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LATTICE REPRESENTATION OF COMPLETELY 0-SIMPLE SEMIGROUPS

Alanka Thomas and P. G. Romeo

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Abstract K.S.S. Nambooripad introduced a group lattice as a lattice with group action [5, 6]. Here we extend this idea to the action of semigroups on lattices. In 1971 Donald B. McAlister proved that certain linear representations of a group extend to a linear representation of completely 0-simple semigroups [4]. The main theorem here is analogous to this observation.

1 Introduction

An action of a group G on any algebraic structure gives a way to represent G by automorphisms of the respective structure. K.S.S. Nambooripad introduced the action of a group on a lattice, and he described the resulting structure as a group lattice. In particular, Nambooripad was interested in studying group actions on arguesian geomodular lattices, and he obtained exciting results on the action of groups on such lattices [5].

A semigroup lattice is a generalization of a group lattice since semigroup actions are always an extension of group actions to a broad domain. The action of a semigroup on a lattice affords the structure called a semigroup lattice. The action of a semigroup S on any algebraic structure represents S by endomorphisms of the structure. Here one can observe a one-one correspondence between S-lattices and representations of S by lattice endomorphisms. In this paper, we discuss the semigroup lattices of completely 0-simple semigroups, which is a semigroup with zero whose only ideals are $\{0\}$ and itself and has a primitive idempotent. Rees theorem characterizes completely 0-simple semigroups by regular Rees matrix semigroups $\mathcal{M}^0(G; I, \Lambda; P)$, which is a matrix semigroup having entries from $G \cup \{0\}$.

In 1971, Donald B. McAlister studied linear representations of completely 0-simple semigroups, and he observed that every linear representation of $S = \mathcal{M}^0(G; I, \Lambda; P)$ induces a linear representation of G. He also noted that certain linear representations of G extend to a linear representation of S, and he called them extendable representations [4]. We observe the same thing while studying S-lattices over completely 0-simple semigroups. That is, every S-lattice of $S = \mathcal{M}^0(G; I, \Lambda; P)$ induces a G-lattice, and there are certain G-lattices that can be extended to a S-lattices. We illustrate this using the S_3-lattice $L(S_3)$, the lattice of all subgroups of S_3, where S_3 acts by conjugation.

This paper consists of three sections. In the second section, we define the semigroup lattice with an example. In the third section, we discuss the semigroup lattices over completely 0-simple semigroups and include the action of a union of the band of groups on lattices as its subsection.

2 Semigroup lattices

Semigroup actions arise as a generalization of group actions. K.S.S. Nambooripad introduced the action of a group on lattices, and he termed the resulting structure a G-lattice [5]. In this section, we extend this concept to S-lattices.

Definition 2.1. Let S be a semigroup and $\langle L, \leq, \vee, \wedge \rangle$ be a lattice. An action of S on L is a map $(s, m) \to sm \in L$ where $s \in S$ and $m \in L$ satisfies the following for each $s, t \in S$ and $m, n \in L$.

(i) $s(tm) = (st)m$

(ii) $m \leq n$ implies $sm \leq sn$

(iii) $s(m \vee n) = sm \vee sn$

(iv) $s(m \wedge n) = sm \wedge sn$

Then L is called an $S - lattice$. A $semigroup\, lattice$ is an $S-$lattice for some semigroup S.

Consider the lattice $L(\mathbb{Z}) = \{n\mathbb{Z} \mid n \in \mathbb{N}\}$ of all principal ideals of $\mathbb{Z}$ with the usual inclusion, $m\mathbb{Z} \vee n\mathbb{Z} = gcd(m, n)\mathbb{Z}$ and $m\mathbb{Z} \wedge n\mathbb{Z} = lcm(m, n)\mathbb{Z}$. The semigroup $\mathbb{N}$ under multiplication acts on $L(\mathbb{Z})$ by $a(m\mathbb{Z}) = (am)\mathbb{Z}$ and satisfies the following.

