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EXISTENCE RESULTS AND NUMERICAL SOLUTION OF A FOURTH-ORDER NONLINEAR DIFFERENTIAL EQUATION WITH TWO INTEGRAL BOUNDARY CONDITIONS

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Abstract In this paper we consider a fourth order nonlinear differential equation with two integral boundary conditions. We establish the existence, uniqueness and positivity of solutions and propose iterative methods both on continuous and on discrete levels for finding the solution. The numerical solution is proved to be of second order accuracy. Many examples demonstrate the validity of the obtained theoretical results, the efficiency of the numerical method and the applicability of it to a wide class of problems.

1 Introduction

In this paper we consider the boundary value problem

$$u''''(t) = f(t, u(t), u'(t), u''(t), u'''(t)), \quad 0 < t < 1, \tag{1.1}$$

$$\begin{aligned} u(0) &= \int_0^1 g(s)u(s)ds,\ u(1) = 0, \\ u''(0) &= \int_0^1 h(s)u''(s)ds,\ u''(1) = 0, \end{aligned} \tag{1.2}$$

where $f : [0,1] \times \mathbb{R}^4 \to \mathbb{R},\ g, h : [0,1] \to \mathbb{R}$ are continuous functions.

This problem is a natural generalization of the problem considered in [6], where instead of the fully nonlinear term it was $f(t, u(t), u''(t))$. In the above-mentioned paper, based upon the Krein-Rutman theorem and the global birfucation techniques, the authors established the existence of at least one positive solution. Many complicated conditions on the behavior of the function $f(t, u, p)$ near zero and at infinity are posed and they are related to the first generalized eigenvalue of the generalized eigenvalue problem associated with the problem under consideration. So, the authors could not show examples, where the required conditions are satisfied.

Boundary value problems with integral boundary conditions arise in many applied fields such as heat conduction, chemical engineering, hydrodynamics, thermoelastisity, and plasma physics. Therefore, recently they have attracted attention from many researchers. Among them are many works concerning fourth order nonlinear differential equations with integral boundary conditions (see, e.g., [1, 6, 7, 8, 11, 9, 10, 12, 13]). In the mentioned works some interesting results on the existence and multiplicity of solutions are obtained by using different fixed theorems and the method of lower and upper solutions. *It should be emphasized that these works concern with only the existence and multiplicity of solutions. The methods for finding solutions are not* *considered in these works. And to the best of our knowledge, in these works the authors only could show the examples, where the assumptions of the theorems of existence of solutions are satisfied but could not show the solutions themselves. Even in the works [6, 11] such examples are absent.*

Very recently, in [5] for the first time we investigated the existence and uniqueness of solutions and iterative method for finding the solution of the problem with one integral boundary condition

$$u''''(t) = f(t, u(t), u'(t), u''(t), u'''(t)), \quad 0 < t < 1,$$
$$u'(0) = u''(0) = u'(1) = 0, \; u(0) = \int_0^1 g(s)u(s)ds,$$

where $f : [0, 1] \times \mathbb{R}^4 \to \mathbb{R}^+, \; g : [0, 1] \to \mathbb{R}^+$ are continuous functions.

The present paper is a further development of the technique used in [5] for the problem (1.1)-(1.2). We establish the existence, uniqueness and positivity of solutions for the problem and study iterative methods at both continuous and discrete levels for finding the solution. Some examples demonstrate the applicability of the obtained theoretical results and the efficiency of the discrete iterative method. Then the numerical method is applied successfully to find solutions of the problem with non-homogeneous boundary conditions and for problems with singular right-hand side.

2 Existence and uniqueness of solution

To investigate the problem (1.1)- (1.2) we set

$$\begin{aligned} \varphi(t) &= f(t, u(t), u'(t), u''(t), u'''(t)), \\ \alpha &= \int_0^1 g(s)u(s)ds, \\ \beta &= \int_0^1 h(s)u''(s)ds. \end{aligned} \tag{2.1}$$

Then the problem becomes

$$\begin{aligned} u''''(t) &= \varphi(t), \quad 0 < t < 1, \\ u(0) &= \alpha, \; u(1) = 0, \\ u''(0) &= \beta, \; u''(1) = 0. \end{aligned} \tag{2.2}$$

This problem has a solution presented in the form

$$u(t) = \int_0^1 G_0(t, s)\varphi(s)ds + P(\alpha, \beta, t), \quad 0 < t < 1, \tag{2.3}$$

where $G_0(t, s)$ is the Green function of the operator $u''''(t) = 0$ associated with the homogeneous boundary conditions $u(0) = u(1) = 0, u''(0) = u''(1) = 0,$

$$G_0(t, s) = -\frac{1}{6} \begin{cases} s(1-t)(t^2 - 2t + s^2), & 0 \le s \le t \le 1 \\ t(1-s)(s^2 - 2s + t^2), & 0 \le t \le s \le 1 \end{cases} \tag{2.4}$$

and

$$P(\alpha, \beta, t) = \alpha(1 - t) - \frac{2\beta t}{3}(2 - 3t + t^2). \tag{2.5}$$

It is easy to see that $G_0(t, s) \ge 0 \; \forall t, s \in [0, 1]$ and $P(\alpha, \beta, t) \ge 0 \; \forall t \in [0, 1]$ for $\alpha \ge 0, \; \beta \le 0$. It is obvious that the solution (2.3) depending on φ, α, β must satisfy the relations (2.1). We shall express this by an operator equation for the triplet $w = (\varphi, \alpha, \beta)^T$. For this reason, we introduce the space $\mathcal{B} = C[0, 1] \times \mathbb{R} \times \mathbb{R}$ of these triplets w and equip it with the norm

$$\|w\|_{\mathcal{B}} = \max(\|\varphi\|, r|\alpha|, r|\beta|), \tag{2.6}$$

where r is a number, $r \geq 1$ and $\|\varphi\| = \max_{0\leq t\leq 1} |\varphi(t)|$. The number r will be chosen later for each particular problem.

Further, define the operator A acting on elements $w \in \mathcal{B}$ by the formula

$$Aw = \begin{pmatrix} f(t, u(t), u'(t), u''(t), u'''(t)) \\ \int_0^1 g(s)u(s)ds \\ \int_0^1 h(s)u''(s)ds \end{pmatrix}, \tag{2.7}$$

where $u(t)$ is the solution of the problem (2.2).

Assuming that the function $f(t, u, y, v, z)$ is continuous and the functions $g(s), h(s)$ are integrable on $[0, 1]$ we have $Aw \in \mathcal{B}$. It is easy to verify the following

Lemma 2.1. *If $w = (\varphi, \alpha, \beta)^T$ is a fixed point of the operator A in the space $\mathcal{B}$, i.e., w is a solution of the operator equation*

$$Aw = w \tag{2.8}$$

in $\mathcal{B}$, then the function $u(t)$ determined from the problem (2.2) solves the original problem (1.1)-(1.2). Conversely, if $u(t)$ is a solution of the problem (1.1)-(1.2), then the triplet (φ, α, β) defined by (2.1) is a solution of the operator equation (2.8).

Thus, by this lemma, the problem (1.1)-(1.2) is reduced to the fixed point problem for A in the space $\mathcal{B}$.

Now, we study the properties of A. For this purpose we consider the solution of the problem (2.2).

Differentiating both sides of (2.3) we obtain

$$\begin{aligned} u'(t) &= \int_0^1 G_1(t, s)\varphi(s)ds + P'(\alpha, \beta, t), \\ u''(t) &= \int_0^1 G_2(t, s)\varphi(s)ds + P''(\alpha, \beta, t), \\ u'''(t) &= \int_0^1 G_3(t, s)\varphi(s)ds + P'''(\alpha, \beta, t), \end{aligned} \tag{2.9}$$

where

$$G_i(t, s) = \frac{\partial^i G_0(t, s)}{\partial t^i}, \ (i = 1, 2, 3).$$

Notice that

$$G_2(t, s) = \begin{cases} -s(1-t), & 0 \leq s \leq t \leq 1 \\ -t(1-s), & 0 \leq t \leq s \leq 1, \end{cases}$$

and $G_2(t, s) \leq 0 \ \forall t, s \in [0, 1]$.
Now denote

$$\begin{aligned} M_i &= \max_{0\leq t\leq 1} \int_0^1 |G_i(t, s)|ds, \\ P_i &= \|P^{(i)}\|, \ (i = 0, 1, 2, 3). \end{aligned} \tag{2.10}$$

Here $P^{(i)}$ is the derivative of order i in t of the function $P(\alpha, \beta, t)$.
It is easy to verify that

$$\begin{aligned} &M_0 = \frac{5}{384}, \ M_1 = \frac{1}{24}, \ M_2 = \frac{1}{8}, \ M_3 = \frac{1}{2} \\ &P_0 \leq |\alpha| + \frac{1}{6}|\beta|, \ P_1 \leq |\alpha| + \frac{1}{3}|\beta|, \ P_2 = P_3 = |\beta|. \end{aligned} \tag{2.11}$$

Then, from (2.3), (2.9) and (2.10) we obtain the following estimates for the solution of the problem (2.2):

$$
\begin{aligned}
\|u\| &\le M_0\|\varphi\| + P_0,\\
\|u'\| &\le M_1\|\varphi\| + P_1,\\
\|u''\| &\le M_2\|\varphi\| + P_2,\\
\|u'''\| &\le M_3\|\varphi\| + P_3.
\end{aligned}
\tag{2.12}
$$

For any number $M > 0$ define the domain

$$
\begin{aligned}
\mathcal{D}_M =&\{(t,u,y,v,z) \mid 0 \le t \le 1,\ |u| \le (M_0 + \frac{7}{6r})M,\ |y| \le (M_1 + \frac{4}{3r})M,\\
&|v| \le (M_2 + \frac{1}{r})M, |z| \le (M_3 + \frac{1}{r})M\}
\end{aligned}
\tag{2.13}
$$

and as usual, denote by $B[0, M]$ the closed ball in $\mathcal{B}$.
From now on suppose that the function $f(t, u, y)$ is continuous in $\mathcal{D}_M$. Further, denote

$$
\theta_1 = \int_0^1 |g(s)|ds, \quad \theta_2 = \int_0^1 |h(s)|ds. \tag{2.14}
$$

Lemma 2.2. *Suppose that there exists a number $M > 0$ such that the function $f(t, u, y, v, z)$ is continuous and bounded by M in $\mathcal{D}_M$, i.e.,*

$$
|f(t,u,y,v,z)| \le M \quad \forall (t,u,y,v,z) \in \mathcal{D}_M \tag{2.15}
$$

and the condition

$$
q_1 := \max\{r\theta_1(M_0 + \frac{7}{6r}),\ r\theta_2(M_2 + \frac{1}{r})\} \le 1 \tag{2.16}
$$

holds, then the operator A defined by (2.7) maps the closed ball $B[0, M]$ in $\mathcal{B}$ into itself.

Proof. Let $w = (\varphi, \alpha, \beta)^T \in B[0, M]$. Then $\|\varphi\| \le M$ and $|\alpha| \le \frac{M}{r}, |\beta| \le \frac{M}{r}$. Consider the problem (2.2). From the estimates (2.12) for the solution $u(t)$ and its derivatives, and (2.11) we obtain

$$
\begin{aligned}
&\|u\| \le \left(M_0 + \frac{7}{6r}\right)M,\ \|u'\| \le \left(M_1 + \frac{4}{3r}\right)M,\\
&\|u''\| \le (M_2 + \frac{1}{r})M\},\ \|u'''\| \le (M_3 + \frac{1}{r})M.
\end{aligned}
$$

Therefore, $(t, u(t), u'(t), u''(t), u'''(t)) \in \mathcal{D}_M$ for any $t \in [0, 1]$. Hence, by the assumption (2.15) we have

$$
|f(t, u(t), u'(t), u''(t), u'''(t))| \le M.
$$

Next, there hold the estimates

$$
r\left|\int_0^1 g(s)u(s)ds\right| \le r\theta_1\|u\| \le r\theta_1(M_0 + \frac{7}{6r})M \le q_1 M \le M.
$$

Similarly, we have

$$
r\left|\int_0^1 h(s)u''(s)ds\right| \le r\theta_2(M_2 + \frac{1}{r})M \le M.
$$

Therefore,

$$
\|Aw\|_{\mathcal{B}} \le M.
$$

This means that the operator A maps the closed ball $B[0, M] \subset \mathcal{B}$ into itself. $\square$

Lemma 2.3. *The operator A is a compact operator in $\mathcal{B}[0, M]$.*

Proof. The compactness of A follows from the compactness of the integral operators (2.3) and (2.9), the continuity of the function $f(t,u,y,v,z)$ and the compactness of the integral operators $\int_0^1 g(s)u(s)ds$ and $\int_0^1 h(s)u''(s)ds$. □

Theorem 2.4. *Suppose the conditions of Lemma 2.2 are satisfied. Then the problem (1.1)-(1.2) has a solution.*

Proof. By Lemma 2.2 and Lemma 2.3, the operator A is a compact operator mapping the closed ball $\mathcal{B}[0,M]$ in the Banach space $\mathcal{B}$ into itself. Therefore, according to the Schauder fixed point theorem, the operator A has a fixed point in $\mathcal{B}[0,M]$. This fixed point generates a solution of the problem (1.1)-(1.2). □

Now we study the positivity of solutions in the case when $g(s) \geq 0$ $h(s) \geq 0$. In this case we introduce the domain $\mathcal{D}_M^+$ by the formula

$$\begin{aligned}\mathcal{D}_M^+ =\Big\{&(t,u,y,v,z) \mid 0 \leq t \leq 1,\ 0 \leq u \leq (M_0 + \frac{7}{6r})M,\ |y| \leq (M_1 + \frac{4}{3r})M,\\ &-(M_2 + \frac{1}{r})M \leq v \leq 0,\ |z| \leq (M_3 + \frac{1}{r})M\Big\}\end{aligned} \tag{2.17}$$

and instead of the ball $B[0,M]$ we introduce the set

$$S_M = \Big\{(\varphi,\alpha,\beta)^T,\ 0 \leq \varphi \leq M,\ 0 \leq r\alpha \leq M,\ -M \leq r\beta \leq 0\Big\}$$

Theorem 2.5 (Positivity of solution). *Suppose $g(s) \geq 0, h(s) \geq 0$ in $[0,1]$ and*

(i) The function $f(t,u,y,v,z)$ is continuous and

$$0 \leq f(t,u,y,v,z) \leq M \text{ in } \mathcal{D}_M^+, \tag{2.18}$$

(ii) The condition (2.16) is satisfied.

Then the problem (2.2) has a nonnegative solution. Besides, if $f(t,0,0,0,0) \not\equiv 0$ in $(0,1)$ then the solution is positive.

Proof. It is easy to verify that under the conditions of the theorem, the operator A maps S_M into itself. Indeed, for any $w \in S_M, w = (\varphi,\alpha,\beta)^T$ we have $0 \leq \varphi \leq M, 0 \leq r\alpha \leq M, -M \leq r\beta \leq 0$. Since $G_0(t,s) \geq 0$ for $0 \leq t,s \leq 1$, and $P(\alpha,\beta,t) \geq 0$ for $0 \leq t,s \leq 1$, from (2.3) and (2.12) we have

$$0 \leq u(t) \leq (M_0 + \frac{7}{6r})M,\ |u'(t)| \leq (M_1 + \frac{4}{3r})M$$

for any $0 \leq t \leq 1$. Next, since $G_2(t,s) \leq 0$ for $0 \leq t,s \leq 1$, and $P''(\alpha,\beta,t) \leq 0$ for $0 \leq t \leq 1$ and $\beta \leq 0$ we also have

$$-(M_2 + \frac{1}{r})M \leq u''(t) \leq 0,\ |u'''(t)| \leq (M_3 + \frac{1}{r})M$$

for any $0 \leq t \leq 1$. Therefore, for the solution $u(t)$ of (2.2) we have

$$(t,u(t),u'(t),u''(t),u'''(t)) \in \mathcal{D}_M^+,$$

and by the condition (2.18) we obtain

$$0 \leq f(t,u(t),u'(t),u''(t),u'''(t)) \leq M.$$

Due to (2.16) we have also

$$0 \leq r\int_0^1 g(s)u(s)ds \leq r\theta_1(M_0 + \frac{7}{6r})M \leq M,$$

$$-M \leq -r\theta_2(M_2 + \frac{1}{r})M \leq r\int_0^1 h(s)u''(s)ds \leq 0.$$

Hence, $(f(t,u(t),u'(t),u''(t),u'''(t)), r\int_0^1 g(s)u(s)ds, r\int_0^1 h(s)u''(s)ds)^T \in S_M$, i.e. $A : S_M \to S_M$.

Also, as shown above, A is a compact operator in S, due to this A has a fixed point in S_M, which generates a solution of the problem (2.2). This solution is nonnegative. Due to the condition $f(t,0,0,0,0) \not\equiv 0$ in $(0,1)$ the function $u(t) \equiv 0$ cannot be the solution of the problem. It implies that the solution must be positive. □

Theorem 2.6 (Existence and uniqueness). *Suppose that the functions $h_1(s), h_2(s)$ are integrable on $[0,1]$ and there exist numbers $M > 0$ and $L_0, L_1, L_2, L_3 \geq 0$ such that the following hypotheses are satisfied:*

(H1) *The function $f(t,u,v,v,z)$ is continuous and $|f(t,u,y,v,z)| \leq M$ for any $(t,u,y,v,z) \in \mathcal{D}_M$, where $\mathcal{D}_M$ is defined by (2.13).*

(H2) $|f(t,u_2,y_2,v_2,z_2) - f(t,u_1,y_1,v_1,z_1)| \leq L_0|u_2-u_1| + L_1|y_2-y_1| + L_2|v_2-v_1| + L_3|z_2-z_1|,\ \forall (t,u_i,y_i,v_i,z_i) \in \mathcal{D}_M,\ i = 1,2.$

(H3) $q := \max\{q_1,q_2\} < 1$, *where q_1 is defined by (2.16) and*

$$q_2 = L_0(M_0 + \frac{7}{6r}) + L_1(M_1 + \frac{4}{3r}) + L_2(M_2 + \frac{1}{r}) + L_3(M_3 + \frac{1}{r}).$$

Then the problem has a unique solution $u \in C^4[0,1]$.

Proof. Under the hypotheses (H1) and (H3) by Lemma 2.2 the operator A defined by (2.9) is a mapping from the closed ball $B[0,M]$ into itself. Taking in addition the hypothesis (H2) into account it is easy to prove that the operator A is a contraction mapping from $B[0,M]$ into itself with the contraction coefficient q. Therefore, it has a unique fixed point in $B[0,M]$ which corresponds to a unique solution of the problem (2.2). This solution is, of course, belongs to $C^4[0,1]$ □

Analogously as the above theorem, it is easy to prove the following

Theorem 2.7 (Existence and uniqueness of positive solution). *Suppose $g(s) \geq 0, h(s) \geq 0$ in $[0,1]$. If in Theorem 2.6 replace $\mathcal{D}_M$ by $\mathcal{D}_M^+$ defined by (2.17), Hypothesis $(H1)$ by (2.18) then the problem has a unique nonnegative solution $u(t) \in C^4[0,1]$. Besides, if $f(t,0,0,0,0) \not\equiv 0$ in $(0,1)$ then the solution is positive.*

Remark 2.8. If $g(s) = 0$ (or $h(s) = 0$) then $\alpha = 0$ (or $\beta = 0$), therefore, some corresponding changes should be taken into account in the definitions of $\mathcal{D}_M, \mathcal{D}_M^+$, θ_1, θ_2 and q_2 in the above theorems.

3 Solution method

To solve the problem (1.1)-(1.2) consider the following iterative method, which in essence is a realization of the successive iterative method for finding the fixed point of the operator A in the ball $B[0,M]$:

(i) Given

$$\varphi_0(t) = f(t,0,0,0,0),\ \alpha_0 = 0,\ \beta_0 = 0. \tag{3.1}$$

(ii) Knowing $\varphi_k(t)$ and α_k, β_k $(k = 0, 1, ...)$ compute

$$\begin{aligned} u_k(t) &= \int_0^1 G_0(t,s)\varphi_k(s)ds + P(\alpha_k, \beta_k, t), \\ y_k(t) &= \int_0^1 G_1(t,s)\varphi_k(s)ds + P'(\alpha_k, \beta_k, t), \\ v_k(t) &= \int_0^1 G_2(t,s)\varphi_k(s)ds + P''(\alpha_k, \beta_k, t), \\ z_k(t) &= \int_0^1 G_3(t,s)\varphi_k(s)ds + P'''(\alpha_k, \beta_k, t) \end{aligned} \tag{3.2}$$

(iii) Update

$$\begin{aligned} \varphi_{k+1}(t) &= f(t, u_k(t), y_k(t), v_k(t), z_k(t)), \\ \alpha_{k+1} &= \int_0^1 g(s)u_k(s)ds, \\ \beta_{k+1} &= \int_0^1 h(s)v_k(s)ds. \end{aligned} \tag{3.3}$$

This iterative method converges with the rate of geometric progression and there holds the estimate

$$\|w_k - w\|_{\mathcal{B}} \le \frac{q^k}{1-q}\|w_1 - w_0\|_{\mathcal{B}} = p_k d,$$

where $w_k - w = (\varphi_k - \varphi, \alpha_k - \alpha, \beta_k - \beta)^T$ and

$$p_k = \frac{q^k}{1-q}, \; d = \|w_1 - w_0\|_{\mathcal{B}}. \tag{3.4}$$

From the definition of the norm in $\mathcal{B}$ it follows

$$\begin{aligned} \|\varphi_k - \varphi\| &\le \frac{q^k}{1-q}\|w_1 - w_0\|_{\mathcal{B}} = p_k d, \\ |\alpha_k - \alpha| &\le \frac{1}{r}\frac{q^k}{1-q}\|w_1 - w_0\|_{\mathcal{B}} = \frac{1}{r}p_k d. \\ |\beta_k - \beta| &\le \frac{1}{r}\frac{q^k}{1-q}\|w_1 - w_0\|_{\mathcal{B}} = \frac{1}{r}p_k d. \end{aligned}$$

These estimates imply the following result of the convergence of the iterative method (3.1)-(3.3).

Theorem 3.1. *The iterative method* (3.1)-(3.3) *converges and for the approximate solution* $u_k(t)$ *there hold estimates*

$$\|u_k - u\| \le \left(M_0 + \frac{7}{6r}\right)p_k d, \; \|u_k' - u'\| \le \left(M_1 + \frac{4}{3r}\right)p_k d,$$
$$\|u_k'' - u''\| \le \left(M_2 + \frac{1}{r}\right)p_k d, \; \|u_k''' - u'''\| \le \left(M_3 + \frac{1}{r}\right)p_k d,$$

where u *is the exact solution of the problem* (1.1)-(1.2) *,* p_k *and* d *are defined by* (3.4)*, and* r *is the number available in* (2.6)*.*

To numerically realize the above iterative method we construct a corresponding discrete iterative method. For this purpose cover the interval $[0, 1]$ by the uniform grid $\bar{\omega}_h = \{t_i = ih, \; h = 1/N, i = 0, 1, ..., N\}$ and denote by $\Phi_k(t), U_k(t), Y_k(t), V_k(t), Z_k(t)$ the grid functions, which are defined on the grid $\bar{\omega}_h$ and approximate the functions $\varphi_k(t), u_k(t), y_k(t), v_k(t), z_k(t)$ on this grid. We also denote by $\hat{\alpha_k}$ the approximation of α_k and $\hat{\beta_k}$ the approximation of β_k.

Consider now the following discrete iterative method:

(i) Given

$$\Phi_0(t_i) = f(t_i, 0, 0, 0, 0),\ i = 0, ..., N; \quad \hat{\alpha}_0 = 0,\ \hat{\beta}_0 = 0. \tag{3.5}$$

(ii) Knowing $\Phi_k(t_i),\ i = 0, ..., N$ and $\hat{\alpha}_k, \hat{\beta}_k\ (k = 0, 1, ...)$ compute approximately the definite integrals (3.2) by trapezium formulas

$$\begin{aligned}
U_k(t_i) &= \sum_{j=0}^{N} h\rho_j G_0(t_i, t_j)\Phi_k(t_j) + P(\hat{\alpha}_k, \hat{\beta}_k, t_i),\\
Y_k(t_i) &= \sum_{j=0}^{N} h\rho_j G_1(t_i, t_j)\Phi_k(t_j) + P'(\hat{\alpha}_k, \hat{\beta}_k, t_i),\\
V_k(t_i) &= \sum_{j=0}^{N} h\rho_j G_2(t_i, t_j)\Phi_k(t_j) + P''(\hat{\alpha}_k, \hat{\beta}_k, t_i),\\
Z_k(t_i) &= \sum_{j=0}^{N} h\rho_j G_3^*(t_i, t_j)\Phi_k(t_j) + P'''(\hat{\alpha}_k, \hat{\beta}_k, t_i),\\
& i = 0, ..., N,
\end{aligned} \tag{3.6}$$

where ρ_j is the weight of the trapezium formula

$$\rho_j = \begin{cases} 1/2,\ j = 0, N \\ 1,\ j = 1, 2, ..., N-1 \end{cases}$$

and

$$G_3^*(t, s) = \begin{cases} s, & 0 \le s < t \le 1, \\ -1/2 + s, & s = t, \\ -1 + s, & 0 \le t < s \le 1. \end{cases}$$

(iii) Update

$$\begin{aligned}
\Phi_{k+1}(t_i) &= f(t_i, U_k(t_i), Y_k(t_i), U_k(t_i), Y_k(t_i)_k(t_i), Y_k(t_i)), (i = 0, ..., N),\\
\hat{\alpha}_{k+1} &= \sum_{j=0}^{N} h\rho_j g(t_j) U_k(t_j),\\
\hat{\beta}_{k+1} &= \sum_{j=0}^{N} h\rho_j h(t_j) V_k(t_j).
\end{aligned} \tag{3.7}$$

Analogously as done in [5] we obtain the following result of the convergence of the discrete iterative method (3.5)-(3.7):

Theorem 3.2. *Under the assumptions of Theorem 2.6, for the approximate solution of the problem (1.1)-(1.2) obtained by the discrete iterative method on the uniform grid with grid size h there hold the estimates*

$$\begin{aligned}
\|U_k - u\| &\le \left(M_0 + \frac{7}{6r}\right) p_k d + O(h^2),\\
\|Y_k - u'\| &\le \left(M_1 + \frac{4}{3r}\right) p_k d + O(h^2),\\
\|V_k - u''\| &\le \left(M_2 + \frac{1}{r}\right) p_k d + O(h^2),\\
\|Z_k - u'''\| &\le \left(M_3 + \frac{1}{r}\right) p_k d + O(h^2).
\end{aligned} \tag{3.8}$$

4 Examples

In this section we demonstrate the validity of the obtained theoretical results and the efficiency of the numerical method proposed in the previous section. In all numerical examples below we perform the iterative method (3.5)-(3.7) until

$$\max\left\{\|\Phi_k-\Phi_{k-1}\|, r|\hat{\alpha}_k-\hat{\alpha}_{k-1}|, r|\hat{\beta}_k-\hat{\beta}_{k-1}|\right\}\le 10^{-9}.$$

Example 1. Consider the following problem

$$\begin{aligned}-u''''(t)&=\frac{\pi^4}{25}\cos\pi(t-\frac{1}{2})+\frac{\pi}{1250}\sin\pi(t-\frac{1}{2})\\&\quad-\frac{1}{625}\cos^2\pi(t-\frac{1}{2})+(u(t))^2+u(t)u'(t),\quad 0<t<1,\\u(0)&=\int_0^1(s-\frac{1}{2})u(s)ds,\ u(1)=0,\\u''(0)&=\int_0^1(s-\frac{1}{2})u''(s)ds,\ u''(1)=0\end{aligned}\tag{4.1}$$

with the exact solution $u(t)=\frac{1}{25}\cos\pi(t-\frac{1}{2})$.
For this example $g(s)=s-\frac{1}{2}, h(s)=s-\frac{1}{2}$ and $f=f(t,u,y)=\frac{\pi^4}{25}\cos\pi(t-\frac{1}{2})+\frac{\pi}{1250}\sin\pi(t-\frac{1}{2})-\frac{1}{625}\cos^2\pi(t-\frac{1}{2})+u^2+uy$. It is easy to see that $\theta_1=\theta_2=1/4$,. It is possible to verify that with the selection $r=10, M=5$ we have $r\theta_1(M_0+\frac{7}{6r})=0.3242, r\theta_2(M_2+\frac{1}{r})=0.5625$, so $q_1=0.5625$. The function $f(t,u,y)$ satisfies the Lipschitz condition in u,y with the coefficients $L_0=2.1719, L_1=0.6484$ in $\mathcal{D}_M$. Therefore, $q_2=0.3951$, and, consequently, $q=0.5625<1$. Thus, all the assumptions of Theorem 2.6 are satisfied. By this theorem, the problem (4.1) has a unique solution. This is the above exact solution.

