relativity in which Lorentz transformation is given in terms of quaternions. For details regarding quaternions see [13]. In [10] R. Ghiloni et al. have extended the continuous functional calculus in the case of quaternionic Hilbert spaces.

Khokulan, Thirulogasanthar and Srisatkunarajah [4] have defined and studied frames in finite dimensional quaternionic Hilbert spaces. In [14], Sharma and Goel have studied frames in the separable right quaternionic Hilbert spaces. H. Ellouz [2] studied K-frames in right quaternionic Hilbert spaces and studied the invertibility of corresponding frames operators. Recently in [12] Ruchi et al. has defined OPV-frames in right quaternionic Hilbert spaces and woven frames, woven K-frames and K-fusion frames in right quaternionic Hilbert spaces were studied in [17, 5, 18].

In this paper, we have studied and extended the properties of fusion frames and present some of their characterizations in quaternionic Hilbert spaces. Further, we have examined the existence of synthesis, analysis, and frame operators and investigated their properties for fusion frames in quaternionic Hilbert spaces. Furthermore, we have given some perturbation results for fusion frames in quaternionic Hilbert spaces similar to the results in Hilbert spaces. Finally, woven fusion frames in quaternionic Hilbert spaces are studied.

Outline of the paper

The paper is divided into four sections. In Section 2, we have reviewed the definitions of frames and fusion frames in the right quaternionic Hilbert spaces. Further, some know results are stated which will be used to prove subsequent results. In Section 3, a necessary and sufficient condition for a family of subspaces to be fusion frames (Bessel sequences of subspaces) in terms of synthesis operator and the definition of dual fusion frames are given. In Section 4, we will study the perturbation of fusion frames in the right quaternionic Hilbert spaces. In Section 5, we have defined woven fusion frames in the right quaternionic Hilbert spaces, also given the necessary conditions under which two families of fusion frames form woven fusion frames in right quaternionic Hilbert spaces.

2 Preliminaries

Since quaternions are non-commutative we will consider the right quaternionic Hilbert space. Throughout this paper, we will denote $\mathbb{N}$ to be the set of natural numbers and $\mathcal{I}, J$ be any countable index sets and $\mathfrak{Q}$ denotes the quaternionic field, we assume that right quaternionic Hilbert space $\mathbb{H}^R(\mathfrak{Q})$ is separable. The family of right subspaces of $\mathbb{H}^R(\mathfrak{Q})$ is represented by $\{W_i^R\}_{i\in\mathcal{I}}, \{H_i^R\}_{i\in\mathcal{I}}$ and $\{w_i\}_{i\in\mathcal{I}}, \{v_i\}_{i\in\mathcal{I}}$ denotes the family of weights, i.e $w_i > 0, v_i > 0, \forall\, i \in \mathcal{I}$.
Quaternions are four dimensional non commutative real algebra generated by $1, i, j, k$ where i, j, k called imaginary units. As we know that quaternions are extension of complex number($\mathbb{C}$) and operation on $\mathbb{C}$ are those of $\mathfrak{Q}$ restricted over $\mathbb{C}$, for operation and various properties of quaternions see [10], for further details.

Definition 2.1. [10] A *right quaternionic pre-Hilbert space* or *right quaternionic inner product space* $V_R(\mathfrak{Q})$ is a right quaternionic vector space together with the binary mapping $\langle .|. \rangle : V_R(\mathfrak{Q}) \times V_R(\mathfrak{Q}) \to \mathfrak{Q}$ (called the *Hermitian quaternionic inner product*) which satisfies following properties:

(i) $\overline{\langle v_1|v_2\rangle} = \langle v_2|v_1\rangle$ for all $v_1, v_2 \in V_R(\mathfrak{Q})$.

(ii) $\langle v|v\rangle > 0$ if $v \neq 0$.

(iii) $\langle v|v_1 + v_2\rangle = \langle v|v_1\rangle + \langle v|v_2\rangle$ for all $v, v_1, v_2 \in V_R(\mathfrak{Q})$

(iv) $\langle v|uq\rangle = \langle v|u\rangle q$ for all $v, u \in V_R(\mathfrak{Q})$ and $q \in \mathfrak{Q}$.

Let us define a non negative mapping $||.||$ on $V_R(\mathfrak{Q})$, as $||u|| = \sqrt{\langle u|u\rangle}$ it is easy to prove that $||.||$ form a norm on $V_R(\mathfrak{Q})$. Moreover if $V_R(\mathfrak{Q})$ is complete under this norm then it is said to be right quaternionic Hilbert space and denoted by $\mathbb{H}^R(\mathfrak{Q})$.

In [14], authors introduced frames in any separable right quaternionic Hilbert spaces.

Definition 2.2 ([14]). Let $\mathbb{H}^R(\mathfrak{Q})$ be a right quaternionic Hilbert space and $\{u_i\}_{i\in\mathcal{I}}$ be a sequence in $\mathbb{H}^R(\mathfrak{Q})$. Then $\{u_i\}_{i\in\mathcal{I}}$ is said to be a *frame* for $\mathbb{H}^R(\mathfrak{Q})$, if there exist two finite real constants with $0 < r_1 \leq r_2$ such that

$$r_1\|u\|^2 \leq \sum_{i\in\mathcal{I}} |\langle u_i|u\rangle|^2 \leq r_2\|u\|^2, \text{ for all } u \in \mathbb{H}^R(\mathfrak{Q}).$$

The above inequality is called frame inequality and r_1, r_2 are called lower and upper frame bounds respectively. If only upper bound condition hold then $\{u_i\}_{i\in\mathcal{I}}$ is said to form a Bessel sequence with Bessel bound r_2. For details regarding frame operators, perturbation and dual frames see [14].

Definition 2.3. [15] Let $\{W_i^R\}_{i\in\mathcal{I}}$ be a sequence of right closed subspaces of a right separable quaternionic Hilbert space $\mathbb{H}^R(\mathfrak{Q})$ and $\{w_i\}_{i\in\mathcal{I}}$ be a family of weights i.e. $w_i > 0, \forall\, i \in \mathcal{I}$. Then $\{(W_i^R, w_i)\}_{i\in\mathcal{I}}$ is called a fusion frame (frame of subspaces), if there exist constants $0 < r_1 \leq r_2 < \infty$ called fusion frame bounds such that

$$r_1||u||^2 \leq \sum_{i\in\mathcal{I}} w_i^2||\pi_{W_i^R}(u)||^2 \leq r_2||u||^2, \forall\, u \in \mathbb{H}^R(\mathfrak{Q}).$$

Example 2.4. Let $\mathbb{H}^R(\mathfrak{Q})$ be a right quaternionic Hilbert space and $\{e_i\}_{i\in\mathbb{N}}$ be an orthonormal basis for $\mathbb{H}^R(\mathfrak{Q})$. Define $W_1^R = \overline{span}\{e_1\}$ and $W_i^R = \overline{span}\{e_{i-1}\}$, $\forall i \geq 2$, and $w_i = 1$, $\forall i \in \mathbb{N}$. Then $\{(W_i^R, w_i)\}_{i\in\mathbb{N}}$ is a fusion frame for $\mathbb{H}^R(\mathfrak{Q})$ with lower and upper fusion frame bounds $r_1 = 1$ and $r_2 = 2$, respectively.

We can also say the family of subspaces $\{W_i^R\}_{i\in\mathcal{I}}$ form a fusion frame with respect to $\{w_i\}_{i\in\mathcal{I}}$ for $\mathbb{H}^R(\mathfrak{Q})$. Moreover if $r_1 = r_2$ then the family $\{W_i^R\}_{i\in\mathcal{I}}$ is said to be tight fusion frame for $\mathbb{H}^R(\mathfrak{Q})$ and Parseval fusion frame if $r_1 = r_2 = 1$. If $\mathbb{H}^R(\mathfrak{Q}) = \bigoplus_{i\in\mathcal{I}} W_i^R$, then $\{W_i^R\}_{i\in\mathcal{I}}$ is called an orthonormal basis of subspaces and if $w_i = w_j = w$, then $\{W_i^R\}_{i\in\mathcal{I}}$, is called a w-uniform fusion frame. If only upper bound condition hold then we say the family $\{W_i^R\}_{i\in\mathcal{I}}$, is a Bessel sequence of subspaces with respect to $\{w_i\}_{i\in\mathcal{I}}$ with bound r_2 for $\mathbb{H}^R(\mathfrak{Q})$.

Definition 2.5. [17] Let $\mathbb{N}_m$ be the set of first m natural numbers, $\mathbb{H}^R(\mathfrak{Q})$ be a right quaternionic Hilbert space and $\mathfrak{F} = \{\{u_{ij}\}_{i\in\mathbb{N}} : j \in \mathbb{N}_m\}$ be a family of frames for $\mathbb{H}^R(\mathfrak{Q})$. Then $\mathfrak{F}$ is said to be *woven* if there are universal positive real numbers r_1 and r_2 so that for every partition $P = \{\sigma_j\}_{j\in\mathbb{N}_m}$ of $\mathbb{N}$, the family $\mathfrak{F}_P = \{u_{ij}\}_{i\in\sigma_j, j\in\mathbb{N}_m}$ is a frame for $\mathbb{H}^R(\mathfrak{Q})$ with lower and upper frame bounds r_1 and r_2, respectively. Each family $\mathfrak{F}_P$ is called a weaving. If every weaving is a Bessel sequence, then $\mathfrak{F}$ is called a woven Bessel sequence for $\mathbb{H}^R(\mathfrak{Q})$.

Lemma 2.6. *[10] If $\mathbb{H}^R(\mathfrak{Q})$ is a right quaternionic Hilbert space and $\phi \neq A \subset \mathbb{H}^R(\mathfrak{Q})$, then $\mathbb{H}^R(\mathfrak{Q}) = A^\perp \oplus \langle A\rangle$. Where $A^\perp = \{v \in \mathbb{H}^R(\mathfrak{Q}) : \langle v|u\rangle = 0 \ \forall\ u \in A\}$ and $\langle A\rangle$ denotes the right $\mathfrak{Q}$-linear subspace of $\mathbb{H}^R(\mathfrak{Q})$ consisting of all finite right $\mathfrak{Q}$-linear combinations of elements of A.*

Theorem 2.7. *[14](The Reconstruction Formula). Let $\mathbb{H}^R(\mathfrak{Q})$ be a right quaternionic Hilbert space and $\{u_i\}_{i\in\mathcal{I}}$ be a frame for $\mathbb{H}^R(\mathfrak{Q})$ with frame operator S. Then for every $u \in \mathbb{H}^R(\mathfrak{Q})$ can be expressed as $u = \sum\limits_{i\in\mathcal{I}} S^{-1}u_i\langle u_i|u\rangle = \sum\limits_{i\in\mathcal{I}} u_i\langle S^{-1}u_i|u\rangle$.*

Theorem 2.8. *[16] Let $\mathbb{H}^R(\mathfrak{Q})$ be a right quaternionic Hilbert space and $\{u_i\}_{i\in\mathcal{I}}$ be a frame for $\mathbb{H}^R(\mathfrak{Q})$ with the frame operator S. Fix $u \in \mathbb{H}^R(\mathfrak{Q})$, if $u = \sum\limits_{i\in\mathcal{I}} u_i q_i$ for some quaternionic sequence $\{q_i\}_{i\in\mathcal{I}} \in l^2(\mathfrak{Q})$. Then we have*
$$\sum_{i\in\mathcal{I}} |q_i|^2 = \sum_{i\in\mathcal{I}} |\langle S^{-1}u_i|u\rangle|^2 + \sum_{i\in\mathcal{I}} |\langle S^{-1}u_i|u\rangle - q_i|^2.$$

Lemma 2.9. *[10] In right quaternionic Hilbert Space $\mathbb{H}^R(\mathfrak{Q})$, Cauchy-Schwarz inequality hold:*
$|\langle u|v\rangle|^2 \leq \langle u|u\rangle\langle v|v\rangle$ *for all $u, v \in \mathbb{H}^R(\mathfrak{Q})$.*

Lemma 2.10. *[2] Let $\mathbb{H}^R(\mathfrak{Q})$ and $\mathbb{H}_1^R(\mathfrak{Q})$ be two right quaternionic Hilbert spaces and $T \in \mathfrak{B}(\mathbb{H}^R(\mathfrak{Q}), \mathbb{H}_1^R(\mathfrak{Q}))$ be a bounded, right linear operator with closed range then there exist an right linear operator $T^\dagger \in \mathfrak{B}(\mathbb{H}_1^R(\mathfrak{Q}), \mathbb{H}^R(\mathfrak{Q}))$ such that*

$$||T^\dagger||^{-1}||u|| \leq ||T^*u|| \leq ||T||\ ||u||\,, \forall\, u \in R(T).$$

3 Fusion frames in quaternionic Hilbert spaces

Let $\{W_i^R\}_{i\in\mathcal{I}}$ be a family of right subspaces of $\mathbb{H}^R(\mathfrak{Q})$. Then the family $\{W_i^R\}_{i\in\mathcal{I}}$ is said to be complete if $\overline{\text{span}}\{W_i^R\}_{i\in\mathcal{I}} = \mathbb{H}^R(\mathfrak{Q})$. We begin this section with the following equivalent conditions for the fusion frame in a right quaternionic Hilbert spaces.

