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AN ANALYSIS ON THE WEIGHTED PSEUDO ALMOST PERIODIC SOLUTIONS OF HORNNS WITH VARIABLE DELAYS

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Abstract This study belongs to a class of high order recurrent neural networks differential equations with variable delays. With the help of theory of weighted pseudo almost periodic(WPAP) functions and some differential inequalities, we investigate the existence of the solutions of the considered model and the exponential stability of these solutions. An example is given as an application of these results.

1 Introduction

Recurrent neural networks are class of artificial neural networks in which connections between units form a directed loop. RNNs are feed-forward neural networks powered by the addition of edges extending over adjacent time steps, which brings the concept of time to the normal neural network model. Recently, In recent years, the fact that repetitive neural networks (RNNs) have application fields such as signal processing and relational memory in science and engineering have led many researchers to investigate their dynamic properties such as stability, oscillations, periodicity, almost and pseudo almost periodicity of solutions [7, 8, 9, 10, 11, 12, 13, 14, 15, 16, 17, 18]. These types of dynamic behaviors have been extensively studied, especially first order neural networks [[7, 8, 9, 10, 11, 12]]. We will take account of the high-order term that attracts many researchers because of its impressive computing, storage and learning capabilities, stronger approach features and faster convergence rate, and higher error tolerance compared to first order [20, 22, 23, 24]. Diagana [5] introduced to the literature of the weighted pseudo almost Periodic (WPAP) functions class is a wider class of almost periodic and pseudo almost periodic functions. As far as we know, there is no study related to WPAP solutions of the following HORNNs with mixed variable delays and initial condition:

$$\begin{aligned} x'_i(t) &= -\varsigma_i(t)x_i(t) + \sum_{j=1}^{n} b_{ij}(t)B_j(x_j(t)) + \sum_{j=1}^{n} c_{ij}(t)C_j(x_j(q_1(t)) \\ &+ \sum_{j=1}^{n} p_{ij}(t)\int_{-\infty}^{t} K_{i\,j}\,(t\,-\,s)\,q_j(x_j(s))ds + J_i(t) \\ &+ \sum_{j=1}^{n}\sum_{k=1}^{n} W_{ijk}(t)R_k(x_k(t))\widehat{R}_j(x_j(t)) + \sum_{j=1}^{n}\sum_{k=1}^{n} T_{ijk}(t)Q_k(x_k(q_2(t))\widehat{Q}_j(x_j(q_3(t))), \end{aligned} \tag{1.1}$$

where $q_1(t) = t - \tau_{ij}(t)$, $q_2(t) = t - \rho_{ijk}(t)$ and $q_3(t) = t - \sigma_{ijk}(t)$.

$$x_i(s) = \varphi_i(s), \quad s \in (-\infty, 0], \; i = 1, 2, ..., n. \tag{1.2}$$

where $n \geq 2$ defines the number of units in the system, $x_i(t)$ denotes to the neuron i at time t, the ς_i is positive decay rate. In addition, $B_j, C_j, q_j, R_k, \widehat{R}_k, Q_k$ and $\widehat{Q}_k$ are bounded continuous functions and the activation of the i th neurons,$c_{ij}(t)$,$b_{ij}(t)$,$p_{ij}(t)$ and, $T_{ijk}(t)$,$W_{ijk}(t)$can be found in [25], $J_i(t)$ is the external input unit i and $\tau_i(t)$, $\rho_{ijk}(t)$ and $\sigma_{ijk}(t)$are the transmission variable delays at time t. Being motivated with the above discussions, in this paper, we try to get some results for WPAP solutions of system (1.1).

2 Preliminaries

Let $BC(R,R)$ denotes collection of bounded continuous functions, $BC(R,R)$is exact space with norm $\|\omega\|_\infty = \sup_{t\in R}|\omega(t)|$. We use the notations

$$\omega^+ = \sup_{t\in R}|\omega(t)|\,,\ \omega^- = \inf_{t\in R}|\omega(t)|\,,$$

where $\omega(t) \in BC(R,R)$.

We now give some basic information.

