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CYCLIC CONTRACTIONS AND SOME FIXED POINT AND COINCIDENCE POINT RESULTS IN DISLOCATED QUASI RECTANGULAR b-METRIC SPACES

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Abstract In this study, we have pointed out a fatal error in the proofs of the main results of Golhare and Aage [Int. J. Nonlinear Anal. Appl., 12(2021), No. 2, 173-191] to conclude that the paper is wrong. Next in context of the paper we have presented the most generalized form of the concerned results and investigated some coincidence point results as applications of our main results. Our results extend a number of existing results. Moreover, we provide some examples to justify that the generalizations are proper.

1 Introduction

Banach contraction principle [5] is one of the most impressing results in fixed point theory. Because of its simplicity and usefulness it has become a popular tool for solving various problems in nonlinear analysis. Several authors successfully extended this celebrated result in diverse ways (see [2, 7, 8, 18, 19, 21, 22] and references therein). Hitzler and Seda [14] introduced the concept of dislocated metric spaces as a generalization of partial metric spaces and obtained an important characterization of the Banach contraction principle. After that, Amini-Harandi [13] initiated the notion of metric-like spaces. Karapinar et al.[16] noticed that the notions of dislocated metric spaces and metric-like spaces are exactly the same. In 2015, R. George et al.[9] introduced the concept of rectangular b-metric spaces and proved analogue of Banach contraction principle and Kannan type fixed point theorem in rectangular b-metric spaces. Golhare and Aage [11] introduced dislocated quasi rectangular b-metric spaces as a generalization of rectangular b-metric spaces and dislocated quasi metric spaces. Very recently, Golhare et al.[10] proved some fixed point theorems of cyclic contraction mappings in dislocated quasi rectangular b-metric spaces. In this article, we have pointed out a fatal error in the proofs of their main results [10] and presented the most generalized form of the concerned results.

2 Some Basic Concepts

In this section, we recall some basic definitions, notations and crucial results in dislocated quasi rectangular b-metric spaces. Throughout this article, $\mathbb{N}$, $\mathbb{R}$, $\mathbb{R}^+$ denote the set of natural numbers, the set of real numbers and the set of all nonnegative real numbers, respectively.

Definition 2.1. [14] Let X be a nonempty set. A function $\sigma : X \times X \to [0, \infty)$ is said to be a dislocated metric (or a metric-like) on X if for any $x,\ y,\ z \in X$, the following conditions hold:

(σ_1) $\sigma(x, y) = 0 \Longrightarrow x = y$;

(σ_2) $\sigma(x, y) = \sigma(y, x)$;

(σ_3) $\sigma(x, y) \leq \sigma(x, z) + \sigma(z, y)$.

The pair (X, σ) is then called a dislocated metric (or metric-like) space.

It is valuable to note that a partial metric [17] is also a dislocated metric but the converse is not true, in general. A trivial example of a dislocated metric is given by $\sigma(x, y) = \max\{x, y\}$ for all $x, y \geq 0$.

The following example illustrates the above fact.

Example 2.2. [3] Let $X = \{1, 2, 3\}$ and consider the dislocated metric $\sigma : X \times X \to [0, \infty)$ given by

$$\sigma(1,1) = 0,\ \sigma(2,2) = 1,\ \sigma(3,3) = \frac{2}{3},$$

$$\sigma(1,2) = \sigma(2,1) = \frac{9}{10},\ \sigma(2,3) = \sigma(3,2) = \frac{4}{5},$$

$$\sigma(1,3) = \sigma(3,1) = \frac{7}{10}.$$

Since $\sigma(2,2) \neq 0$, σ is not a metric and since $\sigma(2,2) > \sigma(1,2)$, σ is not a partial metric.

Definition 2.3. [4] Let X be a nonempty set and $s \geq 1$ be a given real number. A function $d : X \times X \to \mathbb{R}^+$ is said to be a b-metric on X if the following conditions hold:

(i) $d(x, y) = 0$ if and only if $x = y$;

(ii) $d(x, y) = d(y, x)$ for all $x, y \in X$;

(iii) $d(x, y) \leq s\,(d(x, z) + d(z, y))$ for all $x, y, z \in X$.

The pair (X, d) is called a b-metric space.

Definition 2.4. [20] Let X be a nonempty set. Let $d : X \times X \to [0, \infty)$ be a mapping and $s \geq 1$ be a constant such that

(i) $d(x, y) = 0 = d(y, x)$ if and only if $x = y$;

(ii) $d(x, y) \leq s[d(x, z) + d(z, y)]$ for all $x, y, z \in X$.

Then d is called a quasi b-metric on X and (X, d) is called a quasi b-metric space with coefficient s.

Definition 2.5. [6] Let X be a nonempty set and $d : X \times X \to [0, \infty)$ be a function such that

(i) $d(x, y) = 0$ if and only if $x = y$;

(ii) $d(x, y) = d(y, x)$ for all $x, y \in X$;

(iii) $d(x, y) \leq d(x, u) + d(u, v) + d(v, y)$ for all $x, y \in X$ and all distinct points $u, v \in X \setminus \{x, y\}$.

Then d is called a rectangular metric on X and (X, d) is called a rectangular metric space.

Definition 2.6. [10, 11] Let X be a nonempty set and $d : X \times X \to [0, \infty)$ be a function such that

(i) $d(x, y) = 0 = d(y, x) \Longrightarrow x = y$;

(ii) there exists a real number $s \geq 1$ such that

$$d(x, y) \leq s[d(x, u) + d(u, v) + d(v, y)]$$

for all $x, y \in X$ and all points $u, v \in X \setminus \{x, y\}$.

Then d is called a dislocated quasi or dq-rectangular b-metric on X and (X, d) is called a dislocated quasi or dq-rectangular b-metric space with coefficient s.