Theorem 3.1. *For each $i \in \mathcal{I}$, let $w_i > 0$, $\{u_{ij}\}_{j\in J_i}$ be a frame sequence in $\mathbb{H}^R(\mathfrak{Q})$ with frame bounds r_{1i} and r_{2i}, $W_i^R = \overline{span}_{j\in J_i}\{u_{ij}\}$ and $\{e_{ij}\}_{j\in J_i}$ be an orthonormal basis for the subspace W_i^R. Suppose that*

$$0 < r_1 \left(= \inf_{i\in\mathcal{I}} r_{1i}\right) \leq r_2 \left(= \sup_{i\in\mathcal{I}} r_{2i}\right) < \infty.$$

Then following statements are equivalent:

(i) $\{u_{ij}w_i\}_{i\in\mathcal{I}, j\in J_i}$ *is a frame for* $\mathbb{H}^R(\mathfrak{Q})$.

(ii) $\{e_{ij}w_i\}_{i\in\mathcal{I},j\in J_i}$ *is a frame for* $\mathbb{H}^R(\mathfrak{Q})$.

(iii) $\{({W_i}^R, w_i)\}_{i\in\mathcal{I}}$ *is a fusion frame for* $\mathbb{H}^R(\mathfrak{Q})$.

Proof. Since for $i\in\mathcal{I}$, $\{u_{ij}\}_{j\in J_i}$ is a frame for W_i^R with bounds r_{1i} and r_{2i}, therefore we have

$$\begin{aligned} r_1\sum_{i\in\mathcal{I}} w_i^2||\pi_{W_i^R}(u)||^2 &\leq \sum_{i\in\mathcal{I}} r_{1i}w_i^2||\pi_{W_i^R}(u)||^2 \\ &\leq \sum_{i\in\mathcal{I}}\sum_{j\in J_i} w_i^2|\langle u_{ij}|\pi_{W_i^R}(u)\rangle|^2 \leq r_2\sum_{i\in\mathcal{I}} w_i^2||\pi_{W_i^R}(u)||^2,\ u\in\mathbb{H}^R(\mathfrak{Q}). \end{aligned}$$

Also $\sum_{i\in\mathcal{I}}\sum_{j\in J_i}|\langle u_{ij}w_i|\pi_{W_i^R}(u)\rangle|^2 = \sum_{i\in\mathcal{I}}\sum_{j\in J_i}|\langle u_{ij}w_i|u\rangle|^2, u\in\mathbb{H}^R(\mathfrak{Q})$. Thus (a) and (c) are equivalent. Further

$$w_i^2||\pi_{W_i^R}(u)||^2 = w_i^2||\sum_{j\in J_i} e_{ij}\langle e_{ij}|u\rangle||^2 = \sum_{j\in J_i}|\langle e_{ij}w_i|u\rangle|^2,\ u\in\mathbb{H}^R(\mathfrak{Q}).$$

Hence (b) and (c) are equivalent. □

Lemma 3.2. *If* $\{({W_i}^R, w_i)\}_{i\in\mathcal{I}}$ *is a fusion frame for* $\mathbb{H}^R(\mathfrak{Q})$ *then it is complete.*

Proof. Let $u\in\mathbb{H}^R(\mathfrak{Q})$ be such that $u\perp\overline{\text{span}}\{W_i^R\}_{i\in\mathcal{I}}$, so $\sum_{i\in\mathcal{I}} w_i^2||\pi_{W_i^R}(u)||^2 = 0$ which gives $u=0$. □

In next result we give a necessary and sufficient condition under which a fusion frame form a complete set.

Proposition 3.3. *Let* $\{({W_i}^R, w_i)\}_{i\in\mathcal{I}}$ *be a fusion frame for* $\mathbb{H}^R(\mathfrak{Q})$ *and* $\{e_{ij}\}_{j\in J_i}$ *be an orthonormal basis for* $W_i^R, i\in\mathcal{I}$. *Then* $\{({W_i}^R, w_i)\}_{i\in\mathcal{I}}$ *is complete if and only if* $\{e_{ij}\}_{i\in\mathcal{I},j\in J_i}$ *is complete in* $\mathbb{H}^R(\mathfrak{Q})$.

Proof. Follow from Theorem 3.1 and Lemma 3.2. □

If an element from a frame is removed then either we left with a frame or an incomplete set. In next result we have extended the Theorem 5.4.7 [6] for the case of right quaternionic Hilbert spaces, which will be used to prove subsequent results.

Lemma 3.4. *Let* $\{u_k\}_{k\in\mathcal{I}}$ *be a frame for* $\mathbb{H}^R(\mathfrak{Q})$. *If any element say* u_j *is removed from frame* $\{u_k\}_{k\in\mathcal{I}}$ *then*

(i) If $\langle S^{-1}u_j|u_j\rangle\neq 1$, *then* $\{u_k\}_{k\neq j}$ *is a frame for* $\mathbb{H}^R(\mathfrak{Q})$.

(ii) If $\langle S^{-1}u_j|u_j\rangle = 1$, *then* $\{u_k\}_{k\neq j}$ *is not complete.*

Proof. As $u_j = \sum_{i\in\mathcal{I}} u_i\langle S^{-1}u_i|u_j\rangle$, denote $a_i = \langle S^{-1}u_i|u_j\rangle$, $i\in\mathcal{I}$, as $u_j = \sum_{i\in\mathcal{I}} u_i\delta_{i,j}$ then by Theorem 2.8, we have

$$1 = \sum_{i\in\mathcal{I}}|\delta_{i,j}|^2 = \sum_{i\in\mathcal{I}}|a_i|^2 + \sum_{i\in\mathcal{I}}|a_i-\delta_{i,j}|^2 = \sum_{i\in\mathcal{I}}|a_i|^2 + |a_j-1|^2 + \sum_{i\neq j}|a_i|^2.$$

If $a_j\neq 1$, we have $u_j = \frac{1}{1-a_j}\sum_{i\neq j} u_i a_i$, then for any $u\in\mathbb{H}^R(\mathfrak{Q})$, we compute

$$|\langle u_j|u\rangle|^2 = \left|\frac{1}{1-a_j}\sum_{i\neq j}\overline{a_i}\langle u_i|u\rangle\right|^2 \leq \frac{1}{|1-a_j|^2}\sum_{i\neq j}|a_i|^2\sum_{i\neq j}|\langle u_i|u\rangle|^2 = C\sum_{i\neq j}|\langle u_i|u\rangle|^2.$$

where $C = \frac{1}{|1-a_j|^2}\sum_{i\neq j}|a_i|^2$. Therefore for any $u\in\mathbb{H}^R(\mathfrak{Q})$,

$$r_1||u||^2 \leq \sum_{i\in\mathcal{I}}|\langle u_i|u\rangle|^2 \leq (1+C)\sum_{i\neq j}|\langle u_i|u\rangle|^2.$$

This gives

$$\frac{r_1}{(1+C)} \leq \sum_{i\neq j} |\langle u_i|u\rangle|^2 \leq r_2||u||^2, u \in \mathbb{H}^R(\mathfrak{Q}).$$

If $a_j = 1$ then we have $\sum_{i\neq j} |a_i|^2 = 0$ this implies $\langle S^{-1}u_j|u_i\rangle = 0,\ i \neq j$, since $S^{-1}u_j \neq 0$. Therefore, we have a non-zero element $S^{-1}u_j$ orthogonal to $\{u_i\}_{i\neq j}$. □

Next, we give the following result for fusion frames which is similar to Lemma 3.4 as in case of frames.

Proposition 3.5. *If a subspace from a family of fusion frame is removed then the remaining subspaces either form a fusion frame or it is an incomplete family of subspaces.*

Proof. Let $\{({W_i}^R, w_i)\}_{i\in\mathcal{I}}$ be a fusion frame for $\mathbb{H}^R(\mathfrak{Q})$ and for each $i \in \mathcal{I}$, let $\{e_{ij}\}_{j\in J_i}$ be an orthonormal basis for W_i^R. Then by Theorem 3.1, $\{e_{ij}w_i\}_{i\in\mathcal{I},j\in J_i}$ forms a frame for $\mathbb{H}^R(\mathfrak{Q})$. If $\{e_{ij}w_i\}_{i\in\mathcal{I}\setminus i_0,j\in J_i}$ is a frame therefore again by Theorem 3.1, $\{({W_i}^R, w_i)\}_{i\in\mathcal{I}\setminus i_0}$ forms a fusion frame for $\mathbb{H}^R(\mathfrak{Q})$. Now let $\{e_{ij}w_i\}_{i\in\mathcal{I}\setminus i_0,j\in J_i}$ is not a frame then by Theorem 3.4, it is an incomplete family and hence $\{e_{ij}\}_{i\in\mathcal{I}\setminus i_0,j\in J_i}$ is an incomplete family and hence by Lemma 3.3 $\{({W_i}^R, w_i)\}_{i\in\mathcal{I}\setminus i_0}$ is an incomplete family of subspaces. □

In the following result, we prove that intersection of subspaces of a fusion frame with a right subspace of a right quaternionic Hilbert space forms a fusion frame for the subspace.

Proposition 3.6. *Let $\{({W_i}^R, w_i)\}_{i\in\mathcal{I}}$ be a fusion frame for $\mathbb{H}^R(\mathfrak{Q})$ and V^R be any right subspace of $\mathbb{H}^R(\mathfrak{Q})$, then $\{W_i^R \cap V^R\}_{i\in\mathcal{I}}$ form a fusion frame for V^R with respect to $\{w_i\}_{i\in\mathcal{I}}$ with same fusion frame bounds as of $\{({W_i}^R, w_i)\}_{i\in\mathcal{I}}$.*

Proof. As $\sum_{i\in\mathcal{I}} w_i^2||\pi_{W_i^R}(u)||^2 = \sum_{i\in\mathcal{I}} w_i^2||\pi_{W_i^R\cap V^R}(u)||^2,\ u \in V^R$. Therefore, result holds. □

Let $\{({W_i}^R, w_i)\}_{i\in\mathcal{I}}$ be a Bessel sequence of subspaces for $\mathbb{H}^R(\mathfrak{Q})$. Then the set

$$\left(\sum_{i\in\mathcal{I}} \bigoplus W_i^R\right)_{l^2(\mathfrak{Q})} = \left\{\{u_i\}_{i\in\mathcal{I}} : u_i \in W_i^R, \sum_{i\in\mathcal{I}} ||u_i||^2 < \infty\right\},$$

defines a right quaternionic Hilbert space under the inner product given by

$$\langle\{u_i\}_{i\in\mathcal{I}}|\{p_i\}_{i\in\mathcal{I}}\rangle = \sum_{i\in\mathcal{I}}\langle u_i|p_i\rangle, \quad \{u_i\}_{i\in\mathcal{I}},\ \{p_i\}_{i\in\mathcal{I}} \in \left(\sum_{i\in\mathcal{I}} \bigoplus W_i^R\right)_{l^2(\mathfrak{Q})}$$

Next, we give a Lemma which will be used to define subsequent definitions:

Lemma 3.7. *Let $\{({W_i}^R, w_i)\}_{i\in\mathcal{I}}$ is a Bessel sequence of subspaces then $\sum_{i\in\mathcal{I}} u_iw_i$ converges unconditionally for every $\{u_i\}_{i\in\mathcal{I}} \in \left(\sum_{i\in\mathcal{I}} \bigoplus W_i^R\right)_{l^2(\mathfrak{Q})}$.*

Proof. Let $J \subset \mathcal{I}$ be any finite subset of $\mathcal{I}$ and $u = \sum_{i\in J} u_iw_i$, then we have

$$\left\|\sum_{i\in J} u_iw_i\right\|^4 \leq \left(\sum_{i\in J} w_i||\pi_{W_i^R}(u)||\,||u_i||\right)^2 \leq \sum_{i\in J} w_i^2||\pi_{W_i^R}(u)||^2 \sum_{i\in J} ||u_i||^2 \leq r_2||u||^2 \sum_{i\in J} ||u_i||^2$$

we have $||\sum_{i\in J} u_iw_i||^2 \leq r_2 \sum_{i\in J} ||u_i||^2$, since $\{u_i\}_{i\in\mathcal{I}}$ is a Cauchy sequence and hence series converges unconditionally. □

Definition 3.8. Let $\mathcal{W} = \{({W_i}^R, w_i)\}_{i\in\mathcal{I}}$ be a fusion frame for $\mathbb{H}^R(\mathfrak{Q})$. Then the synthesis operator for $\mathcal{W}$ is a right linear operator $T_{\mathcal{W}} : \left(\sum\limits_{i\in\mathcal{I}} \bigoplus W_i^R\right)_{l^2(\mathfrak{Q})} \to \mathbb{H}^R(\mathfrak{Q})$ given by $T_{\mathcal{W}}(\{u_i\}_{i\in\mathcal{I}}) = \sum\limits_{i\in\mathcal{I}} u_i w_i$.