Definition 2.1. [1] A complex valued function $\gamma(x)$ defined for $-\infty < x < \infty$ is called almost periodic, if for any $\varepsilon > 0$, there exists a trigonometric polynomial $Q_\varepsilon(x)$such that

$$|\gamma(x) - Q_\varepsilon(x)| < \varepsilon.$$

Let Λ denotes the set of functions (weight) $\upsilon(t) \in R$, which are locally integrable functions. If $\upsilon \in \Lambda$, then we set

$$\eta(Q_r) := \int_{Q_r} \upsilon(x)dx, Q := [-r, r].$$

The space of weights Λ_∞ is given by

$$\Lambda_\infty := \left\{\upsilon \in \Lambda : \inf_{x\in R} \upsilon(x) = \upsilon_0 > 0 \text{ and } \lim_{r\to\infty} \upsilon(Q_r) = \infty\right\}.$$

Definition 2.2. [5] Let $\upsilon \in \Lambda_\infty$. A function $f \in BC(R,R)$ is called WPAP function if it can be expressed as

$$f = f_1 + f_2,$$

where $f_1 \in AP(R)$ and $f_2 \in PAP_0(R,R,\upsilon)$. $PAP_0(R,R,\upsilon)$ is defined by

$$PAP_0(R,\upsilon) = \left\{f_2 \in BC(R,R) : \lim_{r\to\infty} \frac{1}{\upsilon([-r,r])} \int_{-r}^{r} \|f_2(t)\|\, \upsilon(t)dt = 0\right\}.$$

Lemma 2.3. *[26] Fix $\upsilon \in \Lambda_\infty$. Assume that for any$s \in R$,*

$$\overline{\lim_{|t|\to\infty}} \frac{\upsilon(s+t)}{\upsilon(t)} < \infty.$$

Then $PAP(R,R,\upsilon)$ is translation invariant.

In view of Lemma 2.3, we give the translation invariant class of WPAP functions as follows:

$$\Lambda_\infty^{\text{Inv}} := \left\{\upsilon \in \Lambda_\infty : \overline{\lim_{|t|\to\infty}} \frac{\upsilon(t+s)}{\upsilon(t)} \text{is finite, ,for all } s \in R\right\}.$$

In the light of this information, for any $\upsilon \in \Lambda_\infty^{\text{Inv}}$, $PAP(R,R,\upsilon)$ is Banach space.

Lemma 2.4. *[22] Let $\upsilon \in \Lambda_\infty^{Inv}$. If, $\vartheta(t) \in C^1(R,R)$, $\iota(t) \in [0,\infty)$, $\iota'(t) \in (-\infty,1]$, $u(t) \in PAP(R,\upsilon)$, then $u(t-\iota(t)) \in PAP(R,R,\upsilon)$.*

The following conditions are given for our main results:

(N_0) $B_j, C_j, R_j, \widehat{R}_j, h_j, Q_j$ and Q_j are global Lipschitz functions with Lipschitz constants respectively $L_j^B, L_j^C, L_j^R, L_j^{\widehat{R}}, L_j^h, L_j^Q$ and $L_j^{\widehat{Q}}$

(N_1) For all $u \in R$, ${\tau'}_{ij}, {\rho'}_{ijk}, {\sigma'}_{ijk} \in (-\infty, 1]$ and

$$M[\varsigma_i] = \lim_{T\to+\infty} \frac{1}{T} \int_t^{t+T} \varsigma_i(u)du > 0,$$

$K_{ij}(t)$ is nonnegative WPAP function and it satisfies

$$\int_0^{+\infty} K_{ij}(s)ds = 1, for\ all\ 1 \leq i, j \leq n.$$

(N_2) $\eta_i > 0$ and $\xi_i > 0$ constants, the following inequality should be achieved

$$\sup_{t\in R} \{-\varsigma_i^-(t) + \left(\sum_{j=1}^n \xi_i^{-1} b_{ij}(t) L_j^B \xi_j^{-1} + \sum_{j=1}^n c_{ij}(t) L_j^C \xi_j^{-1}\right) + \sum_{j=1}^n \xi_i^{-1} p_{ij}(t) L_j^h K_{i\,j} \xi_j^{-1}$$

$$+\xi_i^{-1} \sum_{j=1}^n \sum_{k=1}^n \left(W_{ijk}(t)\xi_j^{-1} L_k^R L_j^{\widehat{R}} + T_{ijk}(t)\xi_k^{-1} L_k^Q L_j^{\widehat{Q}}\right)\} < -\pi_i < 0,$$

(N_3)

$$\sup\nolimits_{T>0} \left\{\int_{-T}^T e^{-\varsigma^-(T+t)} \upsilon(t)dt\right\} < \infty.$$

3 MAIN RESULTS

Lemma 3.1. *For* $i, j, k \in N$ *if* $x_i(t) \in PAP(R, R, \upsilon)$, *then* $\sum\limits_{j=1}^n b_{ij}(t) B_j(x_j(t))$, $\sum\limits_{j=1}^n c_{ij}(t) C_j(x_j(q_1(t)))$, $\sum\limits_{j=1}^n p_{ij}(t) \int_{-\infty}^t K_{i\,j}(t - s) h_j(x_j(t))ds$, $\sum\limits_{j=1}^n \sum\limits_{k=1}^n W_{ijk}(t) R_k(x_k(t)) \widehat{R}_j(x_j(t))$, $\sum\limits_{j=1}^n \sum\limits_{k=1}^n T_{ijk}(t) Q_k(x_k(q_2(t))) \widehat{Q}_j(x_j(q_3(t))) \in PAP(R, R, \upsilon)$.

Lemma 3.2. *2 Define a nonlinear operator* P *as*

$$\begin{aligned}
(P\varphi)(t) &= \int_{-\infty}^t e^{-\int_s^t \varsigma_i(u)du} [\sum_{j=1}^n b_{ij}(v) B_j(\varphi_j(v))\xi_i^{-1} + \sum_{j=1}^n c_{ij}(v) C_j(\varphi_j(q_1(t))\xi_i^{-1} \\
&+ \sum_{j=1}^n \xi_i^{-1} \int_{-\infty}^v K_{i\,j}(s - v) h_j(\varphi_j(s)) p_{ij}(v)ds + \xi_i^{-1} \sum_{j=1}^n \sum_{k=1}^n W_{ijk}(v) R_k \varphi_k(v)) \widehat{R}_j(\varphi_j(v)) \\
&+ \sum_{j=1}^n \sum_{k=1}^n T_{ijk}(v) Q_k(\varphi_k(q_2(t)) \widehat{Q}_j(\varphi_j(q_3(t)) + \xi_i^{-1} J_i(v)] dv \xi_i^{-1}, \varphi \in PAP(R, R, \upsilon).
\end{aligned}$$

Then $P\varphi \in PAP(R, R, \upsilon)$.

Proof. If conditions $(N_0) - (N_3)$ are checked, $\varphi \in BC(R, R)$can be easily appeared. According to Definition 2.2 and Lemma 3.1, there are $A_j^1(t) \in AP(R, R)$and $A_j^2(t) \in PAP_0(R, R, \upsilon)$ such that

$$\begin{aligned} A_j^1(t)+A_j^2(t) &= \xi_i^{-1}\sup_{t\in R}\Bigg|\sum_{j=1}^n b_{ij}(t)B_j(\varphi_j(t))+\sum_{j=1}^n c_{ij}(t)\;C_j(\varphi(q_1(t)) \\ &+ \sum_{j=1}^n p_{ij}(t)\int_{-\infty}^t h_j(\varphi_j(s))K_{i\,j}\,(t\;-\;s)ds+\sum_{j=1}^n\sum_{k=1}^n W_{ijk}(t)R_k\varphi_k(t))\widehat{R}_j(\varphi_j(t)) \\ &\quad+\sum_{j=1}^n\sum_{k=1}^n T_{ijk}(t)Q_k(\varphi_k(q_2(t))\widehat{Q}_j(q_3(t))+J_i(t)\Bigg| \\ &\quad\le\sup_{t\in R}\xi_i^{-1}\left[\left(\sum_{j=1}^n|b_{ij}(t)|\,L_j^B\xi_j^{-1}+\sum_{j=1}^n c_{ij}(t)L_j^C\xi_j^{-1}\right)+\sum_{j=1}^n p_{ij}(t)L_j^h K_{i\,j}\xi_j^{-1}\right. \\ &\quad\left.+\sum_{j=1}^n\sum_{k=1}^n\left(W_{ijk}(t)\xi_j^{-1}L_k^R L_j^{\widehat{R}}+T_{ijk}(t)\xi_k^{-1}L_k^Q L_k^{\widehat{Q}}\right)\right]\in PAP(R,R,\upsilon). \end{aligned}$$

Considering (N_1) and exponential dichotomy in [22], we can arrive

$$\int_{-\infty}^t e^{-\int_s^t \varsigma_i(\xi)d\xi}A_j^1(s)ds\in AP(R,R),$$

which satisfies of the following differential equation

$$z'(t)+\alpha_i(t)z(t)=A_j^1(t),\ i,j\in N.$$

Taking into account [21], one can see

$$\lim_{T\to+\infty}\frac{1}{\int_{-T}^T\rho(t)dt}\int_{-T}^{T}\int_{0}^{+\infty}e^{-(\varsigma_i)^+\xi}\left|A_j^2(t-\xi)\right|d\xi\upsilon(t)dt=0.$$

As a consequence of previous expressions, the following result occurs

$$0\le\lim_{T\to+\infty}\frac{1}{\int_{-T}^T\rho(t)dt}\int_{-T}^{T}\int_{-\infty}^{t}e^{-\int_s^t\varsigma_i(v)dv}\left|A_j^2(t)\right|ds\upsilon(t)dt$$

$$=\lim_{T\to+\infty}\frac{1}{\int_{-T}^T\rho(t)dt}\int_{-T}^{T}\int_{0}^{+\infty}e^{-\varsigma_i^-\xi}\left|A_j^2(t-\xi)\right|d\xi\upsilon(t)dt=0.$$

This shows that

$$\int_{-\infty}^t e^{-\int_s^t \varsigma_i(\xi)d\xi}A_j^2(s)ds\in PAP_0(R,R,\upsilon).$$

Hence $P\varphi\in PAP(R,R,\upsilon)$. □

We give our stability results by the following theorem.

Theorem 3.3. *Assume that* $(N_0)-(N_3)$*, then the (1.1) has a unique WPAP solution* $\overline{x}(t)\in PAP(R,R,\upsilon)$.

Proof. Let $z_i(t)=\xi_i^{-1}x_i(t)$, then system (1.1) turns into the following equation

$$z'_i(t)-\alpha_i(t)u_i(t)=\sum_{j=1}^n\xi_i^{-1}c_{ij}(t)C_j(z_j(q_1(t))\xi_j)+\sum_{j=1}^n\xi_i^{-1}b_{ij}(t)B_j(z_j(t)\xi_j)$$

$$+\xi_i^{-1}\sum_{j=1}^{n}p_{ij}(t)\int_{-\infty}^{t}h_j(z_j(t)\xi_j)K_{i\,j}\,(t\ -\ s)ds+\sum_{j=1}^{n}\sum_{k=1}^{n}W_{ijk}(t)R_k(z_k(t)\xi_k)\widehat{R}_j(z_j(t)\xi_j)\xi_i^{-1}$$

$$+\xi_i^{-1}\sum_{j=1}^{n}\sum_{k=1}^{n}T_{ijk}(t)\widehat{Q}_j(z_j(q_2(t))\xi_j)Q_k(z_k(q_3(t))\xi_k)+J_i(t)\xi_i^{-1}.$$

For $u, v \in PAP(R, R, \upsilon)$, given $(N_0)-(N_3)$, we have

$$\begin{aligned}
&\left|(Eu)_i(t)-(Ev)_i(t)\right|\\
= &\ |\int_{-\infty}^{t}e^{-\int_s^t\varsigma_i(u)du}\{\sum_{j=1}^{n}b_{ij}(s)\left[B_j(u_j(s)\xi_j)-B_j(v_j(s)\xi_j)\right]\xi_i^{-1}\\
+ &\ \sum_{j=1}^{n}c_{ij}(s)\left[C_j(u_j(q_1(s))\xi_j)-C_j(v_j(q_1(s))\xi_j)\right]\xi_i^{-1}\\
+ &\ \sum_{j=1}^{n}p_{ij}(s)\int_{-\infty}^{s}K_{i\,j}\,(s\ -\ r)\left[h_j(u_j(r)\xi_j)-h_j(v_j(r)\xi_j)\right]dr\xi_i^{-1}\\
+ &\ \sum_{j=1}^{n}\sum_{k=1}^{n}W_{ijk}(s)\left[R_k(u_k(s)\xi_k)\widehat{R}_j(u_j(s)\xi_j)-R_k(v_k(s)\xi_k)\widehat{R}_j(v_j(s)\xi_j)\right]\xi_i^{-1}\\
+ &\ \sum_{j=1}^{n}\sum_{k=1}^{n}\xi_i^{-1}T_{ijk}(s)\left[Q_k(u_k(q_2(s))\xi_k)\widehat{Q}_j(u_j(q_3(s))\xi_j)\right]\Big|\,ds\\
- &\ \sum_{j=1}^{n}\sum_{k=1}^{n}\xi_i^{-1}T_{ijk}(s)[Q_k(v_k(q_2(s))\xi_k)\widehat{Q}_j(v_j(q_3(s))\xi_j)]ds\\
= &\ |\int_{-\infty}^{t}e^{-\int_s^t\varsigma_i(\xi)d\xi}\xi_i^{-1}\sum_{j=1}^{n}b_{ij}(s)[B_j(u_j(s)\xi_j)-B_j(v_j(s)\xi_j)]\\
+ &\ \sum_{j=1}^{n}c_{ij}(s)\left[C_j(u_j(q_1(s))\xi_j)-C_j(v_j(q_1(s))\xi_j)\right]\xi_i^{-1}\\
+ &\ \sum_{j=1}^{n}p_{ij}(s)\int_{-\infty}^{s}K_{i\,j}\,(s\ -\ r)\left[h_j(u_j(r)\xi_j)-h_j(v_j(r)\xi_j)\right]dr\xi_i^{-1}\\
+ &\ \sum_{j=1}^{n}\sum_{k=1}^{n}W_{ijk}(t)[R_k(u_k(s)\xi_k)\widehat{R}_j(u_j(s)\xi_j)-R_k(u_k(s)\xi_k)\widehat{R}_j(v_j(s)\xi_j)\\
+ &\ R_k(u_k(s)\xi_k)\widehat{R}_j(v_j(s)\xi_j)-R_k(v_k(s)\xi_k)\widehat{R}_j(v_j(s)\xi_j)]\xi_i^{-1}\\
+ &\ Q_k(u_k(q_2(s))\xi_k)\widehat{Q}_j(v_j(q_3(s))\xi_j)-Q_k(v_k(q_2(s))\xi_k)\widehat{Q}_j(v_j(q_3(s))\xi_j)\}ds|
\end{aligned}$$

$$\begin{aligned}
\leq &\ \int_{-\infty}^{t}e^{-\int_s^t\varsigma_i^-(\xi)d\xi}\{\xi_i^{-1}\left(\sum_{j=1}^{n}\left|\bar{b}_{ij}(s)\right|L_j^B+\sum_{j=1}^{n}|c_{ij}(s)|L_j^C+\sum_{j=1}^{n}|p_{ij}(s)|K_{i\,j}L_j^h\right)\xi_j\\
+ &\ \sum_{j=1}^{n}\sum_{k=1}^{n}|W_{ijk}(s)|\left(M_k^gL_j^R\xi_j+M_j^gL_k^{\widehat{R}}\xi_k\right)\\
+ &\ \sum_{j=1}^{n}\sum_{k=1}^{n}|T_{ijk}(s)|(M_j^gL_k^Q\xi_k+M_k^gL_j^{\widehat{Q}}\xi_j)ds\}\xi_i^{-1}\|u-v\|_\infty
\end{aligned}$$

$$\begin{aligned}
&\leq \int_{-\infty}^{t} e^{-\int_s^t \varsigma_i^-(\xi)d\xi}\|u-v\|_\infty[\alpha_i^-(s)-\pi_i]ds \\
&\leq (\int_{-\infty}^{t} e^{-\int_s^t \varsigma_i^-(\xi)d\xi}\varsigma_i^-(s)ds-\frac{\pi_i}{2}\int_{-\infty}^{t} e^{-\int_s^t \varsigma_i^-(\xi)d\xi}ds)\|u-v\|_\infty \\
&\leq \max_{i\in N}\left\{1-\frac{\pi_i}{2\varsigma_i^+}\right\}\|u-v\|_\infty = c\|u-v\|_\infty.
\end{aligned}$$

Since $c \in (0,1)$, it is clear that the mapping E is a contraction. Thus, the mapping E possesses a unique fixed point $\overline{z} = \{\overline{z}_i(t)\} \in PAP(R,R,\upsilon)$ that is $E\overline{z} = \overline{z}$. Besides, (1.1) has a unique WPAP solution $\overline{x} = \{\overline{x}_i(t)\} = \{\xi_i\overline{z}_i(t)\} \in PAP(R,R,\upsilon)$. □

Theorem 3.4. *Suppose that Theorem 3.3 holds. Then WPAP solution of (1.1) is exponential stable.*

Proof. With a similar proof in Theorem 3.3 of [22], one can pick constants $\lambda \in (0, \min\{\kappa, \underset{i\in N}{min}\,\varsigma_i^-\})$ and $M = \sum_{j=1}^n D_j + 1$ such that

$$\sup_{t\in R}\{\lambda-\varsigma_i(t)+D_i\left(\sum_{j=1}^n |b_{ij}(t)|\,L_j^B+\sum_{j=1}^n |c_{ij}(t)|ML_j^C e^{\lambda\tau_{ij}(t)}\right.$$

$$+\sum_{j=1}^n |p_{ij}(t)|L_j^h e^{\lambda u}dv+\sum_{j=1}^n\sum_{k=1}^n |W_{ijk}(t)|\left(M_k^g L_j^R\xi_j+M_j^g L_k^{\widehat{R}}\xi_k e^{\lambda\sigma_{ijk}(t)}\right)$$

$$\left.\sum_{j=1}^n\sum_{k=1}^n |T_{ijk}(t)|\left(M_k^g L_j^{\widehat{Q}}\xi_j e^{\lambda\rho_{ijk}(t)}+M_j^g L_k^Q\xi_k e^{\lambda\sigma_{ijk}(t)}\right)\right)\}<0,$$

and

$$\|u(t)\| < M\,\|\varphi-x^*\|\,e^{-\lambda t},$$

which proves Theorem 3.4. □

Example 3.5. Consider the following high order recurrent neural network differential equation system

$$\begin{aligned}
x'_1(t) &= -\left(\frac{1}{20}+\frac{3}{40}\cos t\right)x_1(t)+\frac{\cos 2t}{4000}(x_1\left(t-\frac{(1+\cos 2t)}{2}\right)+x_2\left(t-\frac{(1+\cos 2t)}{2}\right)) \\
&+ \frac{1}{2400}\sin 2t(\int_0^{+\infty} e^{-2s}x_1(t-s)ds+\int_0^{+\infty} e^{-2s}x_2(t-s)ds) \\
&+ \frac{1}{14}\sin\sqrt{2}t\frac{1}{(40\pi)^2}(\arctan^2 x_1(t)+\frac{1}{20\pi}\arctan x_1(t)\arctan x_2(t)+\arctan^2 x_2(t)) \\
&+ \frac{1}{14}\frac{1}{(80\pi)^2}\cos\sqrt{2}t(\arctan x_1(t-\frac{1+\cos 3t}{3})\arctan x_1(t-\frac{1+\cos 4t}{3}) \\
&+ \arctan x_1(t-\frac{1+\cos 3t}{3})\arctan x_2(t-\frac{1+\cos 4t}{3}) \\
&+ \arctan x_2(t-\frac{1+\cos 3t}{3})\arctan x_2(t-\frac{1+\cos 4t}{3}) \\
&+ \arctan x_2(t-\frac{1+\cos 3t}{3})\arctan x_1(t-\frac{1+\cos 4t}{3}))+(360+1)\,|\cos 2t|+e^{-t},
\end{aligned}$$

$$\begin{aligned} x'_2(t) &= -\left(\frac{1}{20}+\frac{3}{40}\cos t\right)x_2(t)+\frac{\cos 2t}{4000}(x_1\left(t-\frac{(1+\cos 2t)}{2}\right)+x_2\left(t-\frac{(1+\cos 2t)}{2}\right)) \\ &+ \frac{1}{2400}\sin 2t(\int_0^{+\infty} e^{-2s}x_1(t-s)ds+\int_0^{+\infty} e^{-2s}x_2(t-s)ds) \\ &\times \frac{1}{14}\sin\sqrt{2}t\frac{1}{(40\pi)^2}(\arctan^2 x_1(t)+\frac{1}{20\pi}\arctan x_1(t)\arctan x_2(t)+\arctan^2 x_2(t)) \\ &+ \frac{1}{14}\frac{1}{(80\pi)^2}\cos\sqrt{2}t(\arctan x_1(t-\frac{1+\cos 3t}{3})\arctan(x_1(t-\frac{1+\cos 3t}{3})) \\ &+ \arctan(x_1(t-\frac{1+\cos 3t}{3}))\arctan_2(x_2(t-\frac{1+\cos 4t}{3})) \\ &+ \arctan(x_2(t-\frac{1+\cos 3t}{3}))\arctan(x_2(t-\frac{1+\cos 4t}{3})) \\ &+ \arctan(x_2(t-\frac{1+\cos 3t}{3}))\arctan(x_1(t-\frac{1+\cos 4t}{3}))+(360+2)\left|\cos 2t\right|+e^{-t}. \end{aligned}$$

Clearly, one can take $L_i^B = 0$, $L_i^C = L_i^h = \frac{1}{20}$, $L_i^H = L_i^{\widehat{H}} = \frac{1}{40\pi}$, $L_i^Q = L_i^{\widehat{Q}} = \frac{1}{80\pi}$, $\xi_i = 1, \kappa = 1$, and for $t \geq 0$, $\eta(t) = e^t$, for $t < 0$, $\eta(t) = 1$such that for values in above example, HORNNs initial differential system of (1.1)-(1.2) satisfies all the conditions of Theorem 3.4 Therefore, (1.1) has a unique solution which belongs to $PAP(R, R, \upsilon)$.

References

[1] A. M. Fink, Almost Periodic Differential Equations (Lecture Notes in Mathematics), *Springer-Verlag*, New York, 1974.

[2] Chérif, F., “A various types of almost periodic functions on Banach spaces, *Int. Math. Forum* **6**: 921–952, (2011).

[3] Zhang, C., “Almost Periodic Type Functions and Ergodicity, *Kluwer*, Beijing, China:, 2003

[4] Zhang, C., “Pseudo almost periodic functions and their applications,” Ph.D. thesis, *Dept. Math., Univ. Western Ontario*, London, ON, Canada, 1992.

[5] Diagana, T., “Existence of weighted pseudo almost periodic solutions to some non-autonomous differential equations, *Int. J. Evol. Equ.*,**2**,397–410, (2008).

[6] Diagana, T.,”Weighted pseudo-almost periodic solutions to some differential equations,*Non-linear Anal.*, **8**, 2250-2260,(2008)).

[7] Li, C., Liao, X., “New algebraic conditions for global exponential stability of delayed recurrent neural networks*Neurocomputing*, **64**, 319–333, (2005).

[8] Liu, Y., Wang, Z., Liu X., “Global exponential stability of generalized recurrent neural networks with discrete and distributed delays*Neural Network*, **19**, 667–675, (2006).

[9] Song, Q. “Novel criteria for global exponential periodicity and stability of recurrent neural networks with time-varying delays,*Chaos Solitons Fractals*, **36**, 720–728, (2008).

[10] Song, Q., “Exponential stability of recurrent neural networks with both time-varying delays and general activation functions via LMI approach,” *Neurocomputing*, **71**, 2823–2830, (2008).

[11] Cao, J., Hang, H., Wang, J., “Global exponential stability and periodic solutions of recurrent neural networks with delays,*Phys. Lett.*, **298**, 393–404, (2002).

[12] Huang, X., Cao, J., Ho, D. W. C., “Existence and attractivity of almost periodic solution for recurrent neural networks with unbounded delays and variable coefficients,*Nonlin. Dyn.*, **45**, 337–351, (2006).

[13] Cao, J., Chen, A., Huang, X., “Almost periodic attraction of delayed neural networks with variable coefficients,*Phys. Lett.*, **340**, 104–120, (2005).

[14] Liu, Z., Chen, A., Cao, J., Huang, L., “Existence and global exponential stability of almost periodic solutions of BAM neural networks with continuously distributed delays,*Phys. Lett.*, **319**, 305–316, (2003).

[15] Liu, B., “Almost periodic solutions for Hopfield neural networks with continuously distributed delays,*Math. Comput. Simul.*, **73**, 327–335, (2007).

[16] Zhao, H., “Existence and global attractivity of almost periodic solution for cellular neural network with distributed delays,*Appl. Math. Comput.*, **154**, 683–695, (2004).

[17] Ammar, B., Cherif, F., Alimi, A.M., “Existence and Uniqueness of Pseudo Almost-Periodic Solutions of Recurrent Neural Networks with Time-Varying Coefficients and Mixed Delays”, *IEEE Transactions on Neural Networks and Learning Systems*, **23**, 109-118, (2012).

[18] Brahmi, H., Ammar, B., Alimi, A. M., Chérif, F., “Pseudo almost periodic solutions of impulsive recurrent neural networks with mixed delays”, 2016 *International Joint Conference on Neural Networks*, (2014).

[19] M’hamdi, M. S., Aouiti, C., Touati, A., Alimi, Adel M., Snasel., “Weighted pseudo almost-periodic solutions of shunting inhibitory cellular neural networks with mixed delays”, *Acta Math. Sci. Ser. B Engl. Ed.*, **36**,1662-1682, (2016).

[20] Zhao, L., Li, Y., “Global exponential stability of weighted pseudo-almost periodic solutions of neutral type high-order Hopfield neural networks with distributed delays”,*Abstr. Appl. Anal.*, **17**, (2014).

[21] Xu, Y., “Weighted pseudo-almost periodic delayed cellular neural networks”, *Neural Comput Appl*, **28**, 1-6, (2017).

[22] Xu, Y., “Exponential stability of weighted pseudo almost periodic solutions for HCNNs with mixed delays”,*Neural Processing Letters*, **1**,1–13, (2017)

[23] Wang, Z., Fang, X. “Liu,Global stability of stochastic high-order neural networks with discrete and distributed delays”,*Chaos, Solitons and Fractals*, **36**, 388-396, (2008).

[24] Brahmi, H., Ammar, B., Cheri£, F., Alimi, A. M., “On the dynamics of the high-order type of neural networks with time varying coefficients and mixed delay”, *Neural Networks*, 2063 - 2070, (2014).

[25] F. Cherif, H. Brahmi, B. Ammar, A. M. Alimi,” Exponential synchronization of a class of rnns with discrete and distributed delays”, ICANN 2013, *LNCS* **7**, 4-81, (2013).

[26] Yazgan, R., Tunç, C., “On the weighted pseudo almost periodic solutions of Nicholson’s blowflies equation”,*Appl. Appl. Math.* **14**, 875–889, (2019).

[27] Yazgan, R., Tunç, C., “On the almost periodic solutions of fuzzy cellular neural networks of high order with multiple time lags”, *Int. J. Math. Comput. Sci.* **15**, 183–198, (2020).

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