Example 2.7. Let us take $X = [0, \infty)$ and let $d : X \times X \to [0, \infty)$ be given by

$$d(x, y) = |x - y|^2 + \frac{x}{9} + \frac{y}{10}, \ \forall x, y \in X.$$

Then, $d(x, y) = 0 = d(y, x) \implies x = y$ and for all $u, v \in X \setminus \{x, y\}$, we have

$$\begin{aligned} d(x,y) &= |x-y|^2 + \frac{x}{9} + \frac{y}{10} \\ &= |x-u+u-v+v-y|^2 + \frac{x}{9} + \frac{y}{10} \\ &\leq 3[|x-u|^2 + |u-v|^2 + |v-y|^2] + \frac{x}{9} + \frac{y}{10} \\ &\leq 3[d(x,u) + d(u,v) + d(v,y)]. \end{aligned}$$

Therefore, (X, d) is a dislocated quasi rectangular b-metric space with coefficient $s = 3$. Note that $d(1, 1) = \frac{19}{90} \neq 0$ and hence d is not a metric on X.

In a dislocated quasi rectangular b-metric space (X, d), we define an open ball $B_r(x)$ for $x \in X$ and $r > 0$ as follows:

$$B_r(x) = \{y \in X : \max\{| \ d(x, y) - d(x, x) \ |, | \ d(y, x) - d(x, x) \ |\} < r\}.$$

Definition 2.8. [10] Let (X, d) be a dislocated quasi rectangular b-metric space with coefficient s. A subset $U \subseteq X$ is said to be open if for every $x \in U$ there exists $r > 0$ such that $B_r(x) \subseteq U$.
A subset $V \subseteq X$ is said to be closed if $X \setminus V$ is open.
The family of all open subsets of X will be denoted by τ_{d_q}.

Theorem 2.9. *τ_{d_q} defines a topology on (X, d).*

Remark 2.10. Let (X, d) be a dislocated quasi rectangular b-metric space, (x_n) be a sequence in X and $x \in X$. Then (x_n) converges to x with respect to(w.r.t.) τ_{d_q} if $\lim_{n\to\infty} d(x_n, x) = \lim_{n\to\infty} d(x, x_n) = d(x, x)$.

Suppose that $\lim_{n\to\infty} d(x_n, x) = \lim_{n\to\infty} d(x, x_n) = d(x, x)$. We shall show that $x_n \to x$ w.r.t. τ_{d_q}. Let $U \in \tau_{d_q}$ and $x \in U$. Then there exists $\epsilon > 0$ such that $B_\epsilon(x) \subseteq U$. By hypothesis, there exists $n_0 \in \mathbb{N}$ such that $| \ d(x_n, x) - d(x, x) \ | < \epsilon$ and $| \ d(x, x_n) - d(x, x) \ | < \epsilon$ for all $n \geq n_0$. This ensures that $x_n \in B_\epsilon(x)$ for all $n \geq n_0$ and hence $x_n \in U$ for all $n \geq n_0$. Therefore, (x_n) converges to x w.r.t. τ_{d_q} on X.

Definition 2.11. [11] Let (X, d) be a dislocated quasi rectangular b-metric space with coefficient s and let (x_n) be a sequence in X. Then

(i) (x_n) converges to a point $x \in X$ if $\lim_{n\to\infty} d(x_n, x) = \lim_{n\to\infty} d(x, x_n) = d(x, x)$. This will be denoted as $\lim_{n\to\infty} x_n = x$ or $x_n \to x (n \to \infty)$.

(ii) (x_n) is called a Cauchy sequence if $\lim_{n\to\infty} d(x_{n+i}, x_n)$ and $\lim_{n\to\infty} d(x_n, x_{n+i})$ exist and are finite for all $i \in \mathbb{N}$.

(iii) (X, d) is said to be complete if every Cauchy sequence in X is convergent in X.

Lemma 2.12. *[10] A subset A of a dislocated quasi rectangular b-metric space (X, d) is closed if and only if the following statement holds:*
If (x_n) is a sequence of points in A converging to some point $x \in X$ implies that $x \in A$.

Definition 2.13. Let (X, d) be a dislocated quasi rectangular b-metric space with coefficient s and let (x_n) be a sequence in X. Then,

(i) (x_n) is called a 0-Cauchy sequence if

$$\lim_{n\to\infty} d(x_n, x_{n+p}) = \lim_{n\to\infty} d(x_{n+p}, x_n) = 0, \ \forall \ p \in \mathbb{N}.$$

(ii) (X, d) is said to be 0-complete if every 0-Cauchy sequence in X converges to a point $x \in X$ such that $d(x, x) = 0$.

It is to be noted that if a dislocated quasi rectangular b-metric space (X, d) is complete, then it is 0-complete. The converse assertion is not true, in general. The following example supports the above remark.

Example 2.14. Let $X = [0, \infty)$ and $d : X \times X \to [0, \infty)$ be defined by

$$d(x, y) = |x - y|^2 + \frac{x}{9} + \frac{y}{10}, \ \forall x, \, y \in X.$$

Then (X, d) is a dislocated quasi rectangular b-metric space with coefficient $s = 3$. Let $Y = [0, \infty) \cap \mathbb{Q}$ and $d_Y : Y \times Y \to [0, \infty)$ be defined by

$$d_Y(x, y) = d(x, y), \ \forall x, \, y \in Y.$$

Then (Y, d_Y) is a dislocated quasi rectangular b-metric subspace of (X, d). We now show that (Y, d_Y) is 0-complete but it is not complete.
Let (x_n) be a 0-Cauchy sequence in (Y, d_Y). Then,

$$\lim_{n \to \infty} d_Y(x_n, x_{n+p}) = \lim_{n \to \infty} d_Y(x_{n+p}, x_n) = 0, \ \forall p \in \mathbb{N}.$$