The adjoint operator $T^*_{\mathcal{W}}$ is called analysis operator for $\mathcal{W}$, in the next result the expression for $T^*_{\mathcal{W}}$ is given.

Proposition 3.9. *Let $\mathcal{W} = \{({W_i}^R, w_i)\}_{i\in\mathcal{I}}$ be a fusion frame for $\mathbb{H}^R(\mathfrak{Q})$. Then the analysis operator $T^*_{\mathcal{W}} : \mathbb{H}^R(\mathfrak{Q}) \longrightarrow \left(\sum\limits_{i\in\mathcal{I}} \bigoplus W_i^R\right)_{l^2(\mathfrak{Q})}$ is given by $T^*_{\mathcal{W}}(u) = \{\pi_{W_i^R}(u)w_i\}_{i\in\mathcal{I}}$.*

Proof. Let $u \in \mathbb{H}^R(\mathfrak{Q})$ and $\{u_i\}_{i\in\mathcal{I}} \in \left(\sum\limits_{i\in\mathcal{I}} \bigoplus W_i^R\right)_{l^2(\mathfrak{Q})}$, consider

$$\begin{aligned}
\langle T^*_{\mathcal{W}}(u)|\{u_i\}_{i\in\mathcal{I}}\rangle &= \langle u|T_{\mathcal{W}}(\{u_i\}_{i\in\mathcal{I}})\rangle \\
&= \sum_{i\in\mathcal{I}} w_i \langle \pi_{W_i^R}(u)|u_i\rangle \\
&= \sum_{i\in\mathcal{I}} \langle \pi_{W_i^R}(u)w_i|u_i\rangle = \langle \{\pi_{W_i^R}(u)w_i\}_{i\in\mathcal{I}}|\{u_i\}_{i\in\mathcal{I}}\rangle.
\end{aligned}$$ □

In next result, we give a necessary and sufficient condition under which a family of subspaces form a Bessel sequence of subspaces.

Proposition 3.10. *Let $\{W_i^R\}_{i\in\mathbb{N}}$ be a sequence of right closed subspaces of a right separable quaternionic Hilbert space $\mathbb{H}^R(\mathfrak{Q})$ and $\{w_i\}_{i\in\mathbb{N}}$ be a family of weights. Then $\mathcal{W} = \{({W_i}^R, w_i)\}_{i\in\mathcal{I}}$ is a Bessel sequence of subspaces for $\mathbb{H}^R(\mathfrak{Q})$ with a bound r_2 if and only if $T_{\mathcal{W}} : \left(\sum\limits_{i\in\mathcal{I}} \bigoplus W_i^R\right)_{l^2(\mathfrak{Q})} \to \mathbb{H}^R(\mathfrak{Q})$ is a well defined, bounded right-linear operator on $\mathbb{H}^R(\mathfrak{Q})$ with $||T_{\mathcal{W}}|| \le r_2$.*

Proof. Forward part holds in view of Lemma 3.7. Conversely, as $||T^*_{\mathcal{W}}|| = ||T_{\mathcal{W}}|| \le \sqrt{r}_2$, therefore for any $u \in \mathbb{H}^R(\mathfrak{Q})$ we have

$$\begin{aligned}
r_2||u||^2 &\ge ||T^*_{\mathcal{W}}(u)||^2 \\
&= \langle \{\pi_{W_i^R}(u)w_i\}_{i\in\mathcal{I}}|\{\pi_{W_i^R}(u)w_i\}_{i\in\mathcal{I}}\rangle \\
&= \sum_{i\in\mathcal{I}} \langle \pi_{W_i^R}(u)w_i|\pi_{W_i^R}(u)w_i\rangle = \sum_{i\in\mathcal{I}} w_i^2 ||\pi_{W_i^R}(u)||^2.
\end{aligned}$$ □

If $\mathcal{W} = \{({W_i}^R, w_i)\}_{i\in\mathcal{I}}$ is a fusion frame for $\mathbb{H}^R(\mathfrak{Q})$. Then the right linear operator $S_{\mathcal{W}} : \mathbb{H}^R(\mathfrak{Q}) \to \mathbb{H}^R(\mathfrak{Q})$ defined by

$$S_{\mathcal{W}}(u) = T_{\mathcal{W}}T^*_{\mathcal{W}}(u) = \sum_{i\in\mathcal{I}} \pi_{W_i^R}(u)w_i^2, \;\; u \in \mathbb{H}^R(\mathfrak{Q})$$

is called the frame operator corresponding to fusion frame $\mathcal{W}$.

In the next result we give some properties of frame operator corresponding to a fusion frame.

Theorem 3.11. *Let $\mathcal{W} = \{({W_i}^R, w_i)\}_{i\in\mathcal{I}}$ be a fusion frame for $\mathbb{H}^R(\mathfrak{Q})$ with lower and upper fusion bounds r_1 and r_2 respectively. Then the frame operator $S_{\mathcal{W}}$ is a positive, bounded, invertible and self adjoint right linear operator on $\mathbb{H}^R(\mathfrak{Q})$.*

Proof. For any $u \in \mathbb{H}^R(\mathfrak{Q})$, we have

$$\langle S_{\mathcal{W}}(u)|u\rangle = \left\langle \sum_{i\in\mathcal{I}} \pi_{W_i^R}(u)w_i^2|u \right\rangle = \sum_{i\in\mathcal{I}} w_i^2 \left\langle \pi_{W_i^R}(u)|\pi_{W_i^R}(u) \right\rangle = \sum_{i\in\mathcal{I}} w_i^2 ||\pi_{W_i^R}(u)||^2.$$

This gives $r_1 I \leq S_{\mathcal{W}} \leq r_2 I$ and hence $S_{\mathcal{W}}$ is a positive and bounded right linear operator on $\mathbb{H}^R(\mathfrak{Q})$. Also $0 \leq I - r_2^{-1} S_{\mathcal{W}} \leq \frac{r_2 - r_1}{r_2} I$ and consequently

$$||I - r_2^{-1} S_{\mathcal{W}}|| = \sup_{||u||=1} |\langle (I - r_2^{-1} S_{\mathcal{W}})(u) | u \rangle| \leq \frac{r_2 - r_1}{r_2} < 1.$$

Hence $S_{\mathcal{W}}$ is an invertible operator, further for any $u_1, u_2 \in \mathbb{H}^R(\mathfrak{Q})$ consider

$$\begin{aligned} \langle S_{\mathcal{W}}(u_1) | u_2 \rangle &= \left\langle \sum_{i \in \mathcal{I}} \pi_{W_i^R}(u_1) w_i^2 | u_2 \right\rangle = \sum_{i \in \mathcal{I}} w_i^2 \left\langle \pi_{W_i^R}(u_1) | u_2 \right\rangle = \sum_{i \in \mathcal{I}} w_i^2 \left\langle u_1 | \pi_{W_i^R}(u_2) \right\rangle \\ &= \left\langle u_1 | \sum_{i \in \mathcal{I}} \pi_{W_i^R}(u_2) w_i^2 \right\rangle = \langle u_1 | S_{\mathcal{W}}(u_2) \rangle . \end{aligned}$$

This implies $S_{\mathcal{W}}^* = S_{\mathcal{W}}$. □

Corollary 3.12. *(The Reconstruction Formula) Let $\mathbb{H}^R(\mathfrak{Q})$ be a right quaternionic Hilbert space and $\mathcal{W} = \{(W_i{}^R, w_i)\}_{i \in \mathcal{I}}$ be a fusion frame for $\mathbb{H}^R(\mathfrak{Q})$, corresponding frame operator is $S_{\mathcal{W}}$. Then for any $u \in \mathbb{H}^R(\mathfrak{Q})$, we have $u = \sum_{i \in \mathcal{I}} S_{\mathcal{W}}^{-1} \pi_{W_i^R}(u) w_i^2$.*

Proof. As $S_{\mathcal{W}}$ is invertible, so for any $u \in \mathbb{H}^R(\mathfrak{Q})$, we have

$$u = S_{\mathcal{W}}^{-1} S_{\mathcal{W}}(u) = \sum_{i \in \mathcal{I}} S_{\mathcal{W}}^{-1} \pi_{W_i^R}(u) w_i^2.$$ □

In next result, we give a necessary and sufficient condition for a family of subspaces to be a fusion frame in terms of their synthesis operator.

Proposition 3.13. *Let $\{W_i^R\}_{i \in \mathcal{I}}$ be a family of right subspaces of $\mathbb{H}^R(\mathfrak{Q})$. Then $\mathcal{W} = \{(W_i{}^R, w_i)\}_{i \in \mathcal{I}}$ is a fusion frame if and only if $T_{\mathcal{W}} : \left(\sum_{i \in \mathcal{I}} \bigoplus W_i^R \right)_{l^2(\mathfrak{Q})} \to \mathbb{H}^R(\mathfrak{Q})$ is a well defined, bounded right linear operator from $\left(\sum_{i \in \mathcal{I}} \bigoplus W_i^R \right)_{l^2(\mathfrak{Q})}$ onto $\mathbb{H}^R(\mathfrak{Q})$.*

Proof. As frame operator $S_{\mathcal{W}} = T_{\mathcal{W}} T_{\mathcal{W}}^*$ is an invertible right linear operator implies $T_{\mathcal{W}}$ is a onto operator. Conversely, by Proposition 3.10 $\mathcal{W}$ is a Bessel sequence of subspaces. Also, in view of Lemma 2.10, there exist a bounded right linear operator $T_{\mathcal{W}}^\dagger : \mathbb{H}^R(\mathfrak{Q}) \to \left(\sum_{i \in \mathcal{I}} \bigoplus W_i^R \right)_{l^2(\mathfrak{Q})}$ such that

$$||T^\dagger||^{-2} ||u||^2 \leq ||T^* u||^2 = \sum_{i \in \mathcal{I}} w_i^2 ||\pi_{W_i^R}(u)||^2, \quad u \in \mathbb{H}^R(\mathfrak{Q}).$$ □

In the next result, we construct a fusion frame with the help of a frame.

