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$n-$absorbing $I-$primary hyperideals in multiplicative hyperrings

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Abstract In this paper, we define the concept $I-$primary hyperideal in multiplicative hyperring R. A proper hyperideal P of R is an $I-$primary hyperideal if for $a, b \in R$ with $ab \subseteq P - IP$ implies $a \in P$ or $b \in \sqrt{P}$. We provide some characterizations of $I-$primary hyperideals. Also we conceptualize and study the notions $2-$absorbing $I-$primary and $n-$absorbing $I-$primary ideals into multiplicative hyperrings as generalizations of prime ideals. A proper hyperideal P of a hyperring R is an $n-$absorbing $I-$primary hyperideal if for $x_1, \cdots, x_{n+1} \in R$ such that $x_1 \cdots x_{n+1} \subseteq P - IP$, then $x_1 \cdots x_n \subseteq P$ or $x_1 \cdots x_{i-1}x_{i+1} \cdots x_{n+1} \subseteq \sqrt{P}$ for some $i \in \{1, \cdots, n\}$. We study some properties of such generalizations. We prove that if P is an $I-$primary hyperideal of a hyperring R, then each of $\frac{P}{J}$, $S^{-1}P$, $f(P)$, $f^{-1}(P)$, $\sqrt{P}$ and $P[x]$ are $I-$primary hyperideals under suitable conditions and suitable hyperideal I, where J is a hyperideal contains P. Also, we characterize $I-$primary hyperideals in the decomposite hyperrings. Moreover, we show that the hyperring with finite number of maximal hyperideals in which every proper hyperideal is $n-$absorbing $I-$primary is a finite product of fields.

1 Introduction

Many concepts in modern algebra was generalized by generalizing their structures to hyperstructure. The French mathematician F. Marty in 1934 introduced the concept hyperstructure or multioperation by returning a set of values instead of a single value [8]. The hyperstructures theory was studied from many points of view and applied to several areas of mathematics especially in computer science and logic. In [8] the author presented the concept hypergroup and after that in 1937, the authors H. S. Wall [11] and M. Keranser [7] also gave their respective definitions of hypergroup as a generalization of groups.

The hyerrings were introduced by many authors. A type of hyperring where the multiplication is a hyperoperation while the addition is just an operation introduced by Rota in 1982 and called a multiplicative hyperring [10]. Another type of hyperring in which addition is a hyperoperation while the multiplication is an operation introduced by M. Krasner in 1983 and called Krasner hyperring [7]. The hyperrings in which the additions and multiplications are hyperoperations where introduced by De Salvo [6]. Procesi and Rota in [9] have conceptualized the notion of primeness of hyperideal in a multiplicative hyperring.

In the recent years many generalizations of prime ideals were introduced. Here state some of them. The authors in [4] and [3] introduced the notions $2-$absorbing and $n-$absorbing ideals in commutative rings. A proper ideal P is called $2-$absorbing (or $n-$absorbing) ideal if whenever the product of three (or $n + 1$) elements of R in P, the product of two (or n) of these elements is in P.

In [1] and [2], the author Akray introduced the notions $I-$prime ideal, $I-$primary ideal and $n-$absorbing $I-$ideal in classical rings as a generalization of prime ideals. For fixed proper ideal

I of a commutative ring R with identity, a proper ideal P of R is an $I-$prime if for $a, b \in R$ with $a.b \in P - IP$, then $a \in P$ or $b \in P$. A proper ideal P of R is an $I-$primary if for $a, b \in R$ with $a.b \in P - IP$, then $a \in P$ or $b \in \sqrt{P}$. A proper ideal P of R is an $n-$absorbing $I-$primary ideal if for $x_1, \cdots, x_{n+1} \in R$ such that $x_1 \cdots x_{n+1} \in P - IP$, then $x_1 \cdots x_n \in P$ or $x_1 \cdots x_{i-1}x_{i+1} \cdots x_{n+1} \in \sqrt{P}$ for some $i \in \{1, 2, \cdots, n\}$.

In this paper all rings are commutative hyperring with identity. Here we want to define the concepts $I-$primary ideal, $2-$absorbing $I-$primary and $n-$absorbing $I-$primary ideal in multiplicative hyperrings. For fixed proper hyperideal I of a multiplicative hyperring R, a proper hyperideal P of R is an $I-$primary if $a, b \in R$ with $a.b \subseteq P - IP$, then $a \in P$ or $b \in \sqrt{P}$. A proper hyperideal P of R is a $2-$absorbing $I-$primary if for $x_1, x_2, x_3 \in R$ such that $x_1x_2x_3 \subseteq P - IP$, then $x_1x_2 \subseteq P$ or $x_1x_3 \subseteq \sqrt{P}$ or $x_2x_3 \subseteq \sqrt{P}$. A proper hyperideal P of a hyperring R is an $n-$absorbing $I-$primary hyperideal if for $x_1, \cdots, x_{n+1} \in R$ such that $x_1 \cdots x_{n+1} \subseteq P - IP$, then $x_1 \cdots x_n \subseteq P$ or $x_1 \cdots x_{i-1}x_{i+1} \cdots x_{n+1} \subseteq \sqrt{P}$ for some $i \in \{1, \cdots, n\}$. A proper hyperideal I of a hyperring R is a radical if $I = \sqrt{I}$. Let I, J be two hyperideals of R, we define $(I : J) = \{r \in R : r \circ J \subseteq I\}$.

The paper is arranged as follows. In Section 2, we define the definition of an $I-$primary hyperideal of a hyperring R. Then we provide some properties of $I-$primary hyperideal, like as this an attractive property; let R be a hyperring and $h : R \longrightarrow R$ be a good epimorphism and P be an $I-$primary hyperideal of R with $Kerh \subseteq P$, then $h(P)$ is an $h(I)-$primary hyperideal. It is shown (Theorem 2.9) that if $h : R \longrightarrow L$ be a good homomorphism from hyperring R into L and assume P is an $I-$primary hyperideal of L, then $h^{-1}(P)$ is an $I-$primary hyperideal of R. It is shown (Theorem 2.16) that if P be a proper hyperideal of a hyperring R, then the following assertions are equivalent:

(i) P is $I-$primary hyperideal;

(ii) for $r \in R - P$, $(P : r) = P \cup (IP : r)$;

(iii) for $r \in R - P$, $(P : r) = P$ or $(P : r) = (IP : r)$;

(iv) for hyperideals J and K of R, $JK \subseteq P$ and $JK \nsubseteq IP$ imply $J \subseteq P$ or $K \subseteq P$.