That is, $\lim_{n \to \infty} [|x_{n+p} - x_n|^2 + \frac{x_{n+p}}{9} + \frac{x_n}{10}] = 0$.
Now,

$$\begin{aligned} 0 \leq |x_{n+p} - x_n| &\leq |x_{n+p}| + |x_n| \\ &\leq 10|x_{n+p}| + 9|x_n| \\ &\leq 90[|x_{n+p} - x_n|^2 + \frac{x_{n+p}}{9} + \frac{x_n}{10}]. \end{aligned}$$

Taking limit as $n \to \infty$, we have

$$\lim_{n \to \infty} |x_{n+p} - x_n| = 0, \ \forall p \in \mathbb{N}.$$

Therefore, (x_n) is a Cauchy sequence in $[0, \infty)$ with usual metric and hence converges in $[0, \infty)$. So, there exists $x \in [0, \infty)$ such that $| \ x_n - x \ | \to 0$ as $n \to \infty$.

Now,

$$\begin{aligned} |d(x_n, x) - d(x, x)| &= ||x_n - x|^2 + \frac{x_n}{9} + \frac{x}{10} - \frac{x}{9} - \frac{x}{10}| \\ &\leq |x_n - x|^2 + | \ \frac{x_n - x}{9}| \\ &\to 0, \ as \ n \to \infty. \end{aligned}$$

This gives that,

$$\lim_{n \to \infty} d(x_n, x) = d(x, x).$$

Similarly, we have

$$\lim_{n \to \infty} d(x, x_n) = d(x, x).$$

Again, for all $p \in \mathbb{N}$

$$\begin{aligned} |d(x_n, x_{n+p}) - d(x_n, x)| &= ||x_n - x_{n+p}|^2 + \frac{x_n}{9} + \frac{x_{n+p}}{10} - |x_n - x|^2 - \frac{x_n}{9} - \frac{x}{10}| \\ &\leq |x_n - x_{n+p}|^2 + |x_n - x|^2 + | \ \frac{x_{n+p} - x}{10}| \\ &\to 0, \ as \ n \to \infty. \end{aligned}$$

This implies that,

$$\lim_{n\to\infty} d(x_n, x_{n+p}) = \lim_{n\to\infty} d(x_n, x),\ \forall p \in \mathbb{N}.$$

Therefore,

$$\lim_{n\to\infty} d_Y(x_n, x_{n+p}) = \lim_{n\to\infty} d(x_n, x) = d(x, x).$$

So, it follows that $d(x,x) = \frac{19x}{90} = 0$ which ensures that $x = 0$. Hence $x \in Y$. Thus, we obtain that

$$\lim_{n\to\infty} d_Y(x_n, x) = \lim_{n\to\infty} d_Y(x, x_n) = d_Y(x, x) = 0.$$

This proves that (Y, d_Y) is 0-complete.

We now show that (Y, d_Y) is not complete.

Let us consider the sequence $x_n = (1 + \frac{1}{n})^n$ in (Y, d_Y). Then, $\mid x_n - e \mid \to 0$ as $n \to \infty$. Obviously, $e \in X$ but $e \notin Y$, as e is not a rational number.

For all $p \in \mathbb{N}$, we have

$$\begin{aligned} \mid d(x_n, x_{n+p}) - d(e,e) \mid &= \mid\mid x_n - x_{n+p} \mid^2 + \frac{x_n}{9} + \frac{x_{n+p}}{10} - \frac{e}{9} - \frac{e}{10} \mid \\ &\le \mid x_n - x_{n+p} \mid^2 + \mid \frac{x_n - e}{9} \mid + \mid \frac{x_{n+p} - e}{10} \mid \\ &\to 0,\ as\ n \to \infty. \end{aligned}$$

Thus, $\lim_{n\to\infty} d(x_n, x_{n+p}) = d(e,e) = \frac{19e}{90},\ \forall p \in \mathbb{N}$.
This shows that, $\lim_{n\to\infty} d(x_n, x_{n+p})$ exists and is finite.
Similarly, $\lim_{n\to\infty} d(x_{n+p}, x_n) = d(e,e) = \frac{19e}{90}$ exists and is finite.
Therefore, (x_n) is a Cauchy sequence in (Y, d_Y). It is easy to check that

$$\lim_{n\to\infty} d(x_n, e) = \lim_{n\to\infty} d(e, x_n) = d(e, e).$$

Thus, $(x_n) \subseteq (Y, d_Y)$ converges to $e \in X$ but $e \notin Y$. Therefore, (Y, d_Y) is not complete.
Moreover, the sequence (x_n) with $x_n = (1 + \frac{1}{n})^n$ for each $n \in \mathbb{N}$ is a Cauchy sequence in (Y, d_Y), but it is not a 0-Cauchy sequence in (Y, d_Y).

Theorem 2.15. *Every closed subset of a* 0*-complete dislocated quasi rectangular* b*-metric space* (X, d) *is* 0*-complete.*

Proof. Let Y be a closed subset of (X, d). Let (y_n) be a 0-Cauchy sequence in (Y, d_Y), where $d_Y : Y \times Y \to \mathbb{R}^+$ is defined by $d_Y(u, v) = d(u, v)$ for all $u,\ v \in Y$. Then (y_n) is also a 0-Cauchy sequence in (X, d). As (X, d) is 0-complete, (y_n) converges to a point $x \in X$. By applying Lemma 2.12, it follows that $x \in Y$. Thus, (Y, d_Y) becomes a 0-complete dislocated quasi rectangular b-metric space. □

3 Background and Motivations

Let X be a nonempty set and $A,\ B$ be nonempty subsets of X. A mapping $T : A \cup B \to A \cup B$ is called cyclic if $T(A) \subseteq B$ and $T(B) \subseteq A$. In 2021, Golhare et al.[10] proved the following fixed point theorems of cyclic contraction mappings in dislocated quasi rectangular b-metric spaces.