(i) $a(b(n\mathbb{Z})) = a(bn)\mathbb{Z} = (ab)n\mathbb{Z}$

(ii) $m\mathbb{Z} \subseteq n\mathbb{Z}$ implies $(am)\mathbb{Z} \subseteq (an)\mathbb{Z}$

(iii) $a(m\mathbb{Z} \vee n\mathbb{Z}) = a(gcd(m, n))\mathbb{Z} = gcd(am, an)\mathbb{Z} = am\mathbb{Z} \vee an\mathbb{Z}$

(iv) $a(m\mathbb{Z} \wedge n\mathbb{Z}) = a(lcm(m, n))\mathbb{Z} = lcm(am, an)\mathbb{Z} = am\mathbb{Z} \wedge an\mathbb{Z}$

Hence $L(\mathbb{Z})$ is an $\mathbb{N}$-lattice.

A semigroup action gives a representation of S by endomorphisms of the respective structure. Theorem 2.2 is the analogous result for S-lattices.

Theorem 2.2. *Let S be a semigroup, and L be a lattice. Then L is an S-lattice if and only if there exists a representation Γ of S by endomorphisms on L.*

Proof. Let $\Gamma : S \to End(L)$ be a representation of S by lattice homomorphisms on L. For $s \in S$ and $m \in L$ define $sm = \Gamma(s)(m)$. Since Γ is a representation and each $\Gamma(s)$ is a lattice homomorphism, the above product satisfies properties i to iv of Definition 2.1 and so L is an S-lattice.
Conversely, let L be an S-lattice. For each $s \in S$ define $\Gamma(s) : L \to L$ such that $\Gamma(s)(m) = sm$. Properties i to iv of Definition 2.1 implies that each $\Gamma(s)$ is a lattice homomorphism. Also $\Gamma : S \to End(L)$, which takes each $s \in S$ to $\Gamma(s)$ is a representation of S by lattice endomorphisms. □

3 On the action of a completely 0-simple semigroup on lattices

A Rees matrix semigroup $S = \mathcal{M}^0(G; I, \Lambda; P)$ is a semigroup whose elements are $I \times \Lambda$ matrices over $G \cup \{0\}$ having at most one nonzero entry and the binary operation is sandwich matrix multiplication with $P = [p_{\lambda i}]_{\Lambda \times I}$. The matrix $P = [p_{\lambda i}]_{\Lambda \times I}$ is called the sandwich matrix. A matrix having its unique nonzero entry $g \in G$ at the $(i, \lambda)^{th}$ position can be identified as the triplet (g, i, λ). The binary operation is given by,

$$(g, i, \lambda)(h, j, \mu) = \begin{cases} (gp_{\lambda j}h, i, \mu) & \text{if } p_{\lambda j} \neq 0; \\ 0 & \text{otherwise.} \end{cases}$$

The matrix P is said to be regular if each row and each column of P has at least one nonzero entry. A regular Rees matrix semigroup is a Rees matrix semigroup with a regular sandwich matrix. Rees theorem characterizes completely 0-simple semigroups by regular Rees matrix semigroups [3]. In the following, we consider semigroup lattices of completely 0-simple semigroups.

In 1971 Donald B. McAlister proved that every linear representation Γ of a completely 0-simple semigroup $S = \mathcal{M}^0(G; I, \Lambda; P)$, induce a linear representation γ of G and Γ can be reconstructed in terms of γ. Conversely, there are certain linear representations γ of G which can be extended to a representation of S. He termed γ's with this property as extendable representations [4]. It is already seen that every G-lattice gives a representation of G by lattice automorphisms, and every S-lattice gives a representation of S by lattice endomorphisms. Now we aim to have an observation analogous to that of D. B. McAlister in lattice theoretic terms (Theorem 3.2).

Lemma 3.1. *Let S be a semigroup with a subgroup G and L be an S-lattice. Then there exists a sublattice of L, which is a G-lattice.*

Proof. Let L be an S-lattice. $L_e = \{em \mid m \in L\}$ be the set of all elements in L moved by the identity e in G. L_e is a sublattice of L, where $em \leq en$ whenever $m \leq n$, $em \vee en = e(m \vee n)$ and $em \wedge en = e(m \wedge n)$. The restriction of the action to $G \times L_e$ gives an action of the group G on L_e, so L_e is a G-lattice. □

Let $S = \mathcal{M}^0(G; I, \Lambda; P)$ be a completely 0-simple semigroup. For $p_{\lambda i} \neq 0$, $H_{i\lambda} = \{(g, i, \lambda) \mid g \in G\}$ is a subgroup of S isomorphic with G. Using Lemma 3.1, every S-lattice L gives an $H_{i\lambda}$-lattice on a sublattice of L denoted as $L_{i\lambda}$. As $H_{i\lambda} \cong G$, $L_{i\lambda}$ is a G-lattice.