Proposition 3.14. *Let $\mathfrak{F} = \{u_i\}_{i \in \mathcal{I}}$ be a frame for $\mathbb{H}^R(\mathfrak{Q})$ with lower and upper frame bounds r_1 and r_2 respectively, and frame operator is $S_{\mathfrak{F}}$. Then $\mathcal{W} = \{(span\{u_i\}, ||u_i||)\}_{i \in \mathcal{I}}$ form a fusion frame for $\mathbb{H}^R(\mathfrak{Q})$ with bounds r_1 and r_2 with frame operator $S_{\mathcal{W}} = S_{\mathfrak{F}}$.*

Proof. For any $u \in \mathbb{H}^R(\mathfrak{Q})$,

$$S_{\mathfrak{F}}(u) = \sum_{i \in \mathcal{I}} u_i \langle u_i | u \rangle = \sum_{i \in \mathcal{I}} ||u_i||^2 \frac{u_i}{||u_i||} \left\langle \frac{u_i}{||u_i||} | u \right\rangle = \sum_{i \in \mathcal{I}} ||u_i||^2 \pi_{\text{span}\{u_i\}}(u) = S_{\mathcal{W}}(u).$$ □

In the following result, projection operator of a right subspace is given in term of its frame operator corresponding to fusion frame.

Proposition 3.15. *Let $\mathcal{W} = \{({W_i}^R, w_i)\}_{i\in\mathcal{I}}$ be a fusion frame for a right subspace V^R of $\mathbb{H}^R(\mathfrak{Q})$. Then the map $\pi_{V^R} : \mathbb{H}^R(\mathfrak{Q}) \to V^R$ given by*

$$\pi_{V^R}(u) = \sum_{i\in\mathcal{I}} S_{\mathcal{W}}^{-1}\pi_{W_i^R}(u)w_i^2,\ u \in \mathbb{H}^R(\mathfrak{Q})$$

is an orthogonal projection of V^R.

Proof. Since $S_{\mathcal{W}} : V^R \to V^R$ implies $\pi_{V^R}(u) = 0,\ \forall\, u \in (V^R)^{\perp}$ and by Corollary 3.12, we have $u = \sum_{i\in\mathcal{I}} S_{\mathcal{W}}^{-1}\pi_{W_i^R}(u)w_i^2,\ \forall\, u \in V^R$. This implies $\pi_{V^R}^2 = \pi_{V^R}$. □

Definition 3.16. If $\mathcal{W} = \{({W_i}^R, w_i)\}_{i\in\mathcal{I}}$ be a fusion frame of $\mathbb{H}^R(\mathfrak{Q})$ with frame operator $S_{\mathcal{W}}$. Then $\{S_{\mathcal{W}}^{-1}W_i^R, w_i\}_{i\in\mathcal{I}}$ is called the dual fusion frame corresponding to fusion frame $\mathcal{W} = \{({W_i}^R, w_i)\}_{i\in\mathcal{I}}$.

One can easily observe that the dual fusion frame is also a fusion frame for $\mathbb{H}^R(\mathfrak{Q})$.

Next, a relationship between frame operator of a fusion frame and frame operator of a frame generated by an orthonormal bases for the corresponding right subspaces of $\mathbb{H}^R(\mathfrak{Q})$ is given.

Proposition 3.17. *Let $\mathcal{W} = \{({W_i}^R, w_i)\}_{i\in\mathcal{I}}$ be a fusion frame for $\mathbb{H}^R(\mathfrak{Q})$ and $\{u_{ij}\}_{j\in J_i}$ be a Parseval frame for W_i^R, $i \in \mathcal{I}$. Then the frame operator $S_{\mathcal{W}}$ of $\{({W_i}^R, w_i)\}_{i\in\mathcal{I}}$ is equal to frame operator $S_{\mathfrak{F}}$ of the frame $\mathfrak{F} = \{u_{ij}w_i\}_{i\in\mathcal{I},j\in J_i}$. Further,*

$$\sum_{i\in\mathcal{I}} S_{\mathcal{W}}^{-1}\pi_{W_i^R}(u)w_i^2 = \sum_{i\in\mathcal{I}}\sum_{j\in J_i} S_{\mathfrak{F}}^{-1}(u_{ij}w_i)\langle u_{ij}w_i|u\rangle,\ u \in \mathbb{H}^R(\mathfrak{Q}).$$

Proof. Since $\{u_{ij}\}_{j\in J_i}$ is a Parseval frame for W_i^R, so we have

$$\pi_{W_i^R}(u) = \sum_{j\in J_i} u_{ij}\langle u_{ij}|\pi_{W_i^R}(u)\rangle = \sum_{j\in J_i} u_{ij}\langle u_{ij}|u\rangle, u \in \mathbb{H}^R(\mathfrak{Q}).$$

This gives

$$\begin{aligned} S_{\mathcal{W}}(u) &= \sum_{i\in\mathcal{I}} \pi_{W_i^R}(u)w_i^2 \\ &= \sum_{i\in\mathcal{I}}\sum_{j\in J_i} u_{ij}\langle u_{ij}|u\rangle w_i^2 \\ &= \sum_{i\in\mathcal{I}}\sum_{j\in J_i} u_{ij}w_i\langle u_{ij}w_i|u\rangle = S_{\mathfrak{F}}(u),\ u \in \mathbb{H}^R(\mathfrak{Q}). \end{aligned}$$

Therefore

$$\begin{aligned} \sum_{i\in\mathcal{I}} S_{\mathcal{W}}^{-1}\pi_{W_i^R}(u)w_i^2 &= \sum_{i\in\mathcal{I}} S_{\mathfrak{F}}^{-1}(\sum_{j\in J_i} u_{ij}\langle u_{ij}|u\rangle)w_i^2 \\ &= \sum_{i\in\mathcal{I}}\sum_{j\in J_i} S_{\mathfrak{F}}^{-1}(u_{ij}w_i)\langle u_{ij}w_i|u\rangle, u \in \mathbb{H}^R(\mathfrak{Q}). \end{aligned}$$

□

Proposition 3.18. *Let $\mathcal{W} = \{({W_i}^R, w_i)\}_{i\in\mathcal{I}}$ be a fusion frame for $\mathbb{H}^R(\mathfrak{Q})$ and $\mathfrak{F}_i = \{u_{ij}\}_{j\in J_i}$ be a frame for W_i^R, $i \in \mathcal{I}$ with canonical dual frame $\tilde{\mathfrak{F}}_i = \{\tilde{u}_{ij} = S_{\mathfrak{F}}^{-1}u_{ij}\}_{j\in\ J_i}$. Then $S_{\mathcal{W}} = \sum_{i\in\mathcal{I}} T_{\tilde{\mathfrak{F}}_i}T_{\mathfrak{F}_i}^{*}w_i^2 = \sum_{i\in\mathcal{I}} T_{\mathfrak{F}_i}T_{\tilde{\mathfrak{F}}_i}^{*}w_i^2$, where $T_{\mathfrak{F}_i}, T_{\mathfrak{F}_i}^{*}$ and $T_{\tilde{\mathfrak{F}}_i}, T_{\tilde{\mathfrak{F}}_i}^{*}$ are synthesis and analysis operators for the frames $\{u_{ij}\}_{j\in J_i}$ and $\{S_{\mathfrak{F}}^{-1}u_{ij}\}_{j\in J_i}$ respectively.*

Proof. As for each $u \in \mathbb{H}^R(\mathfrak{Q})$, $S_{\mathcal{W}}(u) = \sum_{i\in\mathcal{I}}\sum_{j\in J_i} \tilde{u}_{ij}\langle u_{ij}|u\rangle w_i^2 = \sum_{i\in\mathcal{I}}\sum_{j\in J_i} u_{ij}\langle \tilde{u}_{ij}|u\rangle w_i^2$, therefore the result follows. □

Lemma 3.19. *Let $\mathfrak{F} = \{u_i\}_{i\in\mathcal{I}}$ be a frame for $\mathbb{H}^R(\mathfrak{Q})$ with frame operator $S_{\mathfrak{F}}$ and if T be a self adjoint, invertible, right linear operator on $\mathbb{H}^R(\mathfrak{Q})$. Then $\{Tu_i\}_{i\in\mathcal{I}}$ be a frame for $\mathbb{H}^R(\mathfrak{Q})$ with frame operator $TS_{\mathfrak{F}}T$ and its canonical dual frame is $\{T^{-1}S_{\mathfrak{F}}^{-1}u_i\}_{i\in\mathcal{I}}$.*

Proof. Straight forward □

Proposition 3.20. *Let $\mathcal{W} = \{(W_i{}^R, w_i)\}_{i\in\mathcal{I}}$ be a fusion frame for $\mathbb{H}^R(\mathfrak{Q})$ with frame operator $S_{\mathcal{W}}$ and let T be a self adjoint, invertible, right linear operator on $\mathbb{H}^R(\mathfrak{Q})$ such that $TT^*(W_i^R) \subset W_i^R$ for all $i \in \mathcal{I}$. Then $\{(TW_i^R, w_i)\}_{i\in\mathcal{I}}$ is a fusion frame for $\mathbb{H}^R(\mathfrak{Q})$ with frame operator $TS_{\mathcal{W}}T^{-1}$.*

Proof. Let $\mathfrak{F}_\mathfrak{i} = \{u_{ij}\}_{j\in J_i}$ is a frame for W_i^R, $i \in \mathcal{I}$. As $T^*T(W_i^R) \subset W_i^R$ and by Lemma 3.19, $\{Tu_{ij}\}_{j\in J_i}$ be a frame for W_i^R with frame operator $TS_{\mathfrak{F}}T$. The canonical dual frame of $\{Tu_{ij}\}_{j\in J_i}$ is $\{T^{-1}S_{\mathfrak{F}_\mathfrak{i}}^{-1}u_{ij}\}_{j\in J_i}$, so by the Proposition 3.18, we obtain

$$\sum_{i\in\mathcal{I}} \pi_{W_i^R}(u)w_i^2 = \sum_{i\in\mathcal{I}} \left(\sum_{j\in J_i} Tu_{ij}\langle T^{-1}S_{\mathfrak{F}_\mathfrak{i}}^{-1}u_{ij}|u\rangle \right) w_i^2 = TS_{\mathcal{W}}T^{-1}u,\ u \in \mathbb{H}^R(\mathfrak{Q}). \quad \square$$

Next, we give an equivalent condition for a fusion frame to be Parseval fusion frame in term of Parseval frame of a right quaternionic Hilbert space.

Proposition 3.21. *For each $i \in \mathcal{I}$, let $w_i > 0$ and $\{u_{ij}\}_{j\in J_i}$ be a Parseval frame sequence for $\mathbb{H}^R(\mathfrak{Q})$. Define $W_i^R = \overline{span}_{j\in J_i}\{u_{ij}\}$, $i \in \mathcal{I}$ and let $\{e_{ij}\}_{j\in J_i}$ is an orthonormal basis for $W_i^R, i \in \mathcal{I}$. Then the following are equivalent.*

(i) $\{u_{ij}w_i\}_{i\in\mathcal{I},j\in J_i}$ *is a Parseval frame for* $\mathbb{H}^R(\mathfrak{Q})$.

(ii) $\{e_{ij}w_i\}_{i\in\mathcal{I},j\in J_i}$ *is a Parseval frame for* $\mathbb{H}^R(\mathfrak{Q})$.

(iii) $\{(W_i{}^R, w_i)\}_{i\in\mathcal{I}}$ *is a Parseval fusion frame for* $\mathbb{H}^R(\mathfrak{Q})$.

Proof. Straight forward □

Proposition 3.22. $\mathcal{W} = \{(W_i{}^R, w_i)\}_{i\in\mathcal{I}}$ *is a Parseval fusion frame for $\mathbb{H}^R(\mathfrak{Q})$ if and only if the frame operator $S_{\mathcal{W}} = I$ on $\mathbb{H}^R(\mathfrak{Q})$.*

Proof. For each $i \in \mathcal{I}$, let $\{e_{ij}\}_{j\in J_i}$ be an orthonormal basis for W_i^R. Let $\{(W_i{}^R, w_i)\}_{i\in\mathcal{I}}$ is a Parseval fusion frame for $\mathbb{H}^R(\mathfrak{Q})$ then by Theorem 3.11 we have $S_{\mathcal{W}} = I$. Conversely, let $S_{\mathcal{W}} = I$ implies

$$u = S_{\mathcal{W}}(u) = \sum_{i\in\mathcal{I}} \pi_{W_i^R}(u)w_i^2 = \sum_{i\in\mathcal{I}}\sum_{j\in J_i} e_{ij}\langle e_{ij}|u\rangle w_i^2, u \in \mathbb{H}^R(\mathfrak{Q}).$$

Consider

$$\begin{aligned} ||u||^2 = \langle u|u\rangle &= \left\langle \sum_{i\in\mathcal{I}}\sum_{j\in J_i} e_{ij}\langle e_{ij}|u\rangle w_i^2 | u \right\rangle \\ &= \sum_{i\in\mathcal{I}}\sum_{j\in J_i} w_i^2\overline{\langle u|e_{ij}\rangle}\langle e_{ij}|u\rangle \\ &= \sum_{i\in\mathcal{I}}\sum_{j\in J_i} w_i^2|\langle e_{ij}|u\rangle|^2 \\ &= \sum_{i\in\mathcal{I}} w_i^2 \sum_{j\in J_i} |\langle e_{ij}|u\rangle|^2 = \sum_{i\in\mathcal{I}} w_i^2||\pi_{W_i^R}(u)||^2, u \in \mathbb{H}^R(\mathfrak{Q}). \end{aligned} \quad \square$$

Proposition 3.23. *Let $\{W_i^R\}_{i\in\mathcal{I}}$ be a family of right subspaces in $\mathbb{H}^R(\mathfrak{Q})$. Then $\{W_i^R\}_{i\in\mathcal{I}}$ is an orthonormal basis of subspaces of $\mathbb{H}^R(\mathfrak{Q})$ if and only if $\{W_i^R\}_{i\in\mathcal{I}}$ is a 1-uniform Parseval fusion frame of $\mathbb{H}^R(\mathfrak{Q})$.*

Proof. For each $i \in \mathcal{I}$, $\{e_{ij}\}_{j\in J_i}$ be an orthonormal basis for W_i^R. Let $\mathbb{H}^R(\mathfrak{Q}) = \bigoplus_{i\in\mathcal{I}} W_i^R$ this gives $\{e_{ij}\}_{i\in\mathcal{I},j\in J_i}$ is an orthonormal basis for $\mathbb{H}^R(\mathfrak{Q})$ and hence

$$||u||^2 = \sum_{i\in\mathcal{I}}\sum_{j\in J_i} |\langle e_{ij}|u\rangle|^2 = \sum_{i\in\mathcal{I}} ||\pi_{W_i^R}(u)||^2,\ \ u \in \mathbb{H}^R(\mathfrak{Q}).$$

Conversely, as $||u||^2 = \sum_{i\in\mathcal{I}} ||\pi_{W_i^R}(u)||^2 = \sum_{i\in\mathcal{I}}\sum_{j\in J_i} |\langle e_{ij}|u\rangle|^2, u \in \mathbb{H}^R(\mathfrak{Q})$. Hence $\{e_{ij}\}_{i\in\mathcal{I},j\in J_i}$ is an orthonormal basis for $\mathbb{H}^R(\mathfrak{Q})$ which implies $\mathbb{H}^R(\mathfrak{Q}) = \bigoplus W_i^R$. □

4 Perturbation of fusion frames in quaternionic Hilbert spaces

In this section, we will study different types of perturbation of fusion frames in quaternionic Hilbert spaces. Casazza and Kutynoik [9], studied a perturbation in context of Hilbert spaces same can discussed in the context of quaternionic Hilbert spaces.

Definition 4.1. Let $\{W_i^R\}_{i\in\mathcal{I}}$ and $\{\widetilde{W}_i^R\}_{i\in\mathcal{I}}$ be family of closed subspaces in $\mathbb{H}^R(\mathfrak{Q})$, $\{w_i\}_{i\in\mathcal{I}}$ be a family of positive real number. Let there exists $0 \leq \lambda_1, \lambda_2 < 1$ and $\epsilon > 0$ such that

$$||(\pi_{W_i^R} - \pi_{\widetilde{W}_i^R})u|| \leq \lambda_1||\pi_{W_i^R}u|| + \lambda_2||\pi_{\widetilde{W}_i^R}u|| + \epsilon||u||,\ \forall u \in \mathbb{H}^R(\mathfrak{Q}),\ \forall i \in \mathcal{I}.$$

then we say $\{(\widetilde{W}_i^R, w_i)\}_{i\in\mathcal{I}}$ is a $(\lambda_1, \lambda_2, \epsilon)$- perturbation of $\{({W_i}^R, w_i)\}_{i\in\mathcal{I}}$

Proposition 4.2. *Let $\{({W_i}^R, w_i)\}_{i\in\mathcal{I}}$ be a fusion frame for $\mathbb{H}^R(\mathfrak{Q})$ with fusion frame bounds r_1 and r_2, suppose $0 \leq \lambda_1 < 1$ and $\epsilon > 0$ be such that $(1-\lambda_1)\sqrt{r}_1-\epsilon(\sum\limits_{i\in\mathcal{I}} w_i^2)^{\frac{1}{2}} > 0$. Let $\{(\widetilde{W}_i^R, w_i)\}_{i\in\mathcal{I}}$ be a $(\lambda_1, \lambda_2, \epsilon)-$perturbation of $\{({W_i}^R, w_i)\}_{i\in\mathcal{I}}$ for some $0 \leq \lambda_2 < 1$. Then $\{(\widetilde{W}_i^R, w_i)\}_{i\in\mathcal{I}}$ form a fusion frame for $\mathbb{H}^R(\mathfrak{Q})$ with fusion frame bounds $\left[\frac{(1-\lambda_1)\sqrt{r}_1-\epsilon(\sum\limits_{i\in\mathcal{I}} w_i^2)^{\frac{1}{2}}}{1+\lambda_2}\right]^2$ and $\left[\frac{\sqrt{r}_2(1+\lambda_1)+\epsilon(\sum\limits_{i\in\mathcal{I}} w_i^2)^{\frac{1}{2}}}{1-\lambda_2}\right]^2$.*

Proof. Similar to proof of Theorem 5.2 in [9]. □

Example 4.3. Let us consider the Example 2.4, and $\widetilde{W}_i^R = \overline{span}\{e_i\}, \forall i \in \mathbb{N}$. Then for any $\epsilon > 0$ such that $\epsilon < (1-\lambda_1)$ and $\lambda_2 = 1-\lambda_1$. The family $\{(\widetilde{W}_i^R, w_i)\}_{i\in\mathcal{I}}$ is a $(\lambda_1, \lambda_2, \epsilon)$-perturbation of $\{({W_i}^R, w_i)\}_{i\in\mathcal{I}}$. Hence by Theorem 4, $\{(\widetilde{W}_i^R, w_i)\}_{i\in\mathcal{I}}$ forms a fusion frame for $\mathbb{H}^R(\mathfrak{Q})$.

Chirstensen [7] proved the Paley-Wiener Theorem for frames in Hilbert spaces. In Theorem 4.1 [15], authors discussed for frames in right quaternionic Hilbert spaces.

Definition 4.4. Let $\{u_i\}_{i\in\mathcal{I}}$ and $\{\tilde{u}_i\}_{i\in\mathcal{I}}$ be two sequences in a right quaternionic Hilbert space $\mathbb{H}^R(\mathfrak{Q})$, we say $\{\tilde{u}_i\}_{i\in\mathcal{I}}$ is a (λ_1, λ_2)-perturbation of $\{u_i\}_{i\in\mathcal{I}}$ for some $0 \leq \lambda_1, \lambda_2 < 1$ if

$$||\sum_{i\in\mathcal{I}}(u_i - \tilde{u}_i)q_i|| \leq \lambda_1||\sum_{i\in\mathcal{I}} u_i q_i|| + \lambda_2||\sum_{i\in\mathcal{I}} \tilde{u}_i q_i||,\ \ \{q_i\}_{i\in\mathcal{I}} \in l^2(\mathfrak{Q}).$$

If $\{u_i\}_{i\in\mathcal{I}}$ is a frame for $\mathbb{H}^R(\mathfrak{Q})$ and $\{\tilde{u}_i\}_{i\in\mathcal{I}}$ is a (λ_1, λ_2)-perturbation of $\{u_i\}_{i\in\mathcal{I}}$ for some $0 \leq \lambda_1, \lambda_2 < 1$, then by Theorem 4.1 [15], $\{\tilde{u}_i\}_{i\in\mathcal{I}}$ is a frame for $\mathbb{H}^R(\mathfrak{Q})$. Using this we can extended Proposition 5.4 [9], in case of right quaternionic Hilbert space.

Proposition 4.5. *Let $\{u_i\}_{i\in\mathcal{I}}$ be a frame sequence in $\mathbb{H}^R(\mathfrak{Q})$ and let $\{\tilde{u}_i\}_{i\in\mathcal{I}}$ is a (λ_1, λ_2)-perturbation of $\{u_i\}_{i\in\mathcal{I}}$ for some $0 \leq \lambda_1, \lambda_2 < 1$. Then the following hold:*

(i) for all $\{q_i\}_{i\in\mathcal{I}} \in l^2(\mathfrak{Q})$ we have

$$\frac{1-\lambda_1}{1+\lambda_2}||\sum_{i\in\mathcal{I}} \tilde{u}_i q_i|| \leq ||\sum_{i\in\mathcal{I}} u_i q_i|| \leq \frac{1+\lambda_2}{1-\lambda_1}||\sum_{i\in\mathcal{I}} \tilde{u}_i q_i||$$

(ii) If we let $W^R = span_{i\in\mathcal{I}}\{u_i\}$ and $\tilde{W}^R = span_{i\in\mathcal{I}}\{\tilde{u}_i\}$, then

$$||\pi_{W^R}(\pi_{\tilde{W}^R}(u))|| \geq \left(\frac{1-\lambda_1}{1+\lambda_2} - \lambda_1\frac{1+\lambda_2}{1-\lambda_1} - \lambda_2\right)||\pi_{\tilde{W}^R}(u)||, u \in \mathbb{H}^R(\mathfrak{Q}).$$

Moreover, π_{W^R} is a right isomorphism on range of $\pi_{\tilde{W}^R}$ provided $\lambda_1, \lambda_2 \leq \frac{1}{5}$.

Proof. (i) First part follows from Theorem 4.1 [15].
(ii) Let $S_{\tilde{\mathfrak{F}}}$ is the frame operator corresponding to frame $\tilde{\mathfrak{F}} = \{\tilde{u}_i\}_{i\in\mathcal{I}}$, then for any $u \in \mathbb{H}^R(\mathfrak{Q})$

$$\begin{aligned} ||\sum_{i\in\mathcal{I}}(u_i - \tilde{u}_i)\langle S_{\tilde{\mathfrak{F}}}^{-1}\tilde{u}_i|u\rangle|| &\leq \lambda_1||\sum_{i\in\mathcal{I}} u_i\langle S_{\tilde{\mathfrak{F}}}^{-1}\tilde{u}_i|u\rangle|| + \lambda_2||\sum_{i\in\mathcal{I}}\tilde{u}_i\langle S_{\tilde{\mathfrak{F}}}^{-1}\tilde{u}_i|u\rangle|| \\ &\leq \lambda_1\frac{1+\lambda_2}{1-\lambda_1}||\sum_{i\in\mathcal{I}}\tilde{u}_i\langle S_{\tilde{\mathfrak{F}}}^{-1}\tilde{u}_i|u\rangle|| + \lambda_2||\sum_{i\in\mathcal{I}}\tilde{u}_i\langle S_{\tilde{\mathfrak{F}}}^{-1}\tilde{u}_i|u\rangle|| \\ &= \left(\lambda_1\frac{1+\lambda_2}{1-\lambda_1} + \lambda_2\right)||\pi_{\tilde{W}^R}(u)||. \end{aligned}$$

Therefore we have

$$\begin{aligned} ||\pi_{W^R}(\pi_{\tilde{W}^R}(u))|| &= ||\pi_{W^R}(\sum_{i\in\mathcal{I}}\tilde{u}_i\langle S_{\tilde{\mathfrak{F}}}^{-1}\tilde{u}_i|u\rangle)|| \\ &\geq ||\sum_{i\in\mathcal{I}}\pi_{W^R}(u_i)\langle S_{\tilde{\mathfrak{F}}}^{-1}\tilde{u}_i|u\rangle|| - ||\sum_{i\in\mathcal{I}}\pi_{W^R}(u_i-\tilde{u}_i)\langle S_{\tilde{\mathfrak{F}}}^{-1}\tilde{u}_i|u\rangle|| \\ &\geq \left(\frac{1-\lambda_1}{1+\lambda_2} - \lambda_1\frac{1+\lambda_2}{1-\lambda_1} - \lambda_2\right)||\pi_{\tilde{W}^R}(u)||,\ u\in\mathbb{H}^R(\mathfrak{Q}). \end{aligned}$$

□

Remark 4.6. In part (ii) of Proposition 4.5, we can change π_{W^R} and $\pi_{\tilde{W}^R}$ without affecting the bounds.

Proposition 4.7. *Let $\{(W_i{}^R, w_i)\}_{i\in\mathcal{I}}$ be a fusion frame for $\mathbb{H}^R(\mathfrak{Q})$ with fusion frame bounds r_1 and r_2 respectively. Further let $\{u_{ij}\}_{j\in J_i}$ is a frame for W_i^R, $i\in\mathcal{I}$ and $\{\tilde{u}_{ij}\}_{j\in J_i}$ be a (λ_1,λ_2)-perturbation of $\{u_{ij}\}_{j\in J_i}$ for some $0\leq\lambda_1,\lambda_2<1$. Let $1-\frac{\epsilon^2}{2} = \left(\frac{1-\lambda_1}{1+\lambda_2} - \lambda_1\frac{1+\lambda_2}{1-\lambda_1} - \lambda_2\right)$ and $\sqrt{r_1} - \epsilon(\sum_{i\in\mathcal{I}} w_i^2)^{\frac{1}{2}} > 0$. Then $\{(\tilde{W}_i^R, w_i)\}_{i\in\mathcal{I}}$ is a fusion frame for $\mathbb{H}^R(\mathfrak{Q})$ with fusion frame bounds $\left[\sqrt{r}_1 - \epsilon(\sum_{i\in\mathcal{I}} w_i^2)^{\frac{1}{2}}\right]^2$ and $\left[\sqrt{r}_2 + \epsilon(\sum_{i\in\mathcal{I}} w_i^2)^{\frac{1}{2}}\right]^2$, where $\tilde{W}_i^R = span\{\tilde{u}_{ij}\}_{j\in J_i}, i\in\mathcal{I}$.*

Proof. For any $u\in\mathbb{H}^R(\mathfrak{Q})$ and fixed $i\in\mathcal{I}$ by Proposition 4.5 we have

$$\begin{aligned} ||\pi_{W_i^R}(u)||^2 &\geq ||\pi_{\tilde{W}_i^R}\pi_{W_i^R}(u)||^2 + ||(I-\pi_{\tilde{W}_i^R})\pi_{W_i^R}(u)||^2 \\ &\geq (1-\frac{\epsilon^2}{2})||\pi_{W_i^R}(u)||^2 + ||(I-\pi_{\tilde{W}_i^R})\pi_{W_i^R}(u)||^2. \end{aligned}$$

Hence by Remark 4.6 we have

$$||(I-\pi_{\tilde{W}_i^R})\pi_{W_i^R}(u)||^2 \leq \frac{\epsilon^2}{2}||\pi_{W_i^R}(u)||^2, ||(I-\pi_{W_i^R})\pi_{\tilde{W}_i^R}(u)||^2 \leq \frac{\epsilon^2}{2}||\pi_{\tilde{W}_i^R}(u)||^2, u\in\mathbb{H}^R(\mathfrak{Q}).$$

Now this result follow by Proposition 4.2 and fact that

$$\begin{aligned} ||(\pi_{W_i^R} - \pi_{\tilde{W}_i^R})(u)||^2 &= \langle(\pi_{W_i^R} - \pi_{\tilde{W}_i^R})^2(u)|u\rangle \\ &= \langle(\pi_{W_i^R} - \pi_{\tilde{W}_i^R}\pi_{W_i^R} + \pi_{\tilde{W}_i^R} - \pi_{W_i^R}\pi_{\tilde{W}_i^R})(u)|u\rangle \\ &\leq ||(I-\pi_{\tilde{W}_i^R})(\pi_{W_i^R}(u)) + (I-\pi_{W_i^R})(\pi_{\tilde{W}_i^R}(u))||||u|| \\ &\leq \frac{\epsilon^2}{2}||\pi_{W_i^R}(u)||||u|| + \frac{\epsilon^2}{2}||\pi_{\tilde{W}_i^R}(u)||||u|| \\ &\leq \epsilon^2||u||^2, u\in\mathbb{H}^R(\mathfrak{Q}). \end{aligned}$$

□

5 Woven fusion frames in quaternionic Hilbert spaces

Definition 5.1. Let $\mathcal{H} = \{(H_i{}^R, v_i)\}_{i\in\mathcal{I}}$ and $\mathcal{W} = \{(W_i{}^R, w_i)\}_{i\in\mathcal{I}}$ be two fusion frames for $\mathbb{H}^R(\mathfrak{Q})$. Then $\mathcal{H}$ and $\mathcal{W}$ are said to be woven fusion frame for $\mathbb{H}^R(\mathfrak{Q})$ if there exist constants $0<r_1\leq r_2<\infty$ such that for any subset σ of $\mathcal{I}$, the family $\{(H_i{}^R, v_i)\}_{i\in\sigma}\cup\{(W_i{}^R, w_i)\}_{i\in\sigma^c}$ is a fusion frame for $\mathbb{H}^R(\mathfrak{Q})$ with lower and upper fusion frame bounds r_1 and r_2.

If only upper bound condition is hold then $\mathcal{H}$ and $\mathcal{W}$ are said to form a woven Bessel sequence of subspaces for $\mathbb{H}^R(\mathfrak{Q})$ with bound r_2. The constants r_1 and r_2 are called the universal woven fusion frame bounds.

In order to show the existence of woven fusion frames we have have the following examples:

Example 5.2. Let $\mathbb{H}^R(\mathfrak{Q})$ be a right quaternionic Hilbert space with the orthonormal basis $\{e_i\}_{i\in\mathcal{I}}$, $v_i = 1$ and $w_i = 2,\ i \in \mathcal{I}$. Let $W_i^R = H_i^R = \text{span}\{e_i\}$ then $\{({H_i}^R, v_i)\}_{i\in\mathcal{I}}$ and $\{({W_i}^R, w_i)\}_{i\in\mathcal{I}}$ are fusion frames for $\mathbb{H}^R(\mathfrak{Q})$. Therefore, for any $u \in \mathbb{H}^R(\mathfrak{Q})$ and for any subset $\sigma \subset \mathcal{I}$, we have

$$||u||^2 \le \sum_{i\in\sigma} |\langle e_i|u\rangle|^2 + \sum_{i\in\sigma^c} 4|\langle e_i|u\rangle|^2 = ||u||^2 + 3\sum_{i\in\sigma^c} \langle e_i|u\rangle|^2 \le 4||u||^2.$$

Thus $\{({H_i}^R, v_i)\}_{i\in\mathcal{I}}$ and $\{({W_i}^R, w_i)\}_{i\in\mathcal{I}}$ form a woven fusion frame for $\mathbb{H}^R(\mathfrak{Q})$.

Example 5.3. Let $\mathbb{H}^R(\mathfrak{Q})$ be a right quaternionic Hilbert space with the orthonormal basis $\{e_i\}_{i\in\mathbb{N}}$, $v_i = 1$ and $w_i = 2,\ i \in \mathcal{I}$. Let $W_1^R = \text{span}\{e_2\}$, and $W_2^R = \text{span}\{e_1\}$,
$W_i^R = \text{span}\{e_i\},\ i \ge 3$ and $H_i^R = \text{span}\{e_i\}, i \in \mathbb{N}$. Clearly $\{({W_i}^R, w_i)\}_{i\in\mathcal{I}}$ and $\{({H_i}^R, v_i)\}_{i\in\mathcal{I}}$ are fusion frames for $\mathbb{H}^R(\mathfrak{Q})$, but they do not form a woven fusion frame for $\mathbb{H}^R(\mathfrak{Q})$. Consider $\sigma = \{2\} \subset \mathbb{N}$, then

$$\sum_{j\in\sigma^c} v_j^2||\pi_{H_i^R}(e_2)||^2 + \sum_{j\in\sigma} w_j^2||\pi_{W_i^R}(e_2)||^2 = \sum_{j\in\mathcal{I}/\{2\}} |\langle e_i|e_2\rangle|^2 + 4|\langle e_1|e_2\rangle|^2 = 0.$$

Next, we give a necessary condition for the existence of woven fusion frame.

Proposition 5.4. *Let $\mathbb{H}^R(\mathfrak{Q})$ be a right quaternionic Hilbert space with an orthonormal basis $\{e_i\}_{i\in\mathcal{I}}$. Let $\{({H_i}^R, v_i)\}_{i\in\mathcal{I}}$ and $\{({W_i}^R, w_i)\}_{i\in\mathcal{I}}$ are two fusion frames for $\mathbb{H}^R(\mathfrak{Q})$ such that for each $j \in \mathcal{I}$, $\{\pi_{H_j^R}(e_k)\}_{k\in\mathcal{I}}$ and $\{\pi_{W_j^R}(e_k)\}_{k\in\mathcal{I}}$ are orthogonal sets. Suppose for each $k \in \mathcal{I}$, there exists a constant $M > 0$ and a sequence of purely real quaternions $\{\beta_{ij}^k\}_{i,j\in\mathcal{I}}$ such that $\inf\{|\beta_{kj}^k|^2 : j \in \mathcal{I}\} \ge M > 0$ and $\pi_{W_j^R}(e_k) = \sum_{i\in\mathcal{I}} \pi_{H_j^R}(e_i)\frac{v_j}{w_j}\beta_{ij}^k$, $\forall j \in \mathcal{I}$. Then $\{({H_i}^R, v_i)\}_{i\in\mathcal{I}}$ and $\{({W_i}^R, w_i)\}_{i\in\mathcal{I}}$ form a woven fusion frame for $\mathbb{H}^R(\mathfrak{Q})$.*

Proof. Let r_{11}, r_{12} and r_{21}, r_{22} be the lower and upper fusion frame bounds for fusion frames $\{({H_i}^R, v_i)\}_{i\in\mathcal{I}}$ and $\{({W_i}^R, w_i)\}_{i\in\mathcal{I}}$ respectively. Let $\sigma \subset \mathcal{I}$ be any subset and for any $u = \sum_{k\in\mathcal{I}} e_k\gamma_k \in \mathbb{H}^R(\mathfrak{Q})$, then

$$\begin{aligned}(r_{12} + r_{22})||u||^2 &\ge \sum_{j\in\sigma} v_j^2||\pi_{H_i^R}(u)||^2 + \sum_{j\in\sigma^c} w_j^2||\pi_{W_i^R}(u)||^2 \\ &= \sum_{j\in\sigma} v_j^2||\pi_{H_i^R}(u)||^2 + \sum_{j\in\sigma^c} w_j^2||\pi_{W_i^R}(\sum_{k\in\mathcal{I}} e_k\gamma_k)||^2.\end{aligned}$$

Since

$$\left\|\pi_{W_i^R}\left(\sum_{k\in\mathcal{I}} e_k\gamma_k\right)\right\|^2 = \left\langle \pi_{W_i^R}\left(\sum_{k\in\mathcal{I}} e_k\gamma_k\right) \middle| \pi_{W_i^R}\left(\sum_{k\in\mathcal{I}} e_k\gamma_k\right)\right\rangle = \sum_{k\in\mathcal{I}} |\bar{\gamma}_k|^2||\pi_{W_i^R}(e_k)||^2$$

and

$$||\pi_{W_i^R}(e_k)||^2 = \langle \pi_{W_i^R}(e_k)|\pi_{W_i^R}(e_k)\rangle = \sum_{i\in\mathcal{I}} \frac{v_j^2}{w_j^2}|\bar{\beta}_{ij}^k|^2||\pi_{H_i^R}(e_i)||^2.$$

Therefore, for $u \in \mathbb{H}^R(\mathfrak{Q})$

$$r_{11}\min\{1, M\}||u||^2 \le \sum_{j\in\sigma} v_j^2||\pi_{H_i^R}(u)||^2 + \sum_{j\in\sigma^c} w_j^2||\pi_{W_i^R}(u)||^2 \le (r_{12} + r_{22})||u||^2. \qquad \square$$

In the next example, we construct woven fusion frame using Proposition 5.4.

Example 5.5. Let $\mathbb{H}^R(\mathfrak{Q})$ be a right quaternionic Hilbert space with an orthonormal basis $\{e_i\}_{i\in\mathbb{N}}$. Let $H_i^R = W_i^R = \text{span}\{e_i\}$ and $v_i = w_i = 1$, $\forall\, i \in \mathbb{N}$. Then $\{({H_i}^R, v_i)\}_{i\in\mathbb{N}}$ and $\{({W_i}^R, w_i)\}_{i\in\mathbb{N}}$ are fusion frames with fusion frame bounds $r_{11} = r_{12} = 1$ and $r_{21} = r_{22} = 1$. As $\{\pi_{H_i^R}(e_k)\}_{k\in\mathbb{N}}$ and $\{\pi_{W_i^R}(e_k)\}_{k\in\mathbb{N}}$ are orthogonal sets and we define $\{\beta_{ij}^k\}_{i,j\in\mathbb{N}}$ as $\beta_{ij}^k = 1$ if $i = k$ and $\beta_{ij}^k = 0$ otherwise. Then $\inf\{|\beta_{kj}^k|^2 : j \in \mathbb{N}\} = 1 > 0$.

$$\sum_{i\in\mathbb{N}} \pi_{H_j^R}(e_i)\frac{v_j}{w_j}\beta_{ij}^k = \pi_{H_j^R}(e_k)\frac{v_j}{w_j}\beta_{kj}^k = \pi_{H_j^R}(e_k) = \pi_{W_j^R}(e_k).$$

Therefore, by Proposition 5.4, $\{({H_i}^R, v_i)\}_{i\in\mathcal{I}}$ and $\{({W_i}^R, w_i)\}_{i\in\mathcal{I}}$ form a woven fusion frame.

In the following result, we show that family of two Bessel sequence of subspaces are always woven.

Proposition 5.6. *Let $\{({H_i}^R, v_i)\}_{i\in\mathcal{I}}$ and $\{({W_i}^R, w_i)\}_{i\in\mathcal{I}}$ be two Bessel sequences of subspaces for $\mathbb{H}^R(\mathfrak{Q})$ with Bessel bounds r_{12} and r_{22} respectively. Then the family $\{({H_i}^R, v_i)\}_{i\in\mathcal{I}}$ and $\{({W_i}^R, w_i)\}_{i\in\mathcal{I}}$ is a woven Bessel sequence of subspaces with Bessel bound $r_{12} + r_{22}$.*

Proof. For any subset σ of $\mathcal{I}$, any $u \in \mathbb{H}^R(\mathfrak{Q})$,

$$\begin{aligned}\sum_{i\in\sigma} v_i^2||\pi_{H_i^R}(u)||^2 + \sum_{i\in\sigma^c} w_i^2||\pi_{W_i^R}(u)||^2 &\le \sum_{i\in\mathcal{I}} v_i^2||\pi_{H_i^R}(u)||^2 + \sum_{i\in\mathcal{I}} w_i^2||\pi_{W_i^R}(u)||^2 \\ &\le (r_{12}+r_{22})||u||^2.\end{aligned}$$

□

Theorem 5.7. *Suppose $\{({H_i}^R, v_i)\}_{i\in\mathcal{I}}$ and $\{({W_i}^R, w_i)\}_{i\in\mathcal{I}}$ are two fusion frames for $\mathbb{H}^R(\mathfrak{Q})$ with lower and upper fusion frame bounds r_{11}, r_{12} and r_{21}, r_{22} respectively. Let there exist $M > 0$ such that for any subset $J \subset \mathcal{I}$,*

$$\sum_{i\in J} ||\pi_{H_i^R}(u)v_i - \pi_{W_i^R}(u)w_i||^2 \le M \min\left\{\sum_{i\in J} ||\pi_{H_i^R}(u)v_i||^2, \sum_{i\in J} ||\pi_{W_i^R}(u)w_i||^2\right\},\ u \in \mathbb{H}^R(\mathfrak{Q}).$$

Then $\{({H_i}^R, v_i)\}_{i\in\mathcal{I}}$ and $\{({W_i}^R, w_i)\}_{i\in\mathcal{I}}$ form a woven fusion frame with universal woven fusion frame bounds $\frac{(r_{11}+r_{21})}{(2M+3)}$ and $(r_{12} + r_{22})$.

Proof. For any subset σ of $\mathcal{I}$, consider

$$\begin{aligned}(r_{11}+r_{21})||u||^2 &\le \sum_{i\in\mathcal{I}} v_i^2||\pi_{H_i^R}(u)||^2 + \sum_{i\in\mathcal{I}} w_i^2||\pi_{W_i^R}(u)||^2 \\ &\le \sum_{i\in\sigma} ||\pi_{H_i^R}(u)v_i||^2 + 2(\sum_{i\in\sigma^c} ||\pi_{H_i^R}(u)v_i - \pi_{W_i^R}(x)w_i||^2 \\ &+ \sum_{i\in\sigma^c} ||\pi_{W_i^R}(u)w_i||^2) + 2(\sum_{i\in\sigma} ||\pi_{W_i^R}(u)w_i - \pi_{H_i^R}(u)v_i||^2 \\ &+ \sum_{i\in\sigma} ||\pi_{H_i^R}(u)v_i||^2) \sum_{i\in\sigma^c} ||\pi_{W_i^R}(u)w_i||^2 \\ &\le (2M+3)(\sum_{i\in\sigma} ||\pi_{H_i^R}(u)v_i||^2 + \sum_{i\in\sigma^c} ||\pi_{W_i^R}(u)w_i||^2),\ u \in \mathbb{H}^R(\mathfrak{Q}).\end{aligned}$$

and the upper bound condition follow by Proposition 5.6. □

In the next result, we give a necessary condition for fusion frames to be woven in terms of existence of a positive right linear operator.

Theorem 5.8. *Let $\{({H_i}^R, v_i)\}_{i\in\mathcal{I}}$ and $\{({W_i}^R, w_i)\}_{i\in\mathcal{I}}$ be two fusion frames for $\mathbb{H}^R(\mathfrak{Q})$. For any subset $J \subseteq \mathcal{I}$, suppose that the operator $U_J : \mathbb{H}^R(\mathfrak{Q}) \to \mathbb{H}^R(\mathfrak{Q})$ given by*

$$U_J(u) = \sum_{i\in J}[\pi_{W_i^R}(u)w_i^2 - \pi_{H_i^R}(u)v_i^2],\ u \in \mathbb{H}^R(\mathfrak{Q})$$

is a positive right linear operator on $\mathbb{H}^R(\mathfrak{Q})$. Then $\{({H_i}^R, v_i)\}_{i\in\mathcal{I}}$ and $\{({W_i}^R, w_i)\}_{i\in\mathcal{I}}$ form a woven fusion frames for $\mathbb{H}^R(\mathfrak{Q})$.

Proof. Let r_{11}, r_{12} and r_{21}, r_{22} be lower and upper fusion frame bounds for $\{({H_i}^R, v_i)\}_{i\in\mathcal{I}}$ and $\{({W_i}^R, w_i)\}_{i\in\mathcal{I}}$ respectively. Then for any subset $\sigma \subset \mathcal{I}$ and any $u \in \mathbb{H}^R(\mathfrak{Q})$, we have

$$\begin{aligned} r_{11}||u||^2 \leq \sum_{i\in\mathcal{I}} v_i^2||\pi_{H_i^R}(u)||^2 &= \sum_{i\in\sigma} v_i^2||\pi_{H_i^R}(u)||^2 + \langle \sum_{i\in\sigma^c} \pi_{H_i^R}(u)v_i^2|u\rangle \\ &= \sum_{i\in\sigma} v_i^2||\pi_{H_i^R}(u)||^2 + \langle \sum_{i\in\sigma^c} \pi_{W_i^R}(u)w_i^2|u\rangle - \langle U_{\sigma^c}(u)|u\rangle \\ &\leq \sum_{i\in\sigma} v_i^2||\pi_{H_i^R}(u)||^2 + \sum_{i\in\sigma^c} w_i^2||\pi_{W_i^R}(u)||^2 \leq (r_{12}+r_{22})||u||^2. \end{aligned}$$ □

In the next example, we construct woven fusion frame using Theorem 5.8.

Example 5.9. Let $\mathbb{H}^R(\mathfrak{Q})$ has an orthonormal basis $\{e_i\}_{i\in\mathbb{N}}$, then for any $u \in \mathbb{H}^R(\mathfrak{Q})$ we have $u = \sum_{i\in\mathbb{N}} e_i\langle e_i|u\rangle$. Let $H_i^R = \text{span}\{e_i, e_{i+1}\}$ and $W_i^R = \text{span}\{e_i, e_{i+1}, e_{i+2}\}$ and $v_i = w_i = 1, \forall i \in \mathbb{N}$. Then $\{({H_i}^R, v_i)\}_{i\in\mathbb{N}}$ and $\{({W_i}^R, w_i)\}_{i\in\mathbb{N}}$ are fusion frames with fusion frame bounds $1, 2$ and $1, 3$ respectively and for any subset $J \subset \mathbb{N}$ and $u \in \mathbb{H}^R(\mathfrak{Q})$,

$$\begin{aligned} U_J(u) &= \sum_{i\in J}[\pi_{W_i^R}(u)w_i^2 - \pi_{H_i^R}(u)v_i^2] = \sum_{i\in J}\left[\sum_{k=i}^{k=i+2} e_i\langle e_i|u\rangle - \sum_{k=i}^{i+1} e_i\langle e_i|u\rangle\right] \\ &= \sum_{i\in J} e_{i+2}\langle e_{i+2}|u\rangle. \end{aligned}$$

This implies $\langle U_J(u)|u\rangle \geq 0$, $u \in \mathbb{H}^R(\mathfrak{Q})$. Therefore, by Theorem 5.8, $\{({H_i}^R, v_i)\}_{i\in\mathbb{N}}$ and $\{({W_i}^R, w_i)\}_{i\in\mathbb{N}}$ form a woven fusion frame for $\mathbb{H}^R(\mathfrak{Q})$.

For the completion we give two result which can be proved on the similar lines as in the case of complex Hilbert space.

Theorem 5.10. *Let $\{u_i\}_{i\in\mathbb{N}}$ and $\{p_i\}_{i\in\mathbb{N}}$ are frames for $\mathbb{H}^R(\mathfrak{Q})$. Assume that for every two disjoint, finite sets $G, J \subset \mathbb{N}$ and every $\epsilon > 0$, there exists subsets $\sigma, \delta \subset \mathbb{N}\setminus(G\cup J)$ with $\delta = \mathbb{N}\setminus(G\cup J\cup\sigma)$ so that the lower frame bounds of $\{u_i\}_{i\in G\cup\sigma}\cup\{p_i\}_{i\in J\cup\delta}$ is less than ϵ. Then there exist a subset $M \subset \mathbb{N}$ so that $\{f_i\}_{i\in M}\cup\{g_i\}_{i\in M^c}$ is not a frame for $\mathbb{H}^R(\mathfrak{Q})$ and hence $\{u_i\}_{i\in\mathbb{N}}$ and $\{p_i\}_{i\in\mathbb{N}}$ are not a woven frames for $\mathbb{H}^R(\mathfrak{Q})$.*

Proof. Similar to the prove of Lemma 4.3 [19]. □

Theorem 5.11. *Let $\{({H_i}^R, v_i)\}_{i\in\mathbb{N}}$ and $\{({W_i}^R, w_i)\}_{i\in\mathbb{N}}$ are fusion frames for $\mathbb{H}^R(\mathfrak{Q})$. Assume that for every two disjoint, finite sets $G, J \subset \mathbb{N}$ and every $\epsilon > 0$, there exists subsets $\sigma, \delta \subset \mathbb{N}\setminus(G\cup J)$ with $\delta = \mathbb{N}\setminus(G\cup J\cup\sigma)$ so that the lower fusion frame bounds of $\{({H_i}^R, v_i)\}_{i\in G\cup\sigma}\cup\{({W_i}^R, w_i)\}_{i\in J\cup\delta}$ is less than ϵ. Then there exist a subset $M \subset \mathbb{N}$ so that $\{({H_i}^R, v_i)\}_{i\in M}\cup\{({W_i}^R, w_i)\}_{i\in M^c}$ is not a fusion frames for $\mathbb{H}^R(\mathfrak{Q})$ and hence $\{({H_i}^R, v_i)\}_{i\in\mathbb{N}}$ and $\{({W_i}^R, w_i)\}_{i\in\mathbb{N}}$ are not a woven fusion frame for $\mathbb{H}^R(\mathfrak{Q})$.*

Proof. Similar to the prove of Lemma 4.3 [19]. □

Next we give perturbation theorems for woven fusion frame in quaternionic Hilbert spaces.

Theorem 5.12. *Let $\{({H_i}^R, v_i)\}_{i\in\mathcal{I}}$ and $\{({W_i}^R, w_i)\}_{i\in\mathcal{I}}$ be fusion frames for $\mathbb{H}^R(\mathfrak{Q})$ with lower and upper fusion bounds r_{11}, r_{12} and r_{21}, r_{22} respectively. Assume that there exist scalars $\mu \geq 0$, $0 \leq \lambda_1 < \frac{1}{2}$, $\lambda_2 \geq 0$, such that $r_{11}(\frac{1}{2}-\lambda_1) - \lambda_2 r_{22} - \mu > 0$ and*

$$\sum_{i\in\mathcal{I}} ||(\pi_{H_i^R}v_i - \pi_{W_i^R}w_i)(u)||^2 \leq \lambda_1\sum_{i\in\mathcal{I}} v_i^2||\pi_{H_i^R}(u)||^2 + \lambda_2\sum_{i\in\mathcal{I}} w_i^2||\pi_{W_i^R}(u)||^2 + \mu||u||^2,\ u \in \mathbb{H}^R(\mathfrak{Q}).$$

Then $\{({H_i}^R, v_i)\}_{i\in\mathcal{I}}$ and $\{({W_i}^R, w_i)\}_{i\in\mathcal{I}}$ form a woven fusion frames for $\mathbb{H}^R(\mathfrak{Q})$.

Proof. Let $\sigma \subseteq I$ be any subset. Consider

$$\begin{aligned}
(r_{12}+r_{22})||u||^2 &\geq \sum_{i\in\sigma} v_i^2||\pi_{H_i^R}(u)||^2 + \sum_{i\in\sigma^c} w_i^2||\pi_{W_i^R}(u)||^2 \\
&\geq \sum_{i\in\sigma} v_i^2||\pi_{H_i^R}(u)||^2 + \frac{1}{2}\sum_{i\in\sigma^c} v_i^2||\pi_{H_i^R}(u)||^2 \\
&- \sum_{i\in\sigma^c} ||\pi_{H_i^R}(u)v_i - \pi_{W_i^R}(u)w_i||^2 \\
&\geq \left((\frac{1}{2}-\lambda_1)r_{11} - \lambda_2 r_{22} - \mu\right)||u||^2,\ u \in \mathbb{H}^R(\mathfrak{Q}).
\end{aligned}$$

□

Next example will demonstrate the Theorem 5.12.

Example 5.13. Let $\{e_i\}_{i\in\mathbb{N}}$ be an orthonormal basis for $\mathbb{H}^R(\mathfrak{Q})$. Let $H_i^R = W_i^R = \text{span}\{e_i\}$, and $v_i = 2, w_i = 1, \forall i \in \mathbb{N}$. Then $\{(H_i{}^R, v_i)\}_{i\in\mathbb{N}}$ and $\{(W_i{}^R, w_i)\}_{i\in\mathbb{N}}$ are fusion frames for $\mathbb{H}^R(\mathfrak{Q})$ with fusion frame bounds $r_{11} = r_{12} = 4$ and $r_{21} = r_{22} = 1$ respectively. Then for $\lambda_1 = \frac{1}{4}$, $\lambda_2 = \frac{1}{8}$ and $\mu = \frac{3}{4}$ we have $r_{11}(\frac{1}{2}-\lambda_1) - \lambda_2 r_{22} - \mu > 0$,

$$\begin{aligned}
\sum_{i\in\mathbb{N}} ||(\pi_{H_i^R}v_i - \pi_{W_i^R}w_i)(u)||^2 = \sum_{i\in\mathbb{N}} |\langle e_i|u\rangle|^2 \leq \lambda_1 \sum_{i\in\mathbb{N}} v_i^2||\pi_{H_i^R}(u)||^2 \\
+ \lambda_2 \sum_{i\in\mathbb{N}} w_i^2||\pi_{W_i^R}(u)||^2 + \mu||u||^2,\ u \in \mathbb{H}^R(\mathfrak{Q}).
\end{aligned}$$

Therefore, by Theorem 5.12, $\{(H_i{}^R, v_i)\}_{i\in\mathbb{N}}$ and $\{(W_i{}^R, w_i)\}_{i\in\mathbb{N}}$ form a woven fusion frame for $\mathbb{H}^R(\mathfrak{Q})$.

Theorem 5.14. *Let $\{(H_i{}^R, v_i)\}_{i\in\mathcal{I}}$ and $\{(W_i^R, v_i)\}_{i\in\mathcal{I}}$ are fusion frames for $\mathbb{H}^R(\mathfrak{Q})$ with fusion frame bounds r_{11}, r_{12} and r_{21}, r_{22} respectively. Let $\{u_{ij}\}_{j\in J_i}$ and $\{p_{ij}\}_{j\in J_i}$ are frames for H_i^R and W_i^R for each $i \in \mathcal{I}$. Suppose $\{p_{ij}\}_{j\in J_i}$ is a (λ_1, λ_2)-perturbation of $\{u_{ij}\}_{j\in J_i}$ for some $0 \leq \lambda_1, \lambda_2 < 1$ and let $\epsilon > 0$, such that*

$$1-\epsilon^2 = \left(\frac{1-\lambda_1}{1+\lambda_2} - \lambda_1\frac{1+\lambda_2}{1-\lambda_1} - \lambda_2\right)^2, \epsilon^2\left(\sum_{i\in\mathcal{I}} v_i^2\right) > (r_{12}+r_{22}) + 2\epsilon(\sqrt{r}_{12}+\sqrt{r}_{22})\left(\sum_{i\in\mathcal{I}} v_i^2\right)^{\frac{1}{2}}.$$

Then $\{(H_i{}^R, v_i)\}_{i\in\mathcal{I}}$ and $\{(W_i^R, v_i)\}_{i\in\mathcal{I}}$ form a woven fusion frame.

Proof. For all $u \in \mathbb{H}^R(\mathfrak{Q})$ and $i \in \mathcal{I}$, by Proposition 4.7 we have $||(\pi_{H_i^R} - \pi_{W_i^R})(u)|| \leq \epsilon||u||$. Let $\sigma \subset \mathcal{I}$ be any subset then, for $u \in \mathbb{H}^R(\mathfrak{Q})$

$$\begin{aligned}
\sum_{i\in\sigma} v_i^2||\pi_{H_i^R}(u)||^2 + \sum_{i\in\sigma^c} v_i^2||\pi_{W_i^R}(u)||^2 &\geq \sum_{i\in\sigma} v_i^2\left(||\pi_{W_i^R}(u)|| - ||\pi_{W_i^R}(u) - \pi_{H_i^R}(u)||\right)^2 \\
&+ \sum_{i\in\sigma^c} v_i^2\left(||\pi_{H_i^R}(u)|| - ||\pi_{H_i^R}(u) - \pi_{W_i^R}(u)||\right)^2 \\
&\geq -r_{22}||u||^2 - r_{12}||u||^2 + \epsilon^2||u||^2(\sum_{i\in\mathcal{I}} v_i^2) \\
&- 2\epsilon||u||\left[(\sum_{i\in\sigma} v_i^2)^{\frac{1}{2}}(\sum_{i\in\sigma} v_i^2||\pi_{W_i^R}(u)||^2)^{\frac{1}{2}} + (\sum_{i\in\sigma^c} v_i^2)^{\frac{1}{2}}(\sum_{i\in\sigma^c} v_i^2||\pi_{H_i^R}(u)||^2)^{\frac{1}{2}}\right] \\
&\geq \left[\epsilon^2(\sum_{i\in\mathcal{I}} v_i^2) - (r_{12}+r_{22}) - 2\epsilon(\sqrt{r}_{12}+\sqrt{r}_{22})(\sum_{i\in\mathcal{I}} v_i^2)^{\frac{1}{2}}\right]||u||^2.
\end{aligned}$$

By Proposition 5.6 we get the upper bound condition. □

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References

[1] A. Bhandari and S. Mukherjee, *“Atomic Subspaces for Operators"*, Indian Journal of Pure and Applied Mathematics, Vol. 51(3), 1039-1052, 2020.

[2] Ellouz, H. *Some Properties of K-Frames in Quaternionic Hilbert Spaces*, Complex Anal. Oper. Theory 14, 8 (2020). https://doi.org/10.1007/s11785-019-00964-5.

[3] I. Daubechies, A. Grossmann and Y. Meyer, *Painless non-orthogonal expansions*, J. Math. Physics, 27 (1986), 1271-1283.

[4] M. Khokulan, K. Thirulogasanthar and S. Srisatkunarajah, *Discrete frames on finite dimensional quaternion Hilbert spaces*, Proceedings of Jaffna University International Research Conference (JUICE 2014).

[5] Nitin Sharma, S.K. Sharma, Khole Timothy Poumai, *Woven K-Frames in Quaternionic Hilbert Spaces*, Poincare Journal of Analysis and Applications, Vol. 9, No. 2 (2022), 377-398.

[6] O. Christensen, *An introduction to Frames and Riesz Bases*, Birkhäuser, 2003.

[7] O. Christensen,*A Paley–Wiener theorem for frames*, Proc. Am. Math. Soc. 123, 2199–2202 (1995).

[8] P.G. Casazza, G. Kutyniok, *Frames of subspaces, in Wavelets, Frames and Operator Theory*, College Park, MD, 2003, in Contemp. Math., vol. 345, Amer. Math. Soc., Providence, RI, 2004, pp. 87–113..

[9] Peter G. Casazza, Gitta Kutyniok, Shidong Li, *Fusion frames and distributed processing*, Applied and Computational Harmonic Analysis, Volume 25, Issue 1, 2008, Pages 114-132, ISSN 1063-5203, https://doi.org/10.1016/j.acha.2007.10.001.

[10] R. Ghiloni, , V. Moretti and A. Perotti, *Continuous slice functional calculus in quaternionic Hilbert spaces*, Rev. Math. Phys. 25 (2013), 1350006 (83 pages).

[11] R.J. Duffin and A.C. Schaeffer, *A class of non-harmonic Fourier series*, Trans. Amer. Math. Soc., 72 (1952), 341-366.

[12] Ruchi Bhardwaj, S.K. Sharma and S.K. Kaushik, *Trace class operators via OPV- Frames*, Filomat 35:13 (2021), 4353–4368,https://doi.org/10.2298/FIL2113353B.

[13] S.L. Adler, *Quaternionic Quantum Mechanics and Quantum Fields*, Oxford University Press, New York, 1995.

[14] Sumit Kumar Sharma and Shanshank Goel, *Frames in Quaternionic Hilbert Spaces*, Journal of Mathematical Physics, Analysis, Geometry, 2017, pp. 1-19, doi:10.15407/mag15.03.395.

[15] S.K. Sharma, A.M.Jarrrah and S.K.Kaushik, *Frame of operators in Quaterninionic Hilbert Spaces*, arXiv:2003.00546.

[16] S.K. Sharma, Ghanshyam Singh and Soniya Sahu, *Duals of a frame in quaternionic Hilbert spaces*, arXiv:1803.05773.

[17] S. K. Sharma, Nitin Sharma, Khole Timothy Poumai, *“Woven Frames in Quaternionic Hilbert Spaces"*, Journal of Mathematics, vol. 2021, Article ID 6694892, 7 pages, 2021.

[18] S.K. Sharma, Nitin Sharma, Khole Timothy Poumai, *K-Fusion Frames in Quaternionic Hilbert Spaces*, Poincare Journal of Analysis and Applications, Vol. 10, No. 1(2023), 119-133.

[19] T. Bemrose, P. G. Casazza, Gröchenig, M. C. Lammers, and R. G. Lynch, R.G., *Weaving frames*, Oper. Matrices 10(4), 1093–1116 (2016).

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