Definition 3.1. [10] Let A and B be nonempty subsets of a dislocated quasi rectangular b-metric space (X, d) with coefficient s, then a cyclic mapping $T : A \cup B \to A \cup B$ is called a dq-rectangular b-cyclic Banach mapping if there exists $\alpha \in [0, \frac{1}{s})$ such that

$$d(Tx, Ty) \le \alpha d(x, y)$$

for all $x \in A,\ y \in B$.

Theorem 3.2. *[10] Let (X, d) be a complete dislocated quasi rectangular b-metric space with coefficient $s > 1$ and A, B be two nonempty closed subsets of X. If $T : A \cup B \to A \cup B$ is a dq-rectangular b-cyclic Banach mapping then T has a unique fixed point in $A \cap B$.*

Definition 3.3. [10] Let A and B be nonempty subsets of a dislocated quasi rectangular b-metric space (X, d) with coefficient s, then a cyclic mapping $T : A \cup B \to A \cup B$ is called a dq-rectangular b-cyclic Kannan mapping if there exists $\gamma \in [0, \frac{1}{2s}]$ such that

$$d(Tx, Ty) \leq \gamma[d(x, Tx) + d(y, Ty)]$$

for all $x \in A$, $y \in B$.

Theorem 3.4. *[10] Let (X, d) be a complete dislocated quasi rectangular b-metric space with coefficient $s > 1$ and A, B be two nonempty closed subsets of X. If $T : A \cup B \to A \cup B$ is a dq-rectangular b-cyclic Kannan mapping then T has a unique fixed point in $A \cap B$.*

Remark 3.5. Though Theorem 3.2 seems to be a significant improvement of Banach contraction theorem in the setting of dislocated quasi rectangular b-metric spaces, but a close look will reveal that there is a serious error in the proof of Theorem 3.2. In Theorem 3.2, the authors assumed that

$$d(Tx, Ty) \leq \alpha d(x, y) \tag{3.1}$$

for all $x \in A,\ y \in B$, where d is a dislocated quasi rectangular b-metric on X.

In the proof of Theorem 3.2 (see p. 177 of [10]), a sequence (x_n) in X has been constructed in such a way that $x_n = Tx_{n-1}$ for all $n \in \mathbb{N}$, $x_n \in A$ if n is even and $x_n \in B$ if n is odd, where A, B are nonempty closed subsets of X. From condition (3.1), they obtained

$$d(x_{n-1}, x_n) = d(Tx_{n-2}, Tx_{n-1}) \leq \alpha d(x_{n-2}, x_{n-1}) \tag{3.2}$$

for all $n \in \mathbb{N}$. Moreover, the inequality (3.2) has been used repeatedly to obtain the following inequality

$$d(x_{n-1}, x_n) \leq \alpha d(x_{n-2}, x_{n-1}) \leq \cdots \leq \alpha^{n-1} d(x_0, x_1) \tag{3.3}$$

for all $n \in \mathbb{N}$. But there is a lacuna in the computations as if we consider n be an even natural number then we can not apply condition (3.1) to obtain inequality (3.2). In fact, for even natural number n, $x_{n-1} \in B$ and $x_n \in A$, so inequality (3.2) can not be obtained from condition (3.1). Consequently, inequality (3.3) does not hold true.

Also, in the proof of Theorem 3.4 (see p. 181 of [10]), there is a same lacuna in the computations.

We close this section by presenting two examples to ensure that Theorems 3.2 and 3.4 are not correct.

Example 3.6. Let $X = \mathbb{N}$ and $d : X \times X \to [0, \infty)$ be given by

$$d(x, y) = \begin{cases} a, & \text{if } x \text{ is odd and } y \text{ is even}, \\ 4a, & \text{if } x \text{ is even but not divisible by 4 and } y \text{ is odd}, \\ \frac{13a}{4}, & \text{if } x \text{ is divisible by 4 and } y \text{ is odd}, \\ \frac{a}{4}, & \text{otherwise}, \end{cases}$$

where $a > 0$ is a constant. Then (X, d) is a complete dislocated quasi rectangular b-metric space with coefficient $s = \frac{16}{15} > 1$. But it is not a dislocated quasi rectangular metric space. Because, for $x = 6$, $y = 3$, $u = v = 8$, we have

$$\begin{aligned} d(6, 3) &= 4a \\ &> [d(6, 8) + d(8, 8) + d(8, 3)] \\ &= \frac{a}{4} + \frac{a}{4} + \frac{13a}{4} = \frac{15a}{4}. \end{aligned}$$

We take $A = \{x \in \mathbb{N} : x \ is \ even\}$ and $B = \{x \in \mathbb{N} : x \ is \ odd\}$. Then, $X = A \cup B = \mathbb{N}$ and A, B are nonempty closed subsets of (X, d). Let $T : X \to X$ be defined as follows:

$$Tx = \begin{cases} x+1, \text{ if } x \ is \ odd, \\ \\ x-1, \text{ if } x \ is \ even. \end{cases}$$

Obviously, $T(A) \subseteq B$ and $T(B) \subseteq A$, i.e., T is cyclic.

For $x \in A$ and $y \in B$, we have the following two cases:

$\mathbf{Case(1)}$: x is even but not divisible by 4 and y is odd. Then,

$$d(Tx, Ty) = d(x-1, y+1) = a < \frac{1}{3}.4a = \alpha d(x, y),$$

where $\alpha = \frac{1}{3} < \frac{15}{16}$.