The results of convergence of the iterative method (3.5)-(3.7) are given in Table 2. In the

Table 1. The convergence in Example 1.

N	h^2	K	$Error$
50	4.0000e-04	10	8.6079e-10
100	1.0000e-04	10	4.7898e-11
200	2.5000e-05	10	1.6051e-11

table $N+1$ is the number of grid points on $[0,1]$, K is the number of iterations performed, $Error=\|U_K-u\|$, u is the exact solution. From this table we see that the accuracy of the iterative method is much better than $O(h^2)$. So, the iterative method is very efficient.

Example 2. Consider the following problem

$$\begin{aligned}u''''(t)&=1+\sin t+u^2(t)+\frac{1}{2}(u'(t))^2+\frac{1}{3+(u''(t))^2+(u'''(t))^2},\quad 0<t<1,\\u(0)&=\int_0^1 s^2u(s)ds,\ u(1)=0,\\u''(0)&=\int_0^1\frac{s}{2}u''(s)ds,\ u''(1)=0\end{aligned}\tag{4.2}$$

In this example $0\le f(t,u,y,v,z)=1+\sin t+u^2+\frac{1}{2}y^2+\frac{1}{3+v^2+z^2}$. We can choose $M=2.6, r=10$ so that in the domain $\mathcal{D}_M^+$ the function $0\le f(t,u,y,v,z)\le M$ and it satisfies the Lipschitz conditions with the coefficients $L_0=0.6744, L_1=0.4550, L_2=0.1300, L_3=0.4043$. We have also $\theta_1=\int_0^1 s^2ds=\frac{1}{3}, \theta_2=\int_0^1 s/2ds=\frac{1}{4}$. Hence, it is easy to see that all the conditions of Theorem 2.7 are satisfied. Therefore, the problem (4.2) has a unique positive solution. The results of convergence of the iterative method for the example are reported in Table 2.

Table 2. The convergence in Example 2.

N	K	α_K	β_K
50	10	0.0053	-0.0411
100	10	0.0054	-0.0411
200	10	0.0054	-0.0411
500	10	0.0054	-0.0411

The approximate solution found by the discrete iterative method after 10 iterations for $N = 100$ is depicted in Figure 1.

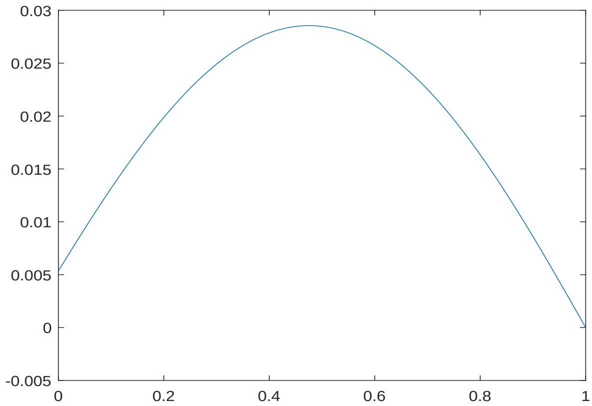

Figure 1. The graph of the approximate solution in Example 2.

Remark 4.1. Consider a generalized version of the problem (1.1)-(1.2), namely, the problem with non-homogeneous boundary conditions

$$u''''(t) = f(t, u(t), u'(t), u''(t), u'''(t)), \quad 0 < t < 1,$$

$$u(0) = \int_0^1 g(s)u(s)ds + c_1,\ u(1) = c_2,$$

$$u''(0) = \int_0^1 h(s)u''(s)ds + c_3,\ u''(1) = c_4,$$

where c_1, c_2, c_3, c_4 are real numbers.
For solving the problem we propose an iterative method similar to the iterative method (3.5)-(3.7) for the problem (1.1)-(1.2) with only a difference that instead of $P(\alpha, \beta, t)$ defined by (2.7) there stands

$$P(\alpha, \beta, t) = a_0 + a_1 t + a_2 t^2 + a_3 t^3,$$

where

$$a_0 = \alpha + c_1,\ a_1 = -\alpha - \frac{1}{3}\beta - c_1 + c_2 - \frac{1}{3}c_3 - \frac{1}{6}c_4,$$
$$a_2 = \frac{1}{2}(\beta + c_3),\ a_3 = \frac{1}{6}c_4 - \frac{1}{6}(\beta + c_3).$$

In all numerical examples below we perform the iterative method (3.5)-(3.7) until

$$\max\left\{\|\Phi_k - \Phi_{k-1}\|, |\hat{\alpha}_k - \hat{\alpha}_{k-1}|, |\hat{\beta}_k - \hat{\beta}_{k-1}|\right\} \leq 10^{-9}.$$

Example 3. Consider the following problem

$$\begin{aligned} u''''(t) &= 24 - (t^4+t)^2 - 12(4t^3+1)t^2 + u^2(t) + u(t)u''(t), \quad 0<t<1,\\ u(0) &= \int_0^1 su(s)ds - \frac{1}{2},\ u(1)=2,\\ u''(0) &= \int_0^1 \frac{1}{2}u''(s)ds - \frac{3}{2},\ u''(1)=12, \end{aligned}$$

This problem has the exact solution $u(t) = t^4 + t$. The results of computation by the iterative method (3.5)-(3.7) for this problem are given in Table 3. In Table 3 α_K and β_K are the computed approximate values of $\alpha = \int_0^1 su(s)ds$ and $\beta = \int_0^1 \frac{s}{2}u''(s)ds$.

Table 3. The convergence in Example 3.

N	h^2	K	α_K	β_K	$Error$
50	4.0000e-04	12	0.5002	1.5006	2.2617e-04
100	1.0000e-04	12	0.5001	1.5002	5.6546e-05
200	2.5000e-05	12	0.5000	1.5000	1.4137e-05
500	4.0000e-06	12	0.5000	1.5000	2.2619e-06

Example 4. Consider the problem

$$\begin{aligned} u''''(t) &= \frac{1}{\sqrt{t}}\Big(\sqrt{1+(u(t)^2)} + \sqrt{1+(u'(t)u''(t))^2}\Big), \quad 0<t<1,\\ u(0) &= \int_0^1 su(s)ds,\ u(1)=0,\\ u''(0) &= \int_0^1 s^2u''(s)ds,\ u''(1)=0. \end{aligned}$$

Notice that since the function

$$\varphi(t) = f(t,u(t),u'(t),u''(t)) = \frac{1}{\sqrt{t}}\Big(\sqrt{1+(u(t)^2)} + \sqrt{1+(u'(t)u''(t))^2}\Big)$$

has a weak singularity at $t=0$ then $\int_0^1 G_l(t,s)\varphi(s)ds$ $(l=0,1,2,3)$ exist. Therefore, the iterative method (3.1)-(3.3) can be carried out. In the discrete version of the method for computing the above integrals by (3.6) we put $\Phi_k(t_1) = \Phi_k(0) = 0$ because $G_l(t_i,0) = 0$.

The results of convergence of the iterative method (3.5)-(3.7) for this example are given in Table 4. The graph of the approximate solution is depicted in Figure 2.

Table 4. The convergence in Example 4.

N	K	α_K	β_K
50	11	0.0176	-0.0807
100	11	0.0176	-0.0808
200	11	0.0176	-0.0808
500	11	0.0176	-0.0809

5 Conclusion

In this paper we have established the existence, uniqueness and positivity of solutions of a fourth order nonlinear differential equation with two integral boundary conditions. The idea of the

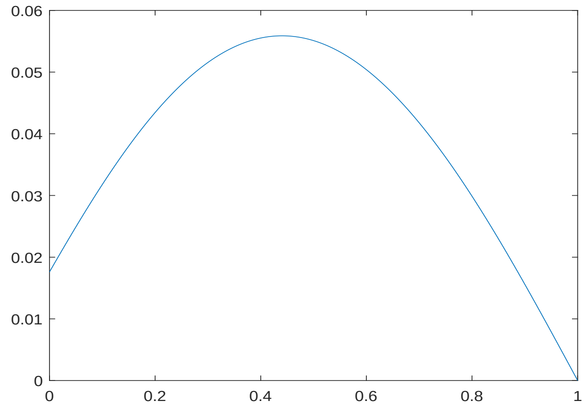

Figure 2. The graph of the approximate solution in Example 4.

method used is to reduce the problem to a fixed point problem for an operator defined on triplets of functions and real numbers. It is a further development of the method applied by ourselves before for other local and nonlocal nonlinear boundary value problems. We also have studied an iterative method for solving the problem at continuous level and a discrete scheme for realizing the continuous iterative method. We have obtained the total error of the approximate discrete solution, which consists of the error of the continuous iterative method and the error of discretization at each iteration. Many examples demonstrate the applicability of the obtained theoretical results and efficiency of the iterative method.

The method used in this paper and in [5] for nonlinear fourth order differential equations with integral boundary conditions can be extended to nonlinear differential equations of any order $n \geq 2$ with $m \leq n$ linear integral boundary conditions provided that the Green functions for them are known.

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