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Curves of constant breadth in a Walker 4-manifold

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Abstract In this paper, we define and study the differential geometry of curves of constant breadth in a Walker 4-manifold and we construct an example of curves of constant breadth in this manifold.

1 Introduction

The study of submanifolds on a given ambiant space is a naturel interesting problem which enriches our knowlegde and understanding of geometry of the space itself. In this study, the ambiant space is a Lorentzian 4-manifold admitting a null parallel vector field called Walker manifold. For more detail see [3], [13].
Many researcher interest to the study of curves of constant breadth since the first introduction of this theory in the plane by L. Euler [1] in 1778. After the work of Euler, Fujivara extended it to space curves. He had put forward a problem based on determining whether there exist space curves of constant breadth or not, and as a solution of the problem, the "breadth" concept for space curves was defined and these curves were shown on a surface of constant breadth in [7]. Furthermore, Blaschke defined the curve of constant breadth on the sphere [4]. Several results were obtained when the ambient space is the Euclidean space and the Lorentz-Minkowski space. For details see [8, 9, 14]. Curves of constant breadth in the three dimensional Walker manifols was studied in [6].
Motivated by the above works, we study the differential geometry of curves of constant breadth in a Walker 4-manifold.
The paper is organised as follow: in section 2, we recall some preliminaries results for Walker 4-manifold $(M, g_{a,b,c})$. In the section 3, we define space curves of constant breadth in a Walker 4-manifold $(M, g_{a,b,c})$ and we give the Characterisations of space curves of constant breadth in the $(M, g_{a,b,c})$. In the last section of this paper, we construct an example of curves of constant breadth in a given Walker 4-manifold to illustrate our results.

2 Preliminaries

A Walker n-dimentional manifold is a pseudo-Riemannian manifold which admit a field of null parallel r-planes with $r \leq \frac{n}{2}$. The canonical forms of the metrics were investigated by A. G. Walker [13]. Walker has derived adapted coordinates to a parallel plan field. Hence, a metric of four-dimensional Walker manifold $(M, g_{a,b,c})$ with coordinates (x_1, x_2, x_3, x_4) is expressed as

$$g_{a,b,c} = 2(dx_1 \circ dx_3 + dx_2 \circ dx_4) + adx_3 \circ dx_3 + bdx_4 \circ dx_4 + 2cdx_3 \circ dx_4 \tag{2.1}$$

and its matrix form as

$$g_{a,b,c} = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & a & c \\ 0 & 1 & c & b \end{bmatrix} \quad with\ inverse\ matrix\ g^{-1}_{a,b,c} = \begin{bmatrix} -a & -c & 1 & 0 \\ -c & -b & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix}$$

where a, b, c are smooth functions on M. An easy computation gives the coefficients of the Levi Civita connection of the metric $g_{a,b,c}$ by :

$$\begin{aligned}
\nabla_{\partial x_1}\partial x_3 &= \frac{1}{2}a_1\partial x_1 + \frac{1}{2}c_1\partial x_2, \qquad \nabla_{\partial x_1}\partial x_4 = \frac{1}{2}c_1\partial x_1 + \frac{1}{2}b_1\partial x_2\\
\nabla_{\partial x_2}\partial x_3 &= \frac{1}{2}a_2\partial x_1 + \frac{1}{2}c_2\partial x_2, \qquad \nabla_{\partial x_2}\partial x_4 = \frac{1}{2}c_2\partial x_1 + \frac{1}{2}b_2\partial x_2\\
\nabla_{\partial x_3}\partial x_3 &= \frac{1}{2}(aa_1 + ca_2 + a_3)\partial x_1 + \frac{1}{2}(ca_1 + ba_2 - a_4 + 2c_3)\partial x_2\\
&\quad - \frac{a_1}{2}\partial x_3 - \frac{a_2}{2}\partial x_4\\
\nabla_{\partial x_3}\partial x_4 &= \frac{1}{2}(a_4 + ac_1 + cc_2)\partial x_1 + \frac{1}{2}(b_3 + cc_1 + bc_2)\partial x_2\\
&\quad - \frac{c_1}{2}\partial x_3 - \frac{c_2}{2}\partial x_4\\
\nabla_{\partial x_3}\partial x_3 &= \frac{1}{2}(ab_1 + cb_2 - b_3 + 2c_4)\partial x_1 + \frac{1}{2}(cb_1 + bb_2 + b_4)\partial x_2\\
&\quad - \frac{b_1}{2}\partial x_3 - \frac{b_2}{2}\partial x_4
\end{aligned} \tag{2.2}$$

If we take the functions $a = b = 0$, the non-zero coefficients of the Christoffel symbole and the non-zero coefficients of the Levi Civita connection are given by

$$\begin{aligned}
&\Gamma^2_{13} = \Gamma^2_{31} = \frac{1}{2}c_1, \quad \Gamma^1_{14} = \Gamma^1_{41} = \frac{1}{2}c_1, \quad \Gamma^2_{23} = \Gamma^2_{32} = \frac{1}{2}c_2,\\
&\Gamma^1_{24} = \frac{1}{2}c_2, \quad \Gamma^2_{33} = c_3, \quad \Gamma^1_{44} = \frac{1}{2}c_4, \quad \Gamma^1_{34} = \frac{1}{2}cc_2\\
&\Gamma^2_{34} = \frac{1}{2}cc_1, \quad \Gamma^3_{34} = -\frac{1}{2}c_1, \quad \Gamma^4_{34} = -\frac{1}{2}c_2;
\end{aligned} \tag{2.3}$$

and

$$\begin{aligned}
\nabla_{\partial x_1}\partial x_3 &= \frac{1}{2}c_1\partial x_2, \qquad \nabla_{\partial x_1}\partial x_4 = \frac{1}{2}c_1\partial x_1\\
\nabla_{\partial x_2}\partial x_3 &= \frac{1}{2}c_2\partial x_2, \qquad \nabla_{\partial x_2}\partial x_4 = \frac{1}{2}c_2\partial x_1\\
\nabla_{\partial x_3}\partial x_3 &= c_3\partial x_2 \qquad \nabla_{\partial x_4}\partial x_4 = \frac{1}{2}c_4\partial x_1\\
\nabla_{\partial x_3}\partial x_4 &= \frac{1}{2}cc_2\partial x_1 + \frac{1}{2}cc_1\partial x_2 - \frac{c_1}{2}\partial x_3 - \frac{c_2}{2}\partial x_4.
\end{aligned} \tag{2.4}$$

Let $\alpha : I \subset \mathbb{R} \to (M, g_{a,b,c})$ be a unit speed curve with parameter s. If $Z(s)$ is a smooth vector field along the curve α, then we denote its covariant derivative by $\frac{DZ}{ds}$. In terme of local coordinates,

$$Z(s) = \sum_{i=1}^{4} Z^i(s)\partial_i, \tag{2.5}$$

where each $Z^i = Z(x^i)$ is a real valued function on I. Then we have the following

$$\frac{DZ}{ds} = \sum_{i=1}^{4}\Big\{\frac{\partial Z^k}{\partial s} + \sum_{i,j}\Gamma^k_{ij}Z^i\frac{\partial x^j}{\partial s}\Big\}\partial_k. \tag{2.6}$$

The Frenet Frame of α is denoted by $\{T, N, B_1, B_2\}$, where T is the tangent vector, N the normal principal, B_1 the first binormal vector and B_2 the second binormal vector. The following Frenet formulae are given in [2]

$$\begin{cases}
T'(s) = \mu_2 k_1 N(s)\\
N'(s) = -\mu_1 k_1 T(s) + \mu_3 k_2 B_1(s)\\
B_1'(s) = -\mu_2 k_2 N(s) + \mu_3 k_3 B_2(s)\\
B_2'(s) = -\mu_3 k_3 B_1(s),
\end{cases} \tag{2.7}$$

where $\mu_1 = g_{a,b,c}(T,T)$, $\mu_2 = g_{a,b,c}(N,N)$, $\mu_3 = g_{a,b,c}(B_1,B_1)$ and $\mu_4 = g_{a,b,c}(B_2,B_2)$ with $\mu_i = \pm 1$, for $i = 1,2,3,4$, and k_1, k_2 and k_3 are first, second and third curvatures of the curve respectively.

3 Characterisations of space curves of constant breadth in M

In this section, we define space curves of constant breadth in a Walker 4-manifold.

Definition 3.1. A curve $\alpha : I \to M$ in the four-dimensional Walker manifold is called a curve of constant breadth if there exists a curve $\beta : I \to M$ such that, at the corresponding point of curves, the parallel tangent vectors of α and β at $\alpha(s)$ and $\beta(s^*)$ at $s, s^* \in I$ are opposite diections and the distance between these points is always constant. In this case (α, β) is called a curve pair of constant breadth.

Assume that α and β be a pair of unit speed curves in M with position vectors $\vec{X}(s)$ and $\vec{X}^*(s^*)$, where s and s^* are of curves α and β, respectively, and let α and β have parallel tangents in opposite directions at the opposite points. The curve β can be written by the following equation

$$\vec{X}(s^*) = X(s) + m_1(s)T(s) + m_2(s)N(s) + m_3(s)B_1(s) + m_4(s)B_2(s) \tag{3.1}$$

where $m_i(s)$, $(1 \le i \le 4)$ are differentiable functions of s.
Differentiating equation (3.1) with respect to s and by using (2.7), we obtain

$$\begin{aligned} T^*\frac{ds^*}{ds} &= (1 + \frac{dm_1}{ds} - m_2\mu_1 k_1)T + (\frac{dm_2}{ds} + m_1\mu_2 k_1 - m_3\mu_2 k_2)N \\ &\quad + (\frac{dm_3}{ds} + m_2\mu_3 k_2 - m_4\mu_3 k_3)B_1 + (\frac{dm_4}{ds} + m_3\mu_4 k_3)B_2, \end{aligned} \tag{3.2}$$

where T^* denotes the tangent vector of β.
Since $T^* = -T$, frome (3.2) we have

$$\begin{cases} 1 + \frac{dm_1}{ds} - m_2\mu_1 k_1 = -\frac{ds^*}{ds} \\ \frac{dm_2}{ds} + m_1\mu_2 k_1 - m_3\mu_2 k_2 = 0 \\ \frac{dm_3}{ds} + m_2\mu_3 k_2 - m_4\mu_3 k_3 = 0 \\ \frac{dm_4}{ds} + m_3\mu_4 k_3 = 0 \end{cases} \tag{3.3}$$

Let us introduce the angle between the tangent vector of α with a given fixed direction, the curvature of α is $k_1(s) = \frac{d\phi}{ds}$ and the curvature of β is $k_1^*(s) = \frac{d\phi}{ds^*}$. By using this fact with (3.3), we obtain

$$\begin{cases} \frac{dm_1}{d\phi} = \mu_1 m_2 - f(\phi) \\ \frac{dm_2}{d\phi} = -\mu_2 m_1 + m_3\mu_2 k_2\rho \\ \frac{dm_3}{d\phi} = -m_2\mu_3 k_2\rho + m_4\mu_3 k_3\rho \\ \frac{dm_4}{d\phi} = -m_3\mu_4 k_3\rho. \end{cases} \tag{3.4}$$

Here $f(\phi) = \rho + \rho^*$ and $\rho = 1/k_1$, $\rho^* = 1/k_1^*$ are the radius of curvatures α and β respectively. Differentiating the first equation of (3.4) with respect to ϕ and by using the second, third and fourth equations of (3.4), we obtain the following diffenrential equation

$$\begin{aligned} &\frac{d}{d\phi}\left[\frac{1}{k_3 k_2\rho^2}\left(\frac{d^3 m_1}{d\phi^3} + \frac{d^2 f}{d\phi^2} + \mu_1\mu_2\frac{dm_1}{d\phi}\right)\right] - \frac{d}{d\phi}\left[\frac{1}{k_3 k_2^2\rho^3}\frac{d(k_2\rho)}{d\phi}\left(\frac{d^2 m_1}{d\phi^2} + \frac{df}{d\phi} + \mu_1\mu_2 m_1\right)\right] \\ &- \mu_2\mu_3\frac{d}{d\phi}\left[\frac{k_2}{k_3}\left(\frac{dm_1}{d\phi} + f\right)\right] + \mu_3\mu_4\frac{k_3}{k_2}\left(\frac{d^2 m_1}{d\phi^2} + \frac{df}{d\phi} + \mu_1\mu_2 m_1\right) = 0. \end{aligned} \tag{3.5}$$

This differential equation is characterization of curves of constant breadth in the Walker 4-manifold M.
If the distance between the opposite points of α and β is constant, then

$$\|\alpha - \beta\|^2 = \mu_1 m_1^2 + \mu_2 m_2^2 + \mu_3 m_3^2 + \mu_4 m_4^2 = k(constant), \tag{3.6}$$

which implies

$$\mu_1 m_1 \frac{dm_1}{d\phi} + \mu_2 m_2 \frac{dm_2}{d\phi} + \mu_3 m_3 \frac{dm_3}{d\phi} + \mu_4 m_4 \frac{dm_4}{d\phi} = 0. \tag{3.7}$$

Using the equations (3.4) and (3.7) we get

$$m_1(\mu_1 \frac{dm_1}{d\phi} - m_2) = 0. \tag{3.8}$$

Then we have $m_1 = 0$ or $\frac{dm_1}{d\phi} = \mu_1 m_2$. Hence we obtain the following system equations

$$\begin{cases} m_1 = 0 \\ \frac{dm_2}{d\phi} = m_3\mu_2 k_2 \rho \\ \frac{dm_3}{d\phi} = -m_2\mu_3 k_2\rho + m_4\mu_3 k_3\rho \\ \frac{dm_4}{d\phi} = -m_3\mu_4 k_3\rho. \end{cases} \tag{3.9}$$

or

$$\begin{cases} \frac{dm_1}{d\phi} = \mu_1 m_2 \\ \frac{dm_2}{d\phi} = -\mu_2 m_1 + m_3\mu_2 k_2 \rho \\ \frac{dm_3}{d\phi} = -m_2\mu_3 k_2\rho + m_4\mu_3 k_3\rho \\ \frac{dm_4}{d\phi} = -m_3\mu_4 k_3\rho. \end{cases} \tag{3.10}$$

Suppose that m_1 is a non-zero constant in the system (3.10). Then we have the following linear differential equations

$$(k_3\rho)\frac{d^2 m_3}{d\phi^2} - \frac{d}{d\phi}(k_3\rho)\frac{dm_3}{d\phi} + m_3(k_3\rho)^3 = 0 \tag{3.11}$$

$$(k_3\rho)\frac{d^2 m_4}{d\phi^2} - \frac{d}{d\phi}(k_3\rho)\frac{dm_4}{d\phi} + m_4(k_3\rho)^3 = 0 \tag{3.12}$$

We can solve the equations (3.11) and (3.12) by considering the new variable $z = z(\phi)$ defined by $\frac{dz}{d\phi} = (k_3\rho)$. So we get

$$\begin{cases} \frac{dm_3}{d\phi} = \frac{dm_3}{dz}(k_3\rho) \\ \frac{d^2 m_3}{d\phi^2} = \frac{d^2 m_3}{dz^2}(k_3\rho)^2 + \frac{dm_3}{dz}\frac{d}{d\phi}(k_3\rho) \end{cases} \tag{3.13}$$

By using (3.13) the equation in (3.11) becomes

$$\frac{d^2 m_3}{dz^2} + m_3 = 0. \tag{3.14}$$

Thus, general solution of m_3 is

$$m_3 = a_1 \cos(\int_0^\phi k_3\rho dt) + a_2 \sin(\int_0^\phi k_3\rho dt). \tag{3.15}$$

Also, if we consider m_4, we obtain

$$m_4 = a_2 \cos(\int_0^\phi k_3\rho dt) - a_1 \sin(\int_0^\phi k_3\rho dt), \tag{3.16}$$

where a_1 and a_2 are arbitrary constants. Thus the general solution is given by

$$\begin{cases} m_1 = a \\ m_2 = 0 \\ m_3 = a_1 \cos(\int_0^\phi k_3\rho dt) + a_2 \sin(\int_0^\phi k_3\rho dt) \\ m_4 = a_2 \cos(\int_0^\phi k_3\rho dt) - a_1 \sin(\int_0^\phi k_3\rho dt) \end{cases} \tag{3.17}$$

where a is also constant.

Theorem 3.2. *Let (α, β) be a pair curve of constant breadth in the Walker 4-manifold M. If β is a curve with $m_2 = 0$ and m_1 non-zero constant, then the curve β has the following form*

$$\begin{aligned}\beta(s) = \alpha(s) + m_1 T(s) + \left[a_1 \cos(\int_0^{\phi} k_3 \rho dt) + a_2 \sin(\int_0^{\phi} k_3 \rho dt)\right] B_1(s) \\ + \left[a_2 \cos(\int_0^{\phi} k_3 \rho dt) - a_1 \sin(\int_0^{\phi} k_3 \rho dt)\right] B_2(s).\end{aligned} \tag{3.18}$$

Assume that $m_1 = 0$, then $f = \mu_1 m_2$ and the differential equation in (3.5) becomes

$$\frac{d}{d\phi}\left(\frac{1}{k_3 k_2 \rho^2} \times \frac{d^2 m_2}{d\phi^2}\right) - \frac{d}{d\phi}\left(\frac{1}{k_3 k_2^2 \rho^3} \times \frac{d(k_2\rho)}{d\phi}\frac{dm_2}{d\phi}\right) - \mu_2\mu_3 \frac{d}{d\phi}\left(\frac{k_2}{k_3} m_2\right) + \mu_3\mu_4 \frac{k_3}{k_2}\left(\frac{dm_2}{d\phi}\right) = 0. \tag{3.19}$$

If m_2 is non-zero constant we have

$$\frac{d}{d\phi}\left(\frac{k_2}{k_3} m_2\right) = m_2 \frac{d}{d\phi}\left(\frac{k_2}{k_3}\right) = 0. \tag{3.20}$$

That is

$$\frac{d}{d\phi}\left(\frac{k_2}{k_3}\right) = 0. \tag{3.21}$$

Thus $\frac{k_2}{k_3}$ is constant.

Theorem 3.3. *The space curve of constant breadth with the tangent component $m_1 = 0$ and the principal normal component m_2 is non-zero constant, is a general helix in the four-dimentional Walker manifold.*

4 Example of curve of constant breadth in Walker 4-manifold

In this section, we construct an example of curve of constant breadth in a given Walker 4-manifold.

Let us consider the Walker 4-manifold $(M, g_{a,b,c})$ with the functions $a(x, y, z, t) = b(x, y, z, t) = 0$ and $c(x, y, z, t) = -\frac{8}{3}e^{3t}$.

We consider the following curve

$$\alpha(s) = (\cosh s, \sinh s, 0, -\frac{1}{2}e^{-s}). \tag{4.1}$$

We have

$$\alpha'(s) = (\sinh s, \cosh s, 0, \frac{1}{2}e^{-s}) \tag{4.2}$$

and

$$g_{0,0,c}(\alpha'(s), \alpha'(s)) = 1, \tag{4.3}$$

so the curve α is spacelike curve in M. By using the formula in (2.6), we get

$$\alpha''(s) = (-\frac{3}{4}e^{s} + \frac{1}{2}e^{-s}, \sinh s, 0, -\frac{1}{2}e^{-s}). \tag{4.4}$$

We take

$$T(s) = (\sinh s, \cosh s, 0, \frac{1}{2}e^{-s}) \tag{4.5}$$

and $\mu_1 = g_{0,0,c}(T(s), T(s)) = 1$ and then

$$N(s) = (-\frac{3}{4}e^{s} + \frac{1}{2}e^{-s}, \sinh s, 0, -\frac{1}{2}e^{-s})). \tag{4.6}$$

By using the first equation of (2.7), we have $g_{0,0,c}(T'(s), N(s)) = \mu_2 k_1 = 1$, that is $\mu_2 = 1$ and $k_1 = 1$. Using the last equation of (2.7), we obtain

$$B_2(s) = (-\frac{3}{2}e^s - e^{-s}, \frac{1}{2}\cosh s, 0, e^{-s}) \tag{4.7}$$

where we suppose $\mu_3 = -1$ and $k_2 = 2$.
Finally by using the third equation of (2.7), get

$$B_1(s) = (4e^s - e^{-s}, 0, 0, e^{-s})$$

by taking $k_3 = 2$. Thus the Frenet apparatus of the curve α is

$$\begin{aligned} T(s) &= (\sinh s, \cosh s, 0, \frac{1}{2}e^{-s}) \\ N(s) &= (-\frac{3}{4}e^s + \frac{1}{2}e^{-s}, \sinh s, 0, -\frac{1}{2}e^{-s}) \\ B_1(s) &= (4e^s - e^{-s}, 0, 0, e^{-s}) \\ B_2(s) &= (-\frac{3}{2}e^s - e^{-s}, \frac{1}{2}\cosh s, 0, e^{-s}) \\ k_1 &= 1 \\ k_2 &= 1 \\ k_3 &= 2 \end{aligned} \tag{4.8}$$

with $\mu_1 = g_{0,0,c}(T(s), T(s)) = 1$, $\mu_2 = g_{0,0,c}(N(s), N(s)) = 1$, $\mu_3 = g_{0,0,c}(B_1(s), B_1(s)) = -1$, $\mu_4 = g_{0,0,c}(B_2(s), B_2(s)) = 1$.
Then by using the formula (21) of the Theorem 3.1, we see that the curves β and α defined us following

$$\begin{aligned} \beta(s) = \alpha(s) + T(s) &+ [\cos(\int_0^\phi 2dt) + \sin(\int_0^\phi 2dt)]B_1(s) \\ &+ [\cos(\int_0^\phi 2dt) - \sin(\int_0^\phi 2dt)]B_2(s). \end{aligned} \tag{4.9}$$

where $m_1 = a_1 = a_2 = 1$ and $\rho = \frac{1}{k_1} = 1$, are curves of constant breadth.

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