$\mathbf{Case(2)}$: x is divisible by 4 and y is odd. Then,

$$d(Tx, Ty) = d(x-1, y+1) = a < \frac{1}{3}.\frac{13a}{4} = \alpha d(x, y).$$

Thus, $T : A \cup B \to A \cup B$ is a dq-rectangular b-cyclic Banach mapping. Therefore, all the conditions of Theorem 3.2 are satisfied. But we observe that T has no fixed point. This proves that Theorem 3.2 is wrong.

Remark 3.7. It is worth mentioning that $T : A \cup B \to A \cup B$ is not a dq-rectangular b-cyclic Banach mapping according to our Definition 4.1.

Example 3.8. Suppose that (X, d), A, B and T are all same as in Example 3.6. For $x \in A$ and $y \in B$, we have the following two cases:

$\mathbf{Case(1)}$: x is even but not divisible by 4 and y is odd. Then,

$$d(x, Tx) + d(y, Ty) = d(x, x-1) + d(y, y+1) = 4a + a = 5a$$

and

$$\begin{aligned} d(Tx, Ty) &= d(x-1, y+1) \\ &= a \\ &< \frac{6}{25}.5a \\ &= \gamma[d(x, Tx) + d(y, Ty)], \end{aligned}$$

where $\gamma = \frac{6}{25} < \frac{15}{32} = \frac{1}{2s}$.

$\mathbf{Case(2)}$: x is divisible by 4 and y is odd. Then,

$$d(x, Tx) + d(y, Ty) = d(x, x-1) + d(y, y+1) = \frac{13a}{4} + a = \frac{17a}{4}$$

and

$$\begin{aligned} d(Tx, Ty) &= d(x-1, y+1) \\ &= a \\ &< \frac{6}{25}.\frac{17a}{4} \\ &= \gamma[d(x, Tx) + d(y, Ty)]. \end{aligned}$$

Thus, $T : A \cup B \to A \cup B$ is a dq-rectangular b-cyclic Kannan mapping. Therefore, all the conditions of Theorem 3.4 are satisfied but T has no fixed point. This proves that Theorem 3.4 is wrong.

Remark 3.9. It is valuable to note that $T : A \cup B \to A \cup B$ is not a dq-rectangular b-cyclic Kannan mapping according to our Definition 4.2.

In this study, our motivation is to present the corrected version of the concerned results.

4 Main Results

We have already pointed out the errors of [10]. So, with the techniques and resources available at present, the only option is to present the corrected version of the concerned results. That's why we are going to define new dq-rectangular b-cyclic Banach mappings and dq-rectangular b-cyclic Kannan mappings in the following manner.

Definition 4.1. Let A and B be nonempty subsets of a dislocated quasi rectangular b-metric space (X, d) with coefficient s. A cyclic mapping $T : A \cup B \to A \cup B$ is called a dq-rectangular b-cyclic Banach mapping if there exists $\alpha \in [0, \frac{1}{s})$ such that

$$\max\{d(Tx, Ty), d(Ty, Tx), d(Tx, Tx), d(Ty, Ty)\} \leq \alpha \min\{d(x, y), d(y, x)\} \tag{4.1}$$

for all $x \in A$, $y \in B$.
If $A = B = X$, then $T : X \to X$ satisfying condition (4.1) is called a dq-rectangular b-Banach mapping.

Definition 4.2. Let A and B be nonempty subsets of a dislocated quasi rectangular b-metric space (X, d) with coefficient s. A cyclic mapping $T : A \cup B \to A \cup B$ is called a dq-rectangular b-cyclic Kannan mapping if there exists $\gamma \in [0, \frac{1}{1+s})$ such that

$$\max \begin{Bmatrix} d(Tx, Ty), d(Ty, Tx), \\ d(Tx, Tx), d(Ty, Ty) \end{Bmatrix} \leq \gamma \min \begin{Bmatrix} d(x, Tx) + d(y, Ty), \\ d(Tx, x) + d(Ty, y) \end{Bmatrix} \tag{4.2}$$

for all $x \in A$, $y \in B$.
If $A = B = X$, then $T : X \to X$ satisfying condition (4.2) is called a dq-rectangular b-Kannan mapping.

Theorem 4.3. *Let (X, d) be a 0-complete dislocated quasi rectangular b-metric space with coefficient $s \geq 1$ and A, B be two nonempty closed subsets of X. If $T : A \cup B \to A \cup B$ is a dq-rectangular b-cyclic Banach mapping, then T has a unique fixed point u in $A \cap B$ with $d(u, u) = 0$.*

Proof. Let us take an arbitrary point $x_0 \in A$. As $Tx_0 \in B$, there exists $x_1 \in B$ such that $x_1 = Tx_0$. Similarly, as $Tx_1 \in A$, there exists $x_2 \in A$ such that $x_2 = Tx_1$. So, we may construct a sequence (x_n) in X such that $x_n = Tx_{n-1}$ for all $n \in \mathbb{N}$. It is to be noted that $x_n \in A$ if n is even and $x_n \in B$ if n is odd. If $x_{n-1} = x_n$ for some $n \in \mathbb{N}$, then $x_{n-1} = Tx_{n-1}$ and hence x_{n-1} becomes a fixed point of T. Therefore, we assume that $x_{n-1} \neq x_n$ for all $n \in \mathbb{N}$.
From condition (4.1), we get

$$\begin{aligned} d(x_{2n-1}, x_{2n}) &= d(Tx_{2n-2}, Tx_{2n-1}) \\ &\leq \max \begin{Bmatrix} d(Tx_{2n-2}, Tx_{2n-1}), d(Tx_{2n-1}, Tx_{2n-2}), \\ d(Tx_{2n-2}, Tx_{2n-2}), d(Tx_{2n-1}, Tx_{2n-1}) \end{Bmatrix} \\ &\leq \alpha \min\{d(x_{2n-2}, x_{2n-1}), d(x_{2n-1}, x_{2n-2})\} \\ &\leq \alpha d(x_{2n-2}, x_{2n-1}). \end{aligned} \tag{4.3}$$