Theorem 3.2. *Let $S = \mathcal{M}^0(G; I, \Lambda; P)$ be a completely 0-simple semigroup, L an S-lattice and $\Gamma : S \to End(L)$ be the corresponding representation. For $p_{\lambda i} \neq 0$, let L' be a G-lattice equivalent to the G-lattice $L_{i\lambda}$, having $\gamma : G \to Aut(L')$ as the representation. Then there exist lattice homomorphisms $R_j : L' \to L$ and $Q_\mu : L \to L'$ such that $Q_\mu R_j = \gamma(p_{\mu j})$ and $\Gamma(g, j, \mu) = R_j\gamma(g)Q_\mu$ for all $(g, j, \mu) \in S$.*

Conversely, let $\gamma : G \to Aut(L')$ be the representation corresponding to a G-lattice L' and L be any lattice admitting lattice homomorphisms $R_j : L' \to L$ and $Q_\mu : L \to L'$ such that $Q_\mu R_j = \gamma(p_{\mu j})$. Then $\Gamma : S \to End(L)$ such that $\Gamma(g, j, \mu) = R_j\gamma(g)Q_\mu$ for all $(g, j, \mu) \in S$, defines an action of S on L. Further, the G-lattice $L_{i\lambda}$ is equivalent to the G-lattice L'.

Proof. $S = \mathcal{M}^0(G; I, \Lambda; P)$ be a completely 0-simple semigroup and L be an S-lattice. For $p_{\lambda i} \neq 0$ the group $H_{i\lambda}$ is isomorphic with G and by Lemma 3.1, $L_{i\lambda} = \{({p_{\lambda i}}^{-1}, i, \lambda)m \mid m \in L\}$ is a G-lattice. The representation associated with the G-lattice $L_{i\lambda}$ be $\gamma_{i\lambda} : G \to Aut(L_{i\lambda})$, which is the composition of the isomorphism between G and $H_{i\lambda}$ and the restriction of Γ into $H_{i\lambda}$. Let L' be any G-lattice isomorphic with $L_{i\lambda}$ and $\gamma : G \to Aut(L')$ be the associated representation of G. Then there exists an isomorphism $\alpha : L_{i\lambda} \to L'$ such that $\gamma_{i\lambda}(g) = \alpha^{-1}\gamma(g)\alpha$ for all $g \in G$. Let $s = (g, j, \mu)$ be any element in S. If e denotes the identity in G,

$$s = (e, j, \lambda)({p_{\lambda i}}^{-1}g, i, \lambda)({p_{\lambda i}}^{-1}, i, \mu).$$

So

$$\begin{aligned}
\Gamma(s) &= \Gamma(e, j, \lambda)\Gamma({p_{\lambda i}}^{-1}g, i, \lambda)\Gamma({p_{\lambda i}}^{-1}, i, \mu) \\
&= \Gamma(e, j, \lambda)\gamma_{i\lambda}(g)\Gamma({p_{\lambda i}}^{-1}, i, \mu) \\
&= \Gamma(e, j, \lambda)\alpha^{-1}\gamma(g)\alpha\Gamma({p_{\lambda i}}^{-1}, i, \mu) \\
&= R_j\gamma(g)Q_\mu
\end{aligned}$$

where $R_j = \Gamma(e, j, \lambda)\alpha^{-1} : L' \to L$ and $Q_\mu = \alpha\Gamma({p_{\lambda i}}^{-1}, i, \mu) : L \to L'$. Then

$$\begin{aligned}
Q_\mu R_j &= \alpha\Gamma({p_{\lambda i}}^{-1}, i, \mu)\Gamma(e, j, \lambda)\alpha^{-1} \\
&= \alpha\Gamma({p_{\lambda i}}^{-1}p_{\mu j}, i, \lambda)\alpha^{-1} \\
&= \alpha\gamma_{i\lambda}(p_{\mu j})\alpha^{-1} = \gamma(p_{\mu j}).
\end{aligned}$$

Conversely, assume that L' is a G-lattice and $\gamma : G \to Aut(L')$ be the corresponding representation. Also assume that, for each $j \in I$ and $\mu \in \Lambda$, $R_j : L' \to L$ and $Q_\mu : L \to L'$ are lattice morphisms such that such that $Q_\mu R_j = \gamma(p_{\mu j})$ for some lattice L. Since P is a regular matrix, each row and each column of P has at least one nonzero entry. Hence, for each R_j, there exists a Q_μ such that $Q_\mu R_j$ is invertible, so R_j is injective. Using the same argument, Q_μ is surjective. Define $\Gamma : S \to End(L)$ such that, for each $s = (g, j, \mu)$ in S,

$$\Gamma(s) = R_j\gamma(g)Q_\mu.$$

Γ is a representation of S by lattice homomorphisms on L and hence L is an S-lattice. The restriction of Γ into $H_{i\lambda}$ gives a G-lattice structure on $L_{i\lambda}$ with $\gamma_{i\lambda} : G \to Aut(L_{i\lambda})$ as the associated representation. We need to show that the G-lattices L' and $L_{i\lambda}$ are isomorphic. We have,

$$L_{i\lambda} = \Gamma({p_{\lambda i}}^{-1}, i, \lambda)(L) = R_i\gamma({p_{\lambda i}}^{-1})Q_\lambda(L) = R_i\gamma({p_{\lambda i}}^{-1})L'$$

since Q_λ is surjective. Since R_i and $\gamma({p_{\lambda i}}^{-1})$ are injective, so is their composition, hence the corestriction of $R_i\gamma({p_{\lambda i}}^{-1})$ into $L_{i\lambda}$ is a lattice isomorphism between L' and $L_{i\lambda}$. Let it be denoted by η. Now,

$$\begin{aligned}\gamma_{i\lambda}(g)\eta &= R_i\gamma({p_{\lambda i}}^{-1}g)Q_\lambda R_i\gamma({p_{\lambda i}}^{-1})\\ &= R_i\gamma({p_{\lambda i}}^{-1}g)\gamma(p_{\lambda i})\gamma({p_{\lambda i}}^{-1})\\ &= R_i\gamma({p_{\lambda i}}^{-1})\gamma(g)\\ &= \eta\gamma(g)\end{aligned}$$

Hence $\eta(gm) = \eta\gamma(g)(m) = \gamma_{i\lambda}(g)\eta(m) = g\eta(m)$ for all $g \in G$ and $m \in L$ and so the G-lattices L' and $L_{i\lambda}$ are isomorphic. □

We illustrate the theorem in the following example.

Example 3.3. Consider S_3 and the lattice of its subgroups $L(S_3)$.

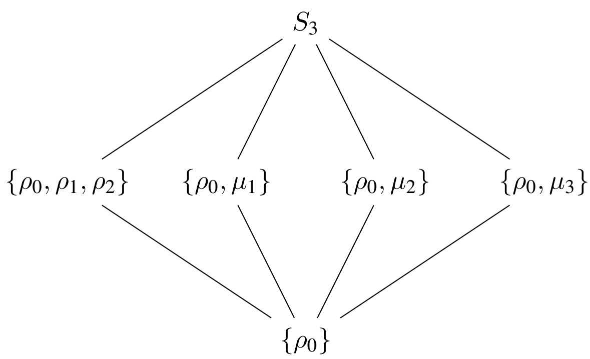

Figure 1. Lattice diagram of $L(S_3)$

It can be seen that $L(S_3)$ is a an S_3-lattice under conjugation. Then the corresponding representation $\gamma : S_3 \to Aut(L(S_3))$ is as follows.

$\circ$	S_3	$\{\rho_0, \rho_1, \rho_2\}$	$\{\rho_0, \mu_1\}$	$\{\rho_0, \mu_2\}$	$\{\rho_0, \mu_3\}$	$\{\rho_0\}$
$\gamma(\rho_0)$	S_3	$\{\rho_0, \rho_1, \rho_2\}$	$\{\rho_0, \mu_1\}$	$\{\rho_0, \mu_2\}$	$\{\rho_0, \mu_3\}$	$\{\rho_0\}$
$\gamma(\rho_1)$	S_3	$\{\rho_0, \rho_1, \rho_2\}$	$\{\rho_0, \mu_3\}$	$\{\rho_0, \mu_1\}$	$\{\rho_0, \mu_2\}$	$\{\rho_0\}$
$\gamma(\rho_2)$	S_3	$\{\rho_0, \rho_1, \rho_2\}$	$\{\rho_0, \mu_2\}$	$\{\rho_0, \mu_3\}$	$\{\rho_0, \mu_1\}$	$\{\rho_0\}$
$\gamma(\mu_1)$	S_3	$\{\rho_0, \rho_1, \rho_2\}$	$\{\rho_0, \mu_1\}$	$\{\rho_0, \mu_3\}$	$\{\rho_0, \mu_2\}$	$\{\rho_0\}$
$\gamma(\mu_2)$	S_3	$\{\rho_0, \rho_1, \rho_2\}$	$\{\rho_0, \mu_3\}$	$\{\rho_0, \mu_2\}$	$\{\rho_0, \mu_1\}$	$\{\rho_0\}$
$\gamma(\mu_3)$	S_3	$\{\rho_0, \rho_1, \rho_2\}$	$\{\rho_0, \mu_2\}$	$\{\rho_0, \mu_1\}$	$\{\rho_0, \mu_3\}$	$\{\rho_0\}$

Table 1. $\gamma : S_3 \to Aut(L(S_3))$

Let $S = \mathcal{M}^0(S_3; 1, 2; P)$ where P is the 2×1 matrix, having both entries equal to ρ_0. Consider $R_1 : L(S_3) \to L(S_3)$ and $Q_1, Q_2 : L(S_3) \to L(S_3)$.

$$R_1 = \begin{pmatrix} S_3 & \{\rho_0, \rho_1, \rho_2\} & \{\rho_0, \mu_1\} & \{\rho_0, \mu_2\} & \{\rho_0, \mu_3\} & \{\rho_0\} \\ S_3 & \{\rho_0, \rho_1, \rho_2\} & \{\rho_0, \mu_2\} & \{\rho_0, \mu_3\} & \{\rho_0, \mu_1\} & \{\rho_0\} \end{pmatrix}$$

$$Q_1 = Q_2 = \begin{pmatrix} S_3 & \{\rho_0, \rho_1, \rho_2\} & \{\rho_0, \mu_1\} & \{\rho_0, \mu_2\} & \{\rho_0, \mu_3\} & \{\rho_0\} \\ S_3 & \{\rho_0, \rho_1, \rho_2\} & \{\rho_0, \mu_3\} & \{\rho_0, \mu_1\} & \{\rho_0, \mu_2\} & \{\rho_0\} \end{pmatrix}$$

Clearly $Q_1R_1 = Q_2R_1 = I_{L(S_3)} = \gamma(\rho_0)$. Define $\Gamma : S \to End(L(S_3))$ such that $\Gamma(s) = R_i\gamma(g)Q_\mu$ for each $s = (g, i, \lambda) \in S$.

$\Gamma(\rho_0, 1, 1) = R_1\gamma(\rho_0)Q_1 = R_1\gamma(\rho_0)Q_2 = \Gamma(\rho_0, 1, 2)$

$$= \begin{pmatrix} S_3 & \{\rho_0, \rho_1, \rho_2\} & \{\rho_0, \mu_1\} & \{\rho_0, \mu_2\} & \{\rho_0, \mu_3\} & \{\rho_0\} \\ S_3 & \{\rho_0, \rho_1, \rho_2\} & \{\rho_0, \mu_1\} & \{\rho_0, \mu_2\} & \{\rho_0, \mu_3\} & \{\rho_0\} \end{pmatrix}$$

$\Gamma(\rho_1, 1, 1) = R_1\gamma(\rho_1)Q_1 = R_1\gamma(\rho_1)Q_2 = \Gamma(\rho_1, 1, 2)$

$$= \begin{pmatrix} S_3 & \{\rho_0, \rho_1, \rho_2\} & \{\rho_0, \mu_1\} & \{\rho_0, \mu_2\} & \{\rho_0, \mu_3\} & \{\rho_0\} \\ S_3 & \{\rho_0, \rho_1, \rho_2\} & \{\rho_0, \mu_3\} & \{\rho_0, \mu_1\} & \{\rho_0, \mu_2\} & \{\rho_0\} \end{pmatrix}$$

$\Gamma(\rho_2, 1, 1) = R_1\gamma(\rho_2)Q_1 = R_1\gamma(\rho_2)Q_2 = \Gamma(\rho_2, 1, 2)$

$$= \begin{pmatrix} S_3 & \{\rho_0, \rho_1, \rho_2\} & \{\rho_0, \mu_1\} & \{\rho_0, \mu_2\} & \{\rho_0, \mu_3\} & \{\rho_0\} \\ S_3 & \{\rho_0, \rho_1, \rho_2\} & \{\rho_0, \mu_2\} & \{\rho_0, \mu_3\} & \{\rho_0, \mu_1\} & \{\rho_0\} \end{pmatrix}$$

$\Gamma(\mu_1, 1, 1) = R_1\gamma(\mu_1)Q_1 = R_1\gamma(\mu_1)Q_2 = \Gamma(\mu_1, 1, 2)$

$$= \begin{pmatrix} S_3 & \{\rho_0, \rho_1, \rho_2\} & \{\rho_0, \mu_1\} & \{\rho_0, \mu_2\} & \{\rho_0, \mu_3\} & \{\rho_0\} \\ S_3 & \{\rho_0, \rho_1, \rho_2\} & \{\rho_0, \mu_3\} & \{\rho_0, \mu_2\} & \{\rho_0, \mu_1\} & \{\rho_0\} \end{pmatrix}$$

$\Gamma(\mu_2, 1, 1) = R_1\gamma(\mu_2)Q_1 = R_1\gamma(\mu_2)Q_2 = \Gamma(\mu_2, 1, 2)$

$$= \begin{pmatrix} S_3 & \{\rho_0, \rho_1, \rho_2\} & \{\rho_0, \mu_1\} & \{\rho_0, \mu_2\} & \{\rho_0, \mu_3\} & \{\rho_0\} \\ S_3 & \{\rho_0, \rho_1, \rho_2\} & \{\rho_0, \mu_2\} & \{\rho_0, \mu_1\} & \{\rho_0, \mu_3\} & \{\rho_0\} \end{pmatrix}$$

$\Gamma(\mu_3, 1, 1) = R_1\gamma(\mu_3)Q_1 = R_1\gamma(\mu_3)Q_2 = \Gamma(\mu_3, 1, 2)$

$$= \begin{pmatrix} S_3 & \{\rho_0, \rho_1, \rho_2\} & \{\rho_0, \mu_1\} & \{\rho_0, \mu_2\} & \{\rho_0, \mu_3\} & \{\rho_0\} \\ S_3 & \{\rho_0, \rho_1, \rho_2\} & \{\rho_0, \mu_1\} & \{\rho_0, \mu_3\} & \{\rho_0, \mu_2\} & \{\rho_0\} \end{pmatrix}$$

It is easy to verify that Γ is a representation of S by lattice morphisms on $L(S_3)$. Consider $H_{11} = \{(g, 1, 1) \mid g \in S_3\}$ and $\gamma_{11} : G \to Aut(L(S_3))$ be the representation of G induced by the restriction of Γ into H_{11}. Then $\eta = R_1\gamma(\rho_0) = R_1$ is a lattice isomorphism. Further, η is an S_3-lattice isomorphism. ie., $\eta(gH) = g\eta(H)$ for all $g \in S_3$ and $H \in L(S_3)$. For consider,

$\gamma_{11}(\rho_0)\eta = \Gamma(\rho_0, 1, 1)\eta$

$$= \begin{pmatrix} S_3 & \{\rho_0, \rho_1, \rho_2\} & \{\rho_0, \mu_1\} & \{\rho_0, \mu_2\} & \{\rho_0, \mu_3\} & \{\rho_0\} \\ S_3 & \{\rho_0, \rho_1, \rho_2\} & \{\rho_0, \mu_2\} & \{\rho_0, \mu_3\} & \{\rho_0, \mu_1\} & \{\rho_0\} \end{pmatrix} = \eta\gamma(\rho_0).$$

Similarly, $\gamma_{11}(g)\eta = \eta\gamma(g)$ for all $g \in S_3$ and so $\eta : L(S_3) \to L(S_3)$ is an isomorphism between the two S_3-lattice structures on $L(S_3)$. So the theorem is verified for this example.

3.1 Action of union of band of groups on lattices

Let S be a semigroup, admitting a decomposition $S = \bigcup_{\alpha\in\Omega} S_\alpha$ by a family $\{S_\alpha \mid \alpha \in \Omega\}$ of disjoint subsemigroups. If, for each α, β in the indexing set Ω there exists $\gamma \in \Omega$ such that $S_\alpha S_\beta \subseteq S_\gamma$, then Ω is a band with respect to a product $\alpha\beta = \gamma$ whenever $S_\alpha S_\beta \subseteq S_\gamma$. S is called union of band Ω of semigroups S_α or simply, a band of semigroups. If each S_α is a group, $S = \bigcup_{\alpha\in\Omega} S_\alpha$ is called a band of groups. If, in addition, Ω is commutative, the band of groups is called a Clifford semigroup.

Remark 3.4. Let $S = \bigcup_{\alpha\in\Omega} G_\alpha$ be a union of band Ω of groups. By Lemma 3.1, each S-lattice gives G_α-lattices for each α.

The converse of the above remark is not valid. That is, the action of G_αs on a lattice need not always generate an action of S. See the following example.

Example 3.5. Let $S_3 = \{\rho_0, \rho_1, \rho_2, \mu_1, \mu_2, \mu_3\}$ be the symmetric group on 3 letters where ρ's are rotations and μ's stand for reflections. Consider $C_3 = \{e, a, a^2\}$, the cyclic group of order 3, generated by a. Since C_3 is isomorphic with the alternating group A_3, there is a homomorphism $\phi : C_3 \to S_3$ such that, $\phi(e) = \rho_0$, $\phi(a) = \rho_1$ and $\phi(a^2) = \rho_2$.

$S = S_3 \cup C_3$ is a semigroup with respect to the following binary operation.

$$s \circ t = \begin{cases} st & \text{if } s, t \in S_3 \text{ or } s, t \in C_3; \\ s\phi(t) & \text{if } s \in S_3 \text{ and } t \in C_3; \\ \phi(s)t & \text{if } s \in C_3 \text{ and } t \in s_3. \end{cases}$$

$\circ$	S_3	C_3
S_3	S_3	S_3
C_3	S_3	C_3

Table 2. The multiplication table of S

S is Clifford semigroup. Consider the lattice $L(S_3)$ of all subgroups of S_3. Let $*_1$ be the trivial action of S_3 on $L(S_3)$, that is $g *_1 H = H$ for all $g \in G$ and $H \in L(S_3)$. Define an action of C_3 on $L(S_3)$ using the homomorphism ϕ as, $g *_2 H = \phi(g)H\phi(g)^{-1}$ for $g \in C_3$ and $H \in L(S_3)$. So, $L(S_3)$ is an S_3-lattice and a C_3-lattice simultaneously.

If we define an action of S on $L(S_3)$, using $*_1$ and $*_2$ as,

$$s * H = \begin{cases} s *_1 H & \text{if } s \in S_3; \\ s *_2 H & \text{if } s \in C_3. \end{cases}$$

$*$ fails to satisfy the property i of Definition 2.1. For, consider $\mu_1 \in S_3$, $a \in C_3$ and $\{\rho_0, \mu_2\} \leq S_3$.

$$\mu_1 * (a * \{\rho_0, \mu_2\}) = \mu_1 *_1 (a *_2 \{\rho_0, \mu_2\}) = \mu_1 *_1 \{\rho_0, \mu_1\} = \{\rho_0, \mu_1\}$$

$$(\mu_1 \circ a) * \{\rho_0, \mu_2\} = \mu_3 *_1 \{\rho_0, \mu_2\} = \{\rho_0, \mu_2\}$$

So

$$\mu_1 * (a * \{\rho_0, \mu_2\}) \neq (\mu_1 \circ a) * \{\rho_0, \mu_2\}$$

and $*$ is not an action of S on $L(S_3)$.

However, the converse of Remark 3.4 holds under certain conditions, which is proved as the following theorem.

Theorem 3.6. *Let $S = \bigcup_{\alpha \in \Omega} G_\alpha$ be a band of groups and L be G_α-lattice for all $\alpha \in \Omega$, with the action $g *_\alpha m \in L$ for $g \in G_\alpha$ and $m \in L$. If,*

$$s *_\alpha (t *_\beta m) = (st) *_\gamma m \tag{3.1}$$

for all $s \in G_\alpha$, $t \in G_\beta$ with $G_\alpha G_\beta \subseteq G_\gamma$ and $m \in L$, $*_\alpha$'s together give an action of S on L.

Proof. Let $S = \bigcup_{\alpha \in \Omega} G_\alpha$ be a band of groups and L be G_α-lattice for all α such that $s *_\alpha (t *_\beta m) = (st) *_\gamma m$. For $s \in S$ and $m \in L$, define $s * m = s *_\alpha m$ whenever $s \in G_\alpha$. Then for $s \in G_\alpha$, $t \in G_\beta$ such that $G_\alpha G_\beta \subseteq G_\gamma$, we have,

(i) $s * (t * m) = s *_\alpha (t *_\beta m) = st *_\gamma m = st * m$

(ii) $m \leq n$ implies $s *_\alpha m \leq s *_\alpha n$, which means $s * m \leq s * n$

(iii) $s * (m \vee n) = s *_\alpha (m \vee n) = s *_\alpha m \vee s *_\alpha n = s * m \vee s * n$

(iv) $s * (m \wedge n) = s *_\alpha (m \wedge n) = s *_\alpha m \wedge s *_\alpha n = s * m \wedge s * n$

so $*$ is an action of S on L and L is an S-lattice. □

An illustration of the theorem is the following example.

Example 3.7. Consider the Clifford semigroup $S = S_3 \cup C_3$ in Example 3.5. The lattice $L(S_3)$ of all subgroups of S_3 is an S_3-lattice under conjugation [6], given by $g *_3 H = gHg^{-1}$ for $g \in S_3$ and $H \in L(S_3)$. Also, $L(S_3)$ is a C_3-lattice with respect to the action $*_2$ in Example 3.5. Define

$$s * H = \begin{cases} s *_3 H & \text{if } s \in S_3; \\ s *_2 H & \text{if } s \in C_3. \end{cases}$$

For $s, t \in S$, we have the following four cases.

(i) If $s, t \in S_3$,
$s *_3 (t *_3 H) = s(tHt^{-1})s^{-1} = (st)H(st)^{-1} = (s \circ t) *_3 H.$

(ii) If $s, \in S_3$ and $t \in C_3$,
$s *_3 (t *_2 H) = s(\phi(t)H\phi(t)^{-1})s^{-1} = (s \circ t)H(s \circ t)^{-1} = (s \circ t) *_3 H.$

(iii) If $s \in C_3$ and $t \in S_3$,
$s *_2 (t *_3 H) = \phi(s)(tHt^{-1})\phi(s)^{-1} = (s \circ t)H(s \circ t)^{-1} = (s \circ t) *_3 H.$

(iv) If $s, t \in C_3$,
$s *_2 (t *_2 H) = \phi(s)(\phi(t)H\phi(t)^{-1})\phi(s)^{-1} = \phi(st)H\phi(st)^{-1} = st *_2 H = (s \circ t) *_2 H.$

Hence the actions $*_3$ and $*_2$ satisfies Equation: (3.1) of Theorem 3.6. So, they together give an action of S on $L(S_3)$, so that $L(S_3)$ is an S-lattice.

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Author information

Alanka Thomas,
Department of Mathematics
Cochin University of Science and Technology(CUSAT)
Kochi, Kerala, 682022, India.
E-mail: *alankathomas1@gmail.com*

P. G. Romeo,
Department of Mathematics
Cochin University of Science and Technology(CUSAT)
Kochi, Kerala, 682022, India.
E-mail: *romeo_parackal@yahoo.com*

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