At the end of this section we prove that if P is an $I-$primary hyperideal of a hyperring R, then $\sqrt{P}$ is a $\sqrt{I}-$prime hyperideal. In Section 3 firstly we introduce the definition of a $2-$absorbing $I-$primary and $n-$absorbing $I-$primary hyperideals. We prove that if $\sqrt{P}$ is a primary hyperideal of R, then P is a $2-$absorbing $I-$primary hyperideal. Also we show that, if $\sqrt{P}$ is a $2-$absorbing primary hyperideal of R, then P is a $3-$absorbing $I-$primary hyperideal and in general, P is an $(n + 1)-$absorbing $I-$primary hyperideal, whenever $\sqrt{P}$ is an $n-$absorbing primary hyperideal of R. Also we prove that if P be an $n-$absorbing $I-$primary hyperideal of R, then $\sqrt{P}$ is $n-$absorbing $I-$primary hyperideal of R and $x^n \subseteq P$, for each $x \in \sqrt{P}$. Also we prove that, let $\mid Max(R) \mid \geq n + 1 \geq 2$. Then each proper hyperideal of R is a $n-$absorbing $I-$primary hyperideal if and only if each quotient of R is a product of $(n + 1)-$fields. At the same time we give a lot of amazing properties of $n-$absorbing $I-$primary hyperideal. Finally, we show that if P is an $I-$primary hyperideal of R, then $P[x]$ is an $I[x]-$primary hyperideal of polynomial hyperring $R[x]$.

2 $I-$primary hyperideals

At this section, firstly we define the definition of an $I-$primary hyperideal of a hyperring R.

Definition 2.1. Let R be a multiplicative hyperring. A proper hyperideal P of R is called an $I-$prime hyperideal of R if $\alpha \circ \beta \subseteq P - IP$ for $\alpha, \beta \in R$ implies that $\alpha \in P$ or $\beta \in P$.

Definition 2.2. Let R be a multiplicative hyperring. A proper hyperideal P of R is called an $I-$primary hyperideal of R if $\alpha \circ \beta \subseteq P - IP$ for $\alpha, \beta \in R$ implies that $\alpha \in P$ or $\beta \in \sqrt{P}$.

Lemma 2.3. *Let P be a proper hyperideal of a hyperring $(R,+,\circ)$. Then P is an $I-$primary hyperideal if and only if P/IP is $\{0\}-$prime in R/IP.*

Proof. ($\Rightarrow$) Let P be an $I-$primary hyperideal in $(R,+,\circ)$, and $a,b\in R$ with $\{0\}\neq(a+IP)(b+IP)=a\circ b+IP\subseteq P/IP$. Then $a\circ b\subseteq P-IP$ implies $a\in P$ or $b\in\sqrt{P}$, hence $a+IP\in P/IP$ or $b+IP\in P/IP$. So P/IP is $\{0\}-$prime hyperideal in R/IP.
($\Leftarrow$) Suppose that P/IP is $\{0\}-$prime hyperideal in R/IP and take $r,s\in R$ such that $r\circ s\subseteq P-IP$. Then $\{0\}\neq r\circ s+IP=(r+IP)(s+IP)\subseteq P/IP$ so $r+IP\in P/IP$ or $s+IP\in P/IP$. Therefore $r\in P$ or $s\in\sqrt{P}$. Thus P is an $I-$primary hyperideal in R. $\square$

Proposition 2.4. *Let P be an $I-$primary hyperideal of R and K be a subset of R. For any $a\in R, aK\subseteq P$, $aK\nsubseteq IP$ and $a\notin P$ implies that $K\subseteq\sqrt{P}$. (or $aK\subseteq P$ and $K\nsubseteq P$ imply that $a\in\sqrt{P}$).*

Proof. Let $aK\subseteq P$, $aK\nsubseteq IP$ and $a\notin P$ for any $a\in R$. Then we have $aK=\cup ak_i\subseteq P$ for all $k_i\in K$. Hence $ak_i\subseteq P$ and $ak_i\nsubseteq IP$ for all $k_i\in K$. Since P is an $I-$primary hyperideal and $a\notin P$, $k_i\in\sqrt{P}$, $\forall k_i\in K$. Thus $K\subseteq\sqrt{P}$. $\square$

Proposition 2.5. *Let P be an $I-$primary hyperideal of R and A,B be two subsets of R. If $AB\subseteq P$ and $AB\nsubseteq IP$, then $A\subseteq P$ or $B\subseteq\sqrt{P}$.*

Proof. Assume that $AB\subseteq P, AB\nsubseteq IP, A\nsubseteq P$ and $B\nsubseteq\sqrt{P}$. Since $AB=\bigcup a_ib_i\subseteq P$, $a_ib_i\subseteq P$, for $a_i\in A, b_i\in B$ and as $A\nsubseteq P$, $B\nsubseteq\sqrt{P}$, we have $x\notin P$ and $y\notin\sqrt{P}$ for some $x\in A, y\in B$. Then $xy\subseteq AB\subseteq P$ and $xy\nsubseteq IP$. From being P an $I-$primary hyperideal, we have $x\in P$ or $y\in\sqrt{P}$ which is a contradiction. Thus $A\subseteq P$ or $B\subseteq\sqrt{P}$. $\square$

Corollary 2.6. *Let P be a radical $I-$primary hyperideal of R and $A,B\subseteq R$ with $AB\subseteq P, AB\nsubseteq IP$. Then $A\subseteq P$ or $B\subseteq P$.*

Theorem 2.7. *Let R be a hyperring and $h:R\longrightarrow R$ be a good epimorphism and let P be an $I-$primary hyperideal of R with $Kerh\subseteq P$. Then $h(P)$ is an $h(I)-$primary hyperideal.*

Proof. First we have to show that $h(P)$ is hyperideal of R. Let $\bar{r}\in R, y\in h(P)$. Then $y=h(x)$ and $\bar{r}=h(r)$ for some $x\in P$ and $r\in R$ and so $rx\in P$. So $\bar{r}y=h(r)h(x)=h(rx)\subseteq h(P)$. Now let us show that $h(P)$ is an $I-$primary hyperideal. To do this for all $x,y\in R$ with $xy\subseteq h(P)-h(I)h(P)$ there exist $h(P)-h(IP)$, $a,b\in R$ such that $x=h(a)$ $y=h(b)$. So $xy=h(a)h(b)=h(ab)\subseteq h(P)$ that is $ab\subseteq P+Kerh$. Then $a\in P$ or $b\in\sqrt{P}$ by primary of P, that is $x=h(a)\in h(P)$ or $y=h(b)\in\sqrt{h(P)}$, so $h(P)$ is $h(I)-$primary hyperideal of R. $\square$

It is clear that $\sqrt{h(P)}=h(\sqrt{P})$ for any good homomorphism $h:R\longrightarrow L$, and hyperideal P of hyperring R. For the inverse homomorphism we have the following lemma.

Lemma 2.8. *Suppose that $h:R\longrightarrow L$ is a good homomorphism between hyperrings R and L and P is a hyperideal of L. Then $h^{-1}(\sqrt{P})\subseteq\sqrt{h^{-1}(P)}$.*

Proof. Suppose that $a\in h^{-1}(\sqrt{P})$. Then $h(a)\in\sqrt{P}$, and as h a good homomorphism, $(h(a))^n=h\left(a^n\right)\subseteq P$ for some $n\geq 1$, which is equivalent to $a^n\subseteq h^{-1}(P)$ or $a\in\sqrt{h^{-1}(P)}$ and this is equivalent to $h^{-1}(\sqrt{P})\subseteq\sqrt{h^{-1}(P)}$. $\square$

Theorem 2.9. *Let $h:R\longrightarrow L$ be a good homomorphism from hyperring R into L and assume P is an $I-$primary hyperideal of L, then $h^{-1}(P)$ is an $I-$primary hyperideal of R.*

Proof. Assume $ab\subseteq h^{-1}(P)$ for $a,b\in R$. So $h(ab)=h(a)h(b)\subseteq P$ and as P is an $I-$primary hyperideal of L, we obtain $h(a)\subseteq P$ or $h(b)\subseteq\sqrt{P}$. Thus, $a\subseteq h^{-1}(P)$ or $b\subseteq h^{-1}(\sqrt{P})\subseteq\sqrt{h^{-1}(P)}$ by Lemma 2.8. Therefore $h^{-1}(P)$ is an $I-$primary hyperideal of R. $\square$

Proposition 2.10. *(1) Let $I\subseteq J$ be two hyperideals of a multiplicative hyperring R. If P is an $I-$primary hyperideal of R, then it is $J-$primary hyperideal.*

(2) Let R be a commutative multiplicative hyperring and P an I-primary hyperideal that is not primary hyperideal, then $P^2\subseteq IP$. Thus, an $I-$primary hyperideal P with $P^2\nsubseteq IP$ is a primary hyperideal.

Proof. (1) The proof comes from the fact that if $I \subseteq J$, then $P - JP \subseteq P - IP$.

(2) Suppose that $P^2 \nsubseteq IP$. We show that P is a primary hyperideal. Let $ab \subseteq P$ for $a, b \in R$. If $ab \nsubseteq IP$, then P $I-$primary gives $a \in P$ or $b \in \sqrt{P}$. So assume that $ab \subseteq IP$. First, suppose that $aP \nsubseteq IP$; say $ax \nsubseteq IP$ for some $x \in P$. Then $a(x+b) \subseteq P - IP$. So $a \in P$ or $x + b \in P$ and hence $a \in P$ or $b \in \sqrt{P}$. Hence we can assume that $aP \subseteq IP$ and in a similar way we can assume that $bP \subseteq IP$. Since $P^2 \nsubseteq IP$, there exist $y, z \in P$ with $yz \nsubseteq IP$. Then $(a+y)(b+z) \subseteq P - IP$. So P I-primary gives $a + y \in P$ or $b + z \in \sqrt{P}$ and hence $a \in P$ or $b \in \sqrt{P}$. Therefore P is a primary hyperideal of R. □

Corollary 2.11. *Let P be an $I-$primary hyperideal of a hyperring R with $IP \subseteq P^3$. Then P is $\cap_{i=1}^{\infty} P^i-$primary hyperideal.*

Proof. If P is primary hyperideal, then P is $\cap_{i=1}^{\infty} P^i-$primary hyperideal. Assume that P is not primary hyperideal of R. Therefore $P^2 \subseteq IP \subseteq P^3$. Thus $IP = P^n$ for each $n \geq 2$. So $\cap_{i=1}^{\infty} P^i = P \cap P^2 = P^2$ and $(\cap_{i=1}^{\infty} P^i)\, P = P^2 P = P^3 = IP$. From being P an I-primary hyperideal, we have P is $\cap_{i=1}^{\infty} P^i-$primary hyperideal. □

Corollary 2.12. *Let P be an $I-$primary hyperideal of a hyperring R which is not primary hyperideal of. Then $\sqrt{P} = \sqrt{IP}$*

Proof. Since by Proposition 2.10, $P^2 \subseteq IP$, hence $\sqrt{P} = \sqrt{P^2} \subseteq \sqrt{IP}$. The other containment always holds. □

Remark 2.13. Assume that P is an $I-$primary hyperideal of R but not primary. Then by Proposition 2.10, if $IP \subseteq P^2$, then $P^2 = IP$. In particular, if P is a $\{0\}-$prime hyperideal but not primary hyperideal, then $P^2 = 0$. Suppose that $IP \subseteq P^3$. Then $P^2 \subseteq IP \subseteq P^3$; So $P^2 = P^3$ and thus P^2 is an idempotent hyperideal.

Proposition 2.14. *(1) Let R and S be two commutative multiplicative hyperrings and P be $\{0\}-$primary hyperideal of R. Then $P \times S$ is $I-$primary hyperideal of $R \times S$ for each hyperideal I of $R \times S$ with $\cap_{i=1}^{\infty}(P \times S)^i \subseteq I(P \times S) \subseteq P \times S$.*

(2) Let P be a finitely generated proper hyperideal of a commutative hyperring R. Assume P is an $I-$primary hyperideal with $IP \subseteq P^3$. Then either P is $\{0\}-$primary or $P^2 \neq 0$ is idempotent and R decomposes as $T \times S$ where $S = P^2$ and $P = J \times S$ where J is $\{0\}-$primary. Thus P is $I-$primary hyperideal for $\cap_{i=1}^{\infty} P^i \subseteq IP \subseteq P$.

Proof. (1) Let R and S be two commutative hyperrings and P be a $\{0\}-$primary hyperideal of R. Then $P \times S$ need not be a $\{0\}-$primary hyperideal of $R \times S$. In fact, $P \times S$ is a $\{0\}-$primary hyperideal if and only if $P \times S$ is primary hyperideal. However, $P \times S$ is $I-$primary hyperideal for each I with $\cap_{i=1}^{\infty}(P \times S)^i \subseteq I(P \times S)$. If P is a primary hyperideal, then $P \times S$ is a primary hyperideal and thus is $I-$primary for all I. Assume that P is not primary hyperideal. Then $P^2 = 0$ and $(P \times S)^2 = 0 \times S$. Hence $\cap_{i=1}^{\infty}(P \times S)^i = \cap_{i=1}^{\infty} P^i \times S = 0 \times S$. Thus $P \times S - \cap_{i=1}^{\infty}(P \times S)^i = P \times S - 0 \times S = (P - 0) \times S$. Since P is $\{0\}-$primary hyperideal, $P \times S$ is $\cap_{i=1}^{\infty}(P \times S)^i-$primary hyperideal and as $\cap_{i=1}^{\infty}(P \times S)^i \subseteq I(P \times S)$, $P \times S$ is an $I-$primary hyperideal.

(2) If P is a primary hyperideal, then P is $\{0\}-$primary. So we can assume that P is not primary hyperideal. Then $P^2 \subseteq IP$ by Proposition 2.10 and hence $P^2 \subseteq IP \subseteq P^3$. So $P^2 = P^3$. Hence P^2 is idempotent. Since P^2 is a finitely generated, $P^2 =< e >$ for some idempotent $e \in R$. Suppose $P^2 = 0$. Then $IP \subseteq P^3 = 0$. So $IP = 0$ and hence P is $\{0\}-$primary. Assume $P^2 \neq 0$. Put $S = P^2 = Re$ and $T = R(1-e)$, so R decomposes as $T \times S$ where $S = P^2$. Let $J = P(1-e)$, so $P = J \times S$ where $J^2 = (P(1-e))^2 = P^2(1-e)^2 =< e > (1-e) = 0$. To show that J is $\{0\}-$primary hyperideal, let $a \circ b \subseteq J - 0$, so $(a, 1)(b, 1) = (a \circ b, 1) \subseteq J \times S - (J \times S)^2 = J \times S - 0 \times S \subseteq P - IP$. Since $IP \subseteq P^3$, $IP \subseteq P^3 = (J \times S)^3 = 0 \times S$. Hence $(a, 1) \in P$ or $(b, 1) \in P$, so $a \in J$ or $b \in J$. Therefore J is a $\{0\}-$primary hyperideal. □

Corollary 2.15. *Let $(R, +, \circ)$ be an indecomposable commutative hyperring and P a finitely generated $I-$primary hyperideal of $(R, +, \circ)$, where $IP \subseteq P^3$. Then P is a $\{0\}-$primary hyperideal.*

Theorem 2.16. *Let P be a proper hyperideal of a hyperring R. Then the following assertions are equivalent:*

(i) P is $I-$primary hyperideal;

(ii) for $r \in R - P$, $(P : r) = P \cup (IP : r)$;

(iii) for $r \in R - P$, $(P : r) = P$ or $(P : r) = (IP : r)$;

(iv) for hyperideals J and K of R, $JK \subseteq P$ and $JK \nsubseteq IP$ imply $J \subseteq P$ or $K \subseteq P$.

Proof. (1) $\Rightarrow$ (2) Suppose $r \in R - P$ and $s \in (P : r)$. So $rs \subseteq P$. If $rs \subseteq P - IP$, then $s \in P$. If $rs \subseteq IP$, then $s \in (IP : r)$, So $(P : r) \subseteq P \cup (IP : r)$. The other containment always holds.

(2) $\Rightarrow$ (3) Note that if a hyperideal is a union of two hyperideals, then it is equal to one of them.

(3) $\Rightarrow$ (4) Let J and K be two hyperideals of R with $JK \subseteq P$. Assume that $J \nsubseteq P$ and $K \nsubseteq P$. We claim that $JK \subseteq IP$. Suppose $r \in J$. First, let $r \notin P$. Then $rK \subseteq P$ gives $K \subseteq (P : r)$. Now $K \nsubseteq P$, so $(P : r) = (IP : r)$. Thus $rK \subseteq IP$. Next, let $r \in J \cap P$ and choose $s \in J - P$. Then $r + s \in J - P$. By the first case $sK \subseteq IP$ and so $(r + s)K \subseteq IP$. Pick $t \in K$. Then $rt = (r + s)t - st \subseteq IP$ and $rK \subseteq IP$. Hence $JK \subseteq IP$.

(4) $\Rightarrow$ (1) Let $rs \subseteq P - IP$. Then $(r)(s) \subseteq P$. But $(r)(s) \nsubseteq IP$. So $(r) \subseteq P$ or $(s) \subseteq \sqrt{P}$ which means $r \in P$ or $s \in \sqrt{P}$. □

Let P be an $I-$primary hyperideal of a hyperring R and $J \subseteq P$ be a hyperideal of R. Then P/J is $I-$primary hyperideal of R/J. Let $x, y \in R$ with $\bar{x} \circ \bar{y} \subseteq P/J - I(P/J) = P/J - (IP + J)/J$ where $\bar{x}, \bar{y}$ are the images of x, y in R/J. Thus $x \circ y \subseteq P - IP$. So $x \in P$ or $y \in \sqrt{P}$. Therefore $\bar{x} \in P/J$ or $\bar{y} \in \sqrt{P}/J$. So P/J is $I-$primary hyperideal.

Assume R_1 and R_2 are two hyperrings. It is known that the primary hyperideals of $R_1 \times R_2$ have the form $P \times R_2$ or $R_1 \times Q$, where P and Q are primary hyperideals of R_1 and R_2 respectively. We next, generalize this result to $I-$primary hyperideals.

Theorem 2.17. *For $i = 1, 2$ let R_i be hyperring and I_i be a hyperideal of R_i. Let $I = I_1 \times I_2$. Then the $I-$primary hyperideals of $R_1 \times R_2$ have exactly one of the following three types:*
(1) $P_1 \times P_2$, where P_i is a proper hyperideal of R_i with $I_iP_i = P_i$.
(2) $P_1 \times R_2$ where P_1 is an I_1-primary hyperideal of R_1 and $I_2R_2 = R_2$.
(3) $R_1 \times P_2$, where P_2 is an I_2-primary hyperideal of R_2 and $I_1R_1 = R_1$.

Proof. We first prove that a hyperideal of $R_1 \times R_2$ having one of these three types is $I-$primary hyperideal. The first type is clear since $P_1 \times P_2 - I\,(P_1 \times P_2) = P_1 \times P_2 - (I_1P_1 \times I_2P_2) = \phi$. Suppose that P_1 is I_1-primary hyperideal and $I_2R_2 = R_2$. Let $(a, b)(x, y) \subseteq P_1 \times R_2 - (I_1P_1 \times I_2R_2) = P_1 \times R_2 - (I_1P_1 \times R_2) = (P_1 - I_1P_1) \times R_2$. Then $ax \subseteq P_1 - I_1P_1$ implies that $a \in P_1$ or $x \in \sqrt{P_1}$, so $(a, b) \in P_1 \times R_2$ or $(x, y) \in \sqrt{P_1 \times R_2}$. Hence $P_1 \times R_2$ is $I-$primary hyperideal. Similarly we can prove the last case. Next, let $P_1 \times P_2$ be an $I-$primary and $ab \subseteq P_1 - I_1P_1$. Then $(a, 0)(b, 0) = (ab, 0) \subseteq P_1 \times P_2 - I\,(P_1 \times P_2)$, so $(a, 0) \in P_1 \times P_2$ or $(b, 0) \in P_1 \times P_2$, that is, $a \in P_1$ or $b \in P_1$. Hence P_1 is I_1-primary. Likewise, P_2 is I_2-primary.

Assume that $P_1 \times P_2 \neq I_1P_1 \times I_2P_2$, say $P_1 \neq I_1P_1$. Let $x \in P_1 - I_1P_1$ and $y \in P_2$. Then $(x, 1)(1, y) = (x, y) \subseteq P_1 \times P_2$. So $(x, 1) \in P_1 \times P_2$ or $(1, y) \subseteq \sqrt{P_1 \times P_2}$. Thus $P_2 = R_2$ or $P_1 = R_1$. Assume that $P_2 = R_2$. Then $P_1 \times R_2$ is an $I-$primary, where P_1 is an I_1-primary of R_1.

□

Lemma 2.18. *If P is an $I-$primary hyperideal of a hyperring R, then $\sqrt{P}$ is a $\sqrt{I}-$prime hyperideal of R.*

Proof. Let $ab \subseteq \sqrt{P} - \sqrt{I}\sqrt{P} = \sqrt{P} - \sqrt{IP}$ for $a, b \in R$. Then $(ab)^n = a^nb^n \subseteq P$ for some $n \in \mathbb{N}$ and $(ab)^m \nsubseteq IP$ for all $m \in \mathbb{N}$. So $a^nb^n \subseteq P - IP$ and as P is an $I-$primary hyperideal of R, $a^n \subseteq P$ or $b^n \subseteq \sqrt{P}$, that is $a \in \sqrt{P}$ or $b \in \sqrt{P}$ which means that $\sqrt{P}$ is a $\sqrt{I}-$prime hyperideal of R. □

We can contract the localization of a multiplicative hyperring R as follows: Let S be a multiplicative closed subset of R, that is, S is closed under the hypermultiplication and contains the identity. Let $S^{-1}R$ be the set $(R \times S/\sim)$ of equivalence classes where

$$(r_1, s_1) \sim (r_2, s_2) \iff \exists s \in S \text{ such that } ss_1r_2 = ss_2r_1$$

Let r/s be the equivalence class of $(r, s) \in R \times S$ under the equivalence relation $\sim$. The operation addition and the hyperoperation multiplication are defined by

$$\frac{r_1}{s_1} + \frac{r_2}{s_2} = \frac{s_1r_2 + s_2r_1}{s_1s_2} = \{\frac{a+b}{c} : a \in s_1r_2, b \in s_2r_1, c \in s_1s_2\}$$

$$\frac{r_1}{s_1} \cdot \frac{r_2}{s_2} = \frac{r_1r_2}{s_1s_2} = \{\frac{a}{b}, a \in r_1r_2, b \in s_1s_2\}$$

Note that the localization map $f : R \to S^{-1}R$, $f(r) = \frac{r}{1}$ is a homomorphism of hyperrings. It is easy to see that the localization of a hyperideal is a hyperideal.

Theorem 2.19. *Let P be an $I-$primary hyperideal of a hyperring R. Then $S^{-1}P$ is an $S^{-1}I-$primary hyperideal of the hyperring $S^{-1}R$.*

Proof. Let $\frac{r_1}{s_1}, \frac{r_2}{s_2} \in S^{-1}R$ such that $\frac{r_1r_2}{s_1s_2} \subseteq S^{-1}P - S^{-1}IS^{-1}P = S^{-1}P - S^{-1}(IP)$. Then for each $n \in r_1r_2$ and $s \in s_1s_2$, we have $\frac{n}{s} \in S^{-1}P - S^{-1}(IP)$. So, there exists $q \in S$ with $qn \subseteq P - IP$, that is $qr_1r_2 \subseteq P - IP$. As P is an $I-$primary hyperideal, $qr_1 \subseteq P$ or $r_2^n \subseteq P$ for some $n \in \mathbb{N}$, which means $\frac{r_1}{s_1} = \frac{qr_1}{qs_1} \subseteq P$ or $\frac{r_2^n}{s_2^n} \subseteq P$ that is $\frac{r_1}{s_1} \in S^{-1}P$ or $\frac{r_2}{s_2} \in \sqrt{P}$. Thus $S^{-1}P$ is an $S^{-1}I-$primary hyperideal of $S^{-1}R$. □

3 $2-$absorbing $I-$primary and $n-$absorbing $I-$primary hyperideals

In this section, we begin to define the definition of a $2-$absorbing $I-$primary and $n-$absorbing $I-$primary hyperideals of a hyperring R.

Definition 3.1. Let R be a multiplicative hyperring. A proper hyperideal P of R is siad to be a $2-$absorbing $I-$primary hyperideal of R if $x \circ y \circ z \subseteq P - IP$ for $x, y, x \in R$ then $x \circ y \subset P$ or $x \circ z \subseteq \sqrt{P}$ or $y \circ z \subseteq \sqrt{P}$.

Definition 3.2. A proper hyperideal P of a hyperring R is an $n-$absorbing $I-$primary hyperideal if for $x_1, \cdots, x_{n+1} \in R$ such that $x_1 \cdots x_{n+1} \subseteq P - IP$, then $x_1 \cdots x_n \subseteq P$ or $x_1 \cdots x_{i-1}x_{i+1} \cdots x_{n+1} \subseteq \sqrt{P}$ for some $i \in \{1, \cdots, n\}$.

Theorem 3.3. *Let $h : R \to L$ be a bijective good homomorphism of hyperrings R and L, and P be a $2-$absorbing $I-$primary hyperideal of L, then $h^{-1}(P)$ is a $2-$absorbing $h^{-1}(I)-$primary hyperideal of R.*

Proof. Suppose that $abc \subseteq h^{-1}(P) - h^{-1}(I)h^{-1}(P) = h^{-1}(P) - h^{-1}(IP)$, for each $a, b, c \in R$. So $h(abc) = h(a)h(b)h(a) \subseteq P$ and $h(abc) \nsubseteq IP$. From being P a $2-$absorbing $I-$primary hyperideal, we have $h(a)h(b) \subseteq P$ or $h(a)h(c) \subseteq \sqrt{P}$ or $h(b)h(c) \subseteq \sqrt{P}$. That is $h(ab) \subseteq P$ or $h(ac) \subseteq \sqrt{P}$ or $h(bc) \subseteq \sqrt{P}$, which implies $ab \subseteq h^{-1}(P)$ or $ac \subseteq h^{-1}(\sqrt{P})$ or $bc \subseteq h^{-1}(\sqrt{P})$. By Lemma 2.8 gives that $h^{-1}(P)$ is a $2-$absorbing $h^{-1}(I)-$primary hyperideal of R. □

Every $I-$primary hyperideal is $2-$absorbing $I-$primary hyperideal. Let $(ab)c \subseteq P - IP$, implies that $ab \subseteq P$ or $bc \subseteq \sqrt{P}$ or $ac \subseteq \sqrt{P}$. If $ab \nsubseteq P$ then by $I-$primary hyperideal of P, we have $c \in \sqrt{P}$ and so $ac \subseteq \sqrt{P}$ or $bc \subseteq \sqrt{P}$. Hence P is a $2-$absorbing $I-$primary hyperideal of R.

Theorem 3.4. *Suppose that P is a proper hyperideal of hyperring R. If $\sqrt{P}$ is a primary hyperideal, then P is a $2-$absorbing $I-$primary hyperideal of R.*

Proof. Assume $xyz \subseteq P - IP$ and $xy \nsubseteq P$, for each $x, y, z \in R$. So $< xz >< yz >=< xyz^2 > \subseteq P \subseteq \sqrt{P}$ and from being $\sqrt{P}$ a primary hyperideal, we have $xz \subseteq \sqrt{P}$ or $yz \subseteq \sqrt{P}$ by Proposition 2.5. Hence P is a $2-$absorbing $I-$primary hyperideal of R. □

Theorem 3.5. *If $\sqrt{P}$ is a $2-$absorbing primary hyperideal of R, then P is a $3-$absorbing $I-$primary hyperideal of R and in general, P is an $(n+1)-$absorbing $I-$primary hyperideal, whenever $\sqrt{P}$ is an $n-$absorbing primary hyperideal of R.*

Proof. Let $abcd \subseteq P - IP$ and $abc \nsubseteq P$. Then $(ad)bc \subseteq P \subseteq \sqrt{P}$ and by hypothesis we have $(ad)b \subseteq \sqrt{P}$ or $(ad)c \subseteq \sqrt{P}$ or $bc \subseteq \sqrt{P}$. Hence $adb \subseteq \sqrt{P}$ or $adc \subseteq \sqrt{P}$ or $bcd \subseteq \sqrt{P}$, which guarantees the $3-$absorbing primary hyperideal condition of P.

□

Proposition 3.6. *Let P be a hyperideal of a hyperring R and $P_1, P_2, \cdots, P_n$ be $2-$absorbing $I-$primary hyperideals of R, such that $\sqrt{P_i} = P$ for all $i = 1, \cdots,$n. Then $\bigcap_{i=1}^{n} P_i$ is a $2-$absorbing $I-$primary hyperideal of R and $\sqrt{\bigcap_{i=1}^{n} P_i} = P$.*

Proof. Assume $P = \bigcap_{i=1}^{n} P_i$, and $\sqrt{\bigcap_{i=1}^{n} P_i} = \bigcap_{i=1}^{n} \sqrt{P_i} = P$. Let $xyz \subseteq P - IP$ and $xy \nsubseteq P$, for $x, y, z \in R$. Thus $xy \nsubseteq P_i$ for some $i = 1, 2, \cdots, n$. From being P_i is a $2-$absorbing $I-$primary hyperideal and $xyz \subseteq P - IP \subseteq P_i$, we have $xz \subseteq \sqrt{P_i} = P$ or $yz \subseteq \sqrt{P_i} = P$. This implies that $xz \subseteq \sqrt{P}$ or $yz \subseteq \sqrt{P}$, which means that P is a $2-$absorbing $I-$primary hyperideal of R. □

Theorem 3.7. *Suppose that $h : R_1 \to R_2$ is a good homomorphism of multiplicative hyperrings. Then the following statements hold.*
(1) Let P_2 be a n-absorbing $I-$primary hyperideal of R_2, then $h^{-1}(P_2)$ is a n-absorbing $I-$primary hyperideal of R_1.

(2) Let h be an epimorphism, P_1 is a $C-$hyperideal of R_1 and P_1 is an n-absorbing $I-$primary hyperideal of R_1 containing $Kerh$, then $h(P_1)$ is a n-absorbing $I-$primary hyperideal of R_2.

Proof. (1) Let $a_1, \cdots, a_{n+1} \in R_1$ and $a_1 \cdots a_{n+1} \subseteq h^{-1}(P_2) - h^{-1}(I_2) h^{-1}(P_2) = h^{-1}(P_2) - h^{-1}(I_2 P_2)$. Then $h(a_1 \cdots a_{n+1}) = h(a_1) \cdots h(a_{n+1}) \subseteq P_2 - I_2 P_2$. From bing P_2 is $n-$absorbing $I-$primary hyperideal of R_2, then $a_1 \cdots a_n \subseteq P_2$ or $(a_1 \cdots a_{i-1} a_{i+1} \cdots a_{n+1})^r \subseteq P_2$ for some $i \in \{1, \cdots, n\}$ and $r \in \mathbb{N}$. We suppose that $h(a_1) \cdots h(a_n) \subseteq P_2$ or $(h(a_1) \cdots h(a_{i-1}) h(a_{i+1}) \cdots h(a_{n+1}))^r \subseteq P_2$ for some $i \in \{1, \cdots, n\}$ and $r \in \mathbb{N}$ and so $a_1 \cdots a_n \subseteq h^{-1}(P_2)$ or $(a_1 \cdots a_{i-1} a_{i+1} \cdots a_{n+1})^r \subseteq h^{-1}(P_2)$ for some $i \in \{1, \cdots, n\}$ and $r \in \mathbb{N}$. Hence, $h^{-1}(P_2)$ is an $n-$absorbing $I-$primary hyperideal of R_1.

(2) Let $x_1, \cdots, x_{n+1} \in R_2$ and $x_1 \cdots x_{n+1} \subseteq h(P_1) - h(I_2 P_1)$. Then we have $y_1, \cdots, y_{n+1} \in R_1$ such that $h(y_1) = x_1, \cdots, h(y_{n+1}) = x_{n+1}$, and $h(y_1 \cdots y_{n+1}) = x_1 \cdots x_{n+1}$. Here, we pick any element $m \in y_1 \cdots y_{n+1}$. Then we obtain $h(m) \in h(y_1 \cdots y_{n+1}) \subseteq h(P_1)$ and so $h(m) = h(n)$ for some $n \in P_1$. This implies that $h(m - n) = 0$, that is, $m - n \in Ker(h) \subseteq P_1$ and so $m \in P_1$. Since P_1 is a $C-$hyperideal of R_1, then we conclude that $y_1 \cdots y_{n+1} \subseteq P_1$. Since P_1 is an $n-$absorbing $I-$primary hyperideal of R_1, then $y_1 \cdots y_n \subseteq P_1$ or $y_1 \cdots y_{i-1} y_{i+1} \cdots y_{n+1} \subseteq \sqrt{P_1}$ for some $i \in \{1, \cdots, n\}$. Without loss of generality, we may assume that $h(y_1) \cdots h(y_n) \subseteq h(P_1)$ or $h(y_1) \cdots h(y_{i-1}) h(y_{i+1}) \cdots h(y_{n+1}) \subseteq h(\sqrt{P_1})$ for some $i \in \{1, \cdots, n\}$ which is equivalent to $x_1 \cdots x_n \subseteq h(P_1)$ or $x_1 \cdots x_{i-1} x_{i+1} \cdots x_{n+1} \subseteq \sqrt{h(P_1)}$ for some $i \in \{1, \cdots, n\}$. Hence $h(P_1)$ is a $n-$absorbing $I-$primary hyperideal of R_2. □

Theorem 3.8. *Let P be an $n-$absorbing $I-$primary hyperideal of R. Then $\sqrt{P}$ is $n-$absorbing $I-$primary hyperideal of R and $x^n \subseteq P$, for each $x \in \sqrt{P}$.*

Proof. Let $x \in \sqrt{P}$, so $x^r \subseteq P$ for some $r \in \mathbb{N}$. If $r \leq n$, then $x^n \subseteq P$, otherwise, we can use the $n-$absorbing $I-$primary condition on the products $x\, x \cdots x^{r-n-1} \subseteq P$ up to conclude that $x^n \subseteq P$. Now to prove that $\sqrt{P}$ is an $n-$absorbing $I-$primary hyperideal, take $x_1 x_2 \cdots x_{n+1} \subseteq \sqrt{P} - I\sqrt{P}$ for $x_1, x_2, \cdots x_{n+1} \in R$. Then $x_1^n x_2^n \cdots x_{n+1}^n \subseteq P$ and $x_1^n x_2^n \cdots x_{n+1}^n \nsubseteq IP \subseteq I\sqrt{P}$. Being P an $n-$absorbing $I-$primary hyperideal of R gives us $x_1^n \cdots x_{i-1}^n x_{i+1}^n \cdots x_{n+1}^n \subseteq \sqrt{P}$. Thus $x_1 \cdots x_{i-1} x_{i+1} \cdots x_{n+1} \subseteq \sqrt{P}$ and hence $\sqrt{P}$ is an $n-$absorbing $I-$primary hyperideal of R. □

Proposition 3.9. *Let P_i be an n_i-absorbing $I-$primary hyperideal of a hyperring R, for $i = 1, 2, \cdots, m$ and $IP_i = IP_j$, for all $i \neq j$. Then $\cap_{i=1}^m P_i$ is an $n-$absorbing $I-$primary hyperideal of R, where $n = \sum_{i=1}^m n_i$.*

Proof. Let $k > n$ and $x_1 \cdots x_k \subseteq \cap_{i=1}^m P_i - I \cap_{i=1}^m P_i$. Then by the hypothesis for each $i = 1, \cdots, m$ there exists a product of n_i of these $k-$elements in P_i. Let A_i be the collection of these elements and let $A = \cup_{i=1}^k A_i$. Thus A has at most $n-$elements. Now, as P_i is an n_i-absorbing $I-$primary hyperideal of R, the product of all elements of A must be in each P_i so $\cap P_i$ contains a product of at most $n-$elements and therefore it is an $n-$absorbing $I-$primary hyperideal of R. □

Theorem 3.10. *Let $R = \prod_{i=1}^{n+1} R_i$ be a product of hyperrings R_i and P be a proper nonzero hyperideal of R. If P is an $(n+1)-$absorbing $I-$primary hyperideal of R, then $P = P_1 \times P_2 \times \cdots \times P_{n+1}$, for some proper $n-$absorbing I_i-primary hyperideals P_i of R_i, where $I = \prod_{i=1}^{n+1} I_i$ and I_i is an hyperideal of R_i, for $i = 1, \cdots, n+1$.*

Proof. Let $x_1, \cdots, x_{n+1} \in R$ with $x_1 \cdots x_{n+1} \subseteq P_1 - I_1 P_1$ and suppose by contrary that P_1 is not $n-$absorbing I_1-primary hyperideal of R. Set $a_i = (x_i, 1, 1, \cdots, 1)$ for $i = 1, \cdots, n+1$ and $a_{n+2} = (1, 0, 0, \cdots, 0)$. Then we have $a_1 \cdots a_{n+2} = (x_1 x_2 \cdots x_{n+1}, 0, 0, \cdots, 0) \subseteq P - IP$, $a_1 a_2 \cdots a_{n+1} = (x_1 x_2 \cdots x_{n+1}, 1, 1, \cdots, 1) \nsubseteq P$ and $a_1 \cdots a_{i-1} a_{i+1} \cdots a_{n+1} = (x_1 x_2 \cdots x_{i-1} x_{i+1} \cdots x_{n+1}, 0, 0, \cdots, 0) \nsubseteq \sqrt{P}$ for $i = 1, \cdots, n+1$, which is a contradiction with being P an $(n+1)-$absorbing $I-$primary hyperideal of R. By similar arguments, we can show that P_i is an $n-$absorbing I_i-primary hyperideal of R_i for $i = 1, \cdots, n+1$. □

Theorem 3.11. *Let $R = \prod_{i=1}^{n+1} R_i$, where R_i is a hyperring for $i \in \{1, \cdots, n+1\}$. If P is an $n-$absorbing $I-$primary hyperideal of R, then either $P = IP$ or $P = P_1 \times P_2 \times \cdots \times P_{i-1} \times R_i \times P_{i+1} \cdots \times P_{n+1}$ for some $i \in \{1, \cdots, n+1\}$ and if $P_j \neq R_i$ for $j \neq i$, then P_j is an $n-$absorbing I_j-primary hyperideal in R_i.*

Proof. Let $P = \prod_{i=1}^{n+1} P_i$ be an $n-$absorbing $I-$primary hyperideal of R. Then there exists $(x_1, \cdots, x_{n+1}) \in P - IP$, and so $(x_1, 1, \cdots, 1)(1, x_2, 1 \cdots, 1) \cdots (1, 1, \cdots, 1, x_{n+1}) = (x_1, x_2, \cdots, x_{n+1}) \subseteq P$. As P is an $n-$absorbing $I-$primary hyperideal of R, we have $(x_1, x_2, \cdots, x_{i-1}, 1, x_{i+1}, \cdots, x_{n+1}) \subseteq P$ for some $i \in \{1, 2, \cdots, n+1\}$. Thus $(0, 0, \cdots, 0, 1, 0, \cdots, 0) \subseteq P$ and hence $P = P_1 \times P_2 \times \cdots \times P_{i-1} \times R_i \times P_{i+1} \cdots \times P_{n+1}$. If $P_j \neq R_i$ for $j \neq i$, then we have to prove P_j is an $n-$absorbing hyperideal of R_i. Let $i < j$ and take $x_1 x_2 \cdots x_{n+1} \subseteq P_j - I_j P_j$. Then $(0, 0, \cdots, 0, 1, 0, \cdots, 0, x_1 x_2 \cdots x_{n+1}, 0 \cdots, 0) = (0, 0, \cdots, 1, 0, \cdots, 0, x_1, 0 \cdots, 0)$ $(0, 0, \cdots, 1, 0, \cdots, 0, x_2, 0 \cdots, 0) \cdots (0, 0, \cdots, 1, 0, \cdots, 0, x_{n+1}, 0 \cdots, 0) \subseteq P - IP$. Since P is an $n-$absorbing $I-$primary hyperideal, $(0, 0, \cdots, 0, 1, 0, \cdots, 0, x_1 x_2 \cdots x_{k-1} x_{k+1} \cdots x_{n+1}, 0, \cdots, 0) \subseteq \sqrt{P}$ for some $k \in \{1, 2, \cdots, n+1\}$. Thus $x_1 x_2 \cdots x_{k-1} x_{k+1} \cdots x_{n+1} \subseteq \sqrt{P_j}$ and hence P_j is an $n-$absorbing I_j-hyperideal of R_i. We can do similar arguments for the case $i > j$. □

In the following result, we characterize hyperrings in which every proper hyperideal of R is an $n-$absorbing $I-$primary hyperideal.

Theorem 3.12. *Let $| Max(R) | \geq n+1 \geq 2$. Then each proper hyperideal of R is a $n-$absorbing $I-$primary hyperideal if and only if each quotient of R is a product of $(n+1)-$fields.*

Proof. $(\Rightarrow)$ Let P be a proper hyperideal of R. Then $\frac{R}{IP} \cong F_1 \times \cdots \times F_{n+1}$ and $\frac{P}{IP} \cong P_1 \times \cdots \times P_{n+1}$, where P_i is a hyperideal of F_i, for $i = 1, \cdots, n+1$. If $P = IP$, then there is nothing to prove, otherwise we have $P_j = 0$, for at least one $j \in \{1, \cdots, n+1\}$ since $\frac{P}{IP}$ is a proper. So $\frac{P}{IP}$ is an $n-$absorbing $\{0\}-$primary hyperideal of $\frac{R}{IR}$ which means P is an $n-$absorbing $I-$primary hyperideal of R.
$(\Leftarrow)$ Let $m_1, \cdots, m_{n+1}$ be distinct maximal hyperideals of R. Then $m = \prod_{i=1}^{n+1} m_i$ is an $n-$absorbing $I-$primary hyperideal of R. we claim that m is not an $n-$absorbing hyperideal. First, if $m_i \subseteq \cup_{j \neq i} m_j$, then there exists m_j with $m_i \subseteq m_j$ by Prime Avoidance Lemma and this contradicts the maximality of m_i. Hence $m_i \nsubseteq \cup_{j \neq i} m_j$ and so, there exists $x_i \in m_i - \cup_{j \neq i}^{n+1} m_j$ so that $x_1 \cdots x_{n+1} \subseteq m$. If there exists $j \in \{1, \cdots, n+1\}$ with $a = x_1 x_2 \cdots x_{j-1} x_{j+1} \cdots x_{n+1} \subseteq m \subseteq m_j$, then $x_i \in m_j$ for some $i \neq j$ which is contradiction. Hence m is not an $n-$absorbing

hyperideal. and so $m^{n+1} = Im$. Then by the Chinese Reminder Theorem we have $\frac{R}{Im} \simeq \frac{R}{m_1^{n+1}} \times \frac{R}{m_2^{n+1}} \times \cdots \times \frac{R}{m_{n+1}^{n+1}}$. Put $F_i = \frac{R}{m_i}$. If F_i is not a field, then it has a nonzero proper hyperideal H and so $0 \times 0 \times .. \times 0 \times H \times 0 \times \cdots \times 0$ is an $n-$absorbing $\{0\}-$primary hyperideal of $\frac{R}{Im}$. Thus, by Theorem 3.11, we have $H = F_i$ or $H = \{0\}$ which is impossible. Hence F_i is a field.

□

Corollary 3.13. *Suppose* $\mid Max(R) \mid \geq n+1 \geq 2$. *Then each proper hyperideal of* R *is an* $n-$*absorbing* $\{0\}-$*primary hyperideal if and only if* $R \cong F_1 \times .. \times F_{n+1}$, *where* $F_1, .., F_{n+1}$ *are fields.*

Let $(R, +, \circ)$ be a hyperring and x be an indeterminate. Then $(R[x], +, \square)$ is polynomial hyperring by the hyper multiplication.

$$ax^n \square bx^m = (a \circ b)x^{n+m}$$

Theorem 3.14. *Suppose that* P *is an* $I-$*primary hyperideal of* R. *Then* $P[x]$ *is an* $I[x]-$*primary hyperideal of* $R[x]$.

Proof. Let $a(x) \cdot b(x) \subseteq P[x] - I[x] \cdot P[x] = P[x] - (IP)[x]$. Without loss of generality, we suppose $a(x) = cx^n$ and $b(x) = dx^m$. Then $c \cdot dx^{n+m} \subseteq P[x]$, $cd \subseteq P$ and $cdx^{n+m} \nsubseteq (IP[x])$ implies that $cd \nsubseteq IP$. P is an $I-$primary hyperideal gives us $c \in P$ or $d^r \in P$ for some positive integer r. Thus $a(x) = cx^n \in P[x]$ or $(b(x))^r = d^r x^{rm} \in P[x]$ and so $a(x) \in P[x]$ or $b(x) \in \sqrt{P[x]}$. □

Corollary 3.15. *Let* P *be an* $I-$*primary hyperideal. Then* $P[x]$ *is* $I-$*primary hyperideal of* $R[x]$.

In a multiplicative hyperring $(R, +, \circ)$ a non empty subset L of R is called a multiplicative set whenever $a, b \in A \Rightarrow a \circ b \cap A \neq \phi$.

The result [5, Proposition 2.10] is not true for $n-$absorbing $I-$primary hyperideal. For example, assume that $(\mathbb{Z}, +, \cdot)$ is the ring of integers and for any $a, b \in \mathbb{Z}$, we define the hyperoperation $a \circ b = \{2ab, 4ab\}$. Then $(\mathbb{Z}, +, \circ)$ is a multiplicative hyperring. It is clear that $2\mathbb{Z}$ is not prime, since $1 \circ 1 = \{2, 4\} \subseteq 2\mathbb{Z}$ but $1 \notin P$, while $2\mathbb{Z}$ is a $2\mathbb{Z}-$prime, since $1 \circ 1 = \{2, 4\} \nsubseteq 2\mathbb{Z} - 4\mathbb{Z}$. As $2\mathbb{Z}$ is $2\mathbb{Z}-$prime we have $2\mathbb{Z}$ is a $2\mathbb{Z}-$primary implies that $2\mathbb{Z}$ is an $n-$absorbing $2\mathbb{Z}-$primary. The set $\mathbb{Z} - 2\mathbb{Z} = \{2n+1 : n \in \mathbb{Z}\} = \mathbb{Z}_{\text{odd}}$ is not a multiplicative set of $\mathbb{Z}$. For $3, 5 \in \mathbb{Z} - 2\mathbb{Z}$ but $3 \circ 5 = \{30, 60\} \nsubseteq \mathbb{Z} - 2\mathbb{Z}$.

4 Conclusion

We introduced two new generalizations to prime ideals in multiplicative hyperrings called $I-$primary hyperideal and $n-$absorbing $I-$primary hyperideal. We concluded that they inherit many of the prime ideal characterizations and properties. Among the main results that we proved is about the characterizing the hyperrings in which every proper hyperideal is of such types of generalizations that we introduced. Furthermore, we show that under suitable condition such generalize is closed under taking radical, homomorphic image, inverse homomorphic image, product, intersect and adjoining an indeterminate.

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