Again,

$$\begin{aligned} d(x_{2n}, x_{2n+1}) &= d(Tx_{2n-1}, Tx_{2n}) \\ &\leq \max \begin{Bmatrix} d(Tx_{2n}, Tx_{2n-1}), d(Tx_{2n-1}, Tx_{2n}), \\ d(Tx_{2n}, Tx_{2n}), d(Tx_{2n-1}, Tx_{2n-1}) \end{Bmatrix} \\ &\leq \alpha \min\{d(x_{2n}, x_{2n-1}), d(x_{2n-1}, x_{2n})\} \\ &\leq \alpha d(x_{2n-1}, x_{2n}). \end{aligned} \tag{4.4}$$

By repeated use of (4.3) and (4.4), we obtain that

$$d(x_{2n}, x_{2n+1}) \leq \alpha^{2n} d(x_0, x_1) \tag{4.5}$$

and

$$d(x_{2n-1}, x_{2n}) \leq \alpha^{2n-1} d(x_0, x_1). \tag{4.6}$$

Combining (4.5) and (4.6), we get

$$d(x_n, x_{n+1}) \leq \alpha^n d(x_0, x_1), \forall n \in \mathbb{N}. \tag{4.7}$$

Similarly,

$$d(x_{n+1}, x_n) \leq \alpha^n d(x_1, x_0), \forall n \in \mathbb{N}. \tag{4.8}$$

Moreover,

$$\begin{aligned} d(x_n, x_n) &= d(Tx_{n-1}, Tx_{n-1}) \\ &\leq \max \begin{Bmatrix} d(Tx_{n-1}, Tx_n), d(Tx_n, Tx_{n-1}), \\ d(Tx_{n-1}, Tx_{n-1}), d(Tx_n, Tx_n) \end{Bmatrix} \\ &\leq \alpha \min\{d(x_{n-1}, x_n), d(x_n, x_{n-1})\} \\ &\leq \alpha^n d(x_0, x_1), \quad \forall n \in \mathbb{N}. \end{aligned} \tag{4.9}$$

We claim that $x_m \neq x_n$ for all m, $n \in \mathbb{N}$ with $m \neq n$. If possible, suppose that $x_m = x_n$ for some m, $n \in \mathbb{N}$ with $m > n$. Let us put $m - n = p \in \mathbb{N}$. As $x_n \neq x_{n+1}$, it follows that at least one of $d(x_n, x_{n+1})$ and $d(x_{n+1}, x_n)$ is non-zero. We assume that $d(x_n, x_{n+1}) \neq 0$. Now,

$$\begin{aligned} d(x_n, x_{n+1}) = d(x_n, Tx_n) &= d(x_m, Tx_m) \\ &= d(x_m, x_{m+1}) \\ &= d(x_{n+p}, x_{n+p+1}) \\ &\leq \alpha^p \, d(x_n, x_{n+1}). \end{aligned}$$

This gives that $\alpha^p \geq 1$, a contradiction since $0 \leq \alpha < \frac{1}{s}$. Thus, our claim is justified.

We now show that (x_n) is a 0-Cauchy sequence in (X, d). That is, we have to show that

$$\lim_{n\to\infty} d(x_{n+p}, x_n) = 0 = \lim_{n\to\infty} d(x_n, x_{n+p}), \ \forall p \in \mathbb{N}.$$

First we show that $\lim_{n\to\infty} d(x_n, x_{n+p}) = 0, \ \forall p \in \mathbb{N}$.

Case – I : Suppose p is even, that is, $p = 2m$ for some $m(> 1) \in \mathbb{N}$. Then by using conditions (4.7) and (4.9), we get

$$\begin{aligned}
d(x_n, x_{n+p}) &\leq s[d(x_n, x_{n+1}) + d(x_{n+1}, x_{n+2}) + d(x_{n+2}, x_{n+2m})] \\
&\leq s[d(x_n, x_{n+1}) + d(x_{n+1}, x_{n+2})] + s^2[d(x_{n+2}, x_{n+3}) + d(x_{n+3}, x_{n+4})] \\
&\quad +s^3[d(x_{n+4}, x_{n+5}) + d(x_{n+5}, x_{n+6})] + \cdots \\
&\quad +s^{m-1}[d(x_{n-4+2m}, x_{n-3+2m}) + d(x_{n-3+2m}, x_{n-2+2m})] \\
&\quad +s^{m-1}d(x_{n-2+2m}, x_{n+2m}) \\
&\leq s\alpha^n[1 + s\alpha^2 + s^2\alpha^4 + \cdots]d(x_0, x_1) \\
&\quad +s\alpha^{n+1}[1 + s\alpha^2 + s^2\alpha^4 + \cdots]d(x_0, x_1) \\
&\quad +s^{m-1}d(x_{n-2+2m}, x_{n+2m}) \\
&\leq \frac{1+\alpha}{1-s\alpha^2}s\alpha^n d(x_0, x_1) + s^{m-1}.s[d(x_{n-2+2m}, x_{n-1+2m}) \\
&\quad +d(x_{n-1+2m}, x_{n-1+2m}) + d(x_{n-1+2m}, x_{n+2m})] \\
&\leq \frac{1+\alpha}{1-s\alpha^2}s\alpha^n d(x_0, x_1) + s^m[\alpha^{n-2+2m}d(x_0, x_1) \\
&\quad +\alpha^{n-1+2m}d(x_0, x_1) + \alpha^{n-1+2m}d(x_0, x_1)] \\
&= \frac{1+\alpha}{1-s\alpha^2}s\alpha^n d(x_0, x_1) \\
&\quad +(\alpha s)^m[(\alpha^{m+n-2} + \alpha^{m+n-1})d(x_0, x_1) + \alpha^{m+n-1}d(x_0, x_1)] \\
&< \frac{1+\alpha}{1-s\alpha^2}s\alpha^n d(x_0, x_1) \\
&\quad +[(\alpha^{m+n-2} + \alpha^{m+n-1})d(x_0, x_1) + \alpha^{m+n-1}d(x_0, x_1)].
\end{aligned}$$

Passing to the limit as $n \to \infty$, we have

$$\lim_{n\to\infty} d(x_n, x_{n+p}) = 0.$$

If $m = 1$, then

$$\begin{aligned}
d(x_n, x_{n+p}) = d(x_n, x_{n+2}) &\leq s[d(x_n, x_{n+1}) + d(x_{n+1}, x_{n+1}) + d(x_{n+1}, x_{n+2})] \\
&\leq s[\alpha^n d(x_0, x_1) + \alpha^{n+1}d(x_0, x_1) + \alpha^{n+1}d(x_0, x_1)] \\
&\to 0 \; as \; n \to \infty.
\end{aligned}$$

Thus,

$$\lim_{n\to\infty} d(x_n, x_{n+p}) = 0 \text{ for all even } p \in \mathbb{N}. \tag{4.10}$$

Case – II : Suppose p is odd, that is, $p = 2m - 1$ for some $m \in \mathbb{N}$.
If $m = 1$, then

$$d(x_n, x_{n+p}) = d(x_n, x_{n+1}) \leq \alpha^n d(x_0, x_1) \to 0 \; as \, n \to \infty.$$

If $m > 1$, then by using condition (4.8), we get

$$\begin{aligned} d(x_n, x_{n+p}) &\leq s[d(x_n, x_{n+1}) + d(x_{n+1}, x_{n+2}) + d(x_{n+2}, x_{n+2m-1})] \\ &\leq s[d(x_n, x_{n+1}) + d(x_{n+1}, x_{n+2})] \\ &\quad +s^2[d(x_{n+2}, x_{n+3}) + d(x_{n+3}, x_{n+4})] \\ &\quad +s^3[d(x_{n+4}, x_{n+5}) + d(x_{n+5}, x_{n+6})] + \cdots \\ &\quad +s^{m-1}[d(x_{n-4+2m}, x_{n-3+2m}) + d(x_{n-3+2m}, x_{n-2+2m})] \\ &\quad +s^{m-1}d(x_{n-2+2m}, x_{n-1+2m}) \\ &\leq s\alpha^n[1 + s\alpha^2 + s^2\alpha^4 + \cdots]d(x_0, x_1) \\ &\quad +s\alpha^{n+1}[1 + s\alpha^2 + s^2\alpha^4 + \cdots]d(x_0, x_1) \\ &= \frac{1+\alpha}{1-s\alpha^2} s\alpha^n d(x_0, x_1). \end{aligned}$$

Passing to the limit as $n \to \infty$, we have

$$\lim_{n\to\infty} d(x_n, x_{n+p}) = 0 \text{ for all odd } p \in \mathbb{N}. \tag{4.11}$$

It follows from (4.10) and (4.11) that

$$\lim_{n\to\infty} d(x_n, x_{n+p}) = 0, \quad \forall p \in \mathbb{N}. \tag{4.12}$$

We now show that $\lim_{n\to\infty} d(x_{n+p}, x_n) = 0, \quad \forall p \in \mathbb{N}.$

We consider the following two cases:

Case(a) : Suppose p is even, that is, $p = 2m$ for some $m \in \mathbb{N}$.
If $m = 1$, then by using conditions (4.8) and (4.9), we get

$$\begin{aligned} d(x_{n+p}, x_n) = d(x_{n+2}, x_n) &\leq s[d(x_{n+2}, x_{n+1}) + d(x_{n+1}, x_{n+1}) + d(x_{n+1}, x_n)] \\ &\leq s[\alpha^{n+1}d(x_1, x_0) + \alpha^{n+1}d(x_0, x_1) + \alpha^n d(x_1, x_0)] \\ &\to 0 \ as \ n \to \infty. \end{aligned}$$

If $m > 1$, then

$$\begin{aligned} d(x_{n+p}, x_n) &\leq s[d(x_{n+2m}, x_{n+2}) + d(x_{n+2}, x_{n+1}) + d(x_{n+1}, x_n)] \\ &\leq s[d(x_{n+2}, x_{n+1}) + d(x_{n+1}, x_n)] \\ &\quad +s^2[d(x_{n+4}, x_{n+3}) + d(x_{n+3}, x_{n+2})] \\ &\quad +s^3[d(x_{n+6}, x_{n+5}) + d(x_{n+5}, x_{n+4})] + \cdots \\ &\quad +s^{m-1}[d(x_{n-2+2m}, x_{n-3+2m}) + d(x_{n-3+2m}, x_{n-4+2m})] \\ &\quad +s^{m-1}d(x_{n+2m}, x_{n-2+2m}) \\ &\leq s\alpha^{n+1}[1 + s\alpha^2 + s^2\alpha^4 + \cdots]d(x_1, x_0) \\ &\quad +s\alpha^n[1 + s\alpha^2 + s^2\alpha^4 + \cdots]d(x_1, x_0) \\ &\quad +s^{m-1}d(x_{n+2m}, x_{n-2+2m}) \\ &\leq \frac{\alpha+1}{1-s\alpha^2} s\alpha^n d(x_1, x_0) + s^m[d(x_{n+2m}, x_{n-1+2m}) \\ &\quad +d(x_{n-1+2m}, x_{n-1+2m}) + d(x_{n-1+2m}, x_{n-2+2m})] \\ &\leq \frac{\alpha+1}{1-s\alpha^2} s\alpha^n d(x_1, x_0) \\ &\quad +(\alpha s)^m[(\alpha^{m+n-1} + \alpha^{m+n-2})d(x_1, x_0) + \alpha^{m+n-1}d(x_0, x_1)] \\ &< \frac{\alpha+1}{1-s\alpha^2} s\alpha^n d(x_1, x_0) \\ &\quad +[(\alpha^{m+n-1} + \alpha^{m+n-2})d(x_1, x_0) + \alpha^{m+n-1}d(x_0, x_1)], \ as \ \alpha s < 1. \end{aligned}$$

Passing to the limit as $n \to \infty$, we have

$$\lim_{n\to\infty} d(x_{n+p}, x_n) = 0.$$

Thus,

$$\lim_{n\to\infty} d(x_{n+p}, x_n) = 0 \text{ for all even } p \in \mathbb{N}. \tag{4.13}$$

$\mathbf{Case(b)}$: Suppose p is odd, that is, $p = 2m - 1$ for some $m \in \mathbb{N}$.
By an argument similar to that used in $\mathbf{Case - II}$, we can show that

$$\lim_{n\to\infty} d(x_{n+p}, x_n) = 0 \text{ for all odd } p \in \mathbb{N}. \tag{4.14}$$

It follows from (4.13) and (4.14) that

$$\lim_{n\to\infty} d(x_{n+p}, x_n) = 0, \ \forall p \in \mathbb{N}. \tag{4.15}$$

From (4.12) and (4.15), it follows that

$$\lim_{n\to\infty} d(x_n, x_{n+p}) = \lim_{n\to\infty} d(x_{n+p}, x_n) = 0, \ \forall p \in \mathbb{N}.$$

Therefore, (x_n) is a 0-Cauchy sequence in (X, d). As (X, d) is 0-complete, there exists $u \in X$ such that

$$\lim_{n\to\infty} d(u, x_n) = \lim_{n\to\infty} d(x_n, u) = d(u, u) = 0.$$

Obviously, subsequences $(x_{2n}) \subseteq A$ and $(x_{2n-1}) \subseteq B$ are also converge to $u \in X$. As A and B are closed, by Lemma 2.12, it follows that $u \in A \cap B$.

We shall show that u is a fixed point of T.
We assume that $x_n, x_{n+1} \in X \setminus \{u, Tu\}$ for large $n \in \mathbb{N}$. Because if $x_{n+1} = u$, then $x_{n+2} = Tx_{n+1} = Tu$ which implies that $d(u, Tu) = d(x_{n+1}, x_{n+2}) \leq \alpha^{n+1} d(x_0, x_1) \to 0$ as $n \to \infty$. This gives that $d(u, Tu) = 0$. Similarly, we can prove that $d(Tu, u) = 0$. Consequently, it follows that $Tu = u$. On the other hand, if $x_{n+1} = Tu$, then $d(u, Tu) = d(u, x_{n+1}) \to 0$ as $n \to \infty$. Therefore, $d(u, Tu) = 0$. By an argument similar to that used above we can show that $d(Tu, u) = 0$ and hence $Tu = u$.

Now,

$$\begin{aligned} d(u, Tu) &\leq s\left[d(u, x_n) + d(x_n, x_{n+1}) + d(x_{n+1}, Tu)\right] \\ &= s\left[d(u, x_n) + d(x_n, x_{n+1}) + d(Tx_n, Tu)\right] \\ &\leq s[d(u, x_n) + \alpha^n d(x_0, x_1) + \alpha \min\{d(x_n, u), d(u, x_n)\}] \\ &\to 0 \ as \ n \to \infty. \end{aligned}$$

This gives that $d(u, Tu) = 0$. Similarly, we have $d(Tu, u) = 0$. Thus, $Tu = u$. This proves that u is a fixed point of T.

To prove the uniqueness, let $v \in A \cap B$ be another fixed point of T such that $d(v, v) = 0$. Then,

$$\begin{aligned} d(u, v) = d(Tu, Tv) &\leq \max\{d(Tu, Tv), d(Tv, Tu), d(Tu, Tu), d(Tv, Tv)\} \\ &\leq \alpha \min\{d(u, v), d(v, u)\} \\ &\leq \alpha d(u, v), \end{aligned}$$

which gives that $d(u, v) = 0$ since $0 \leq \alpha < \frac{1}{s}$. Similarly, we have $d(v, u) = 0$. Therefore, $u = v$. Hence T has a unique fixed point u in $A \cap B$ with $d(u, u) = 0$. □

Corollary 4.4. *Let (X, d) be a complete metric space and $T : X \to X$ be a mapping such that*

$$d(Tx, Ty) \leq \alpha d(x, y)$$

for all $x, y \in X$, where $\alpha \in [0, 1)$ is a constant. Then T has a unique fixed point in X.

Proof. The proof follows from Theorem 4.3 by considering the metric d as a dislocated quasi rectangular b-metric with coefficient $s = 1$ and $A = B = X$. □

Theorem 4.5. *Let (X, d) be a 0-complete dislocated quasi rectangular b-metric space with coefficient $s \geq 1$ and A, B be two nonempty closed subsets of X. If $T : A \cup B \to A \cup B$ is a dq-rectangular b-cyclic Kannan mapping, then T has a unique fixed point u in $A \cap B$ with $d(u, u) = 0$.*

Proof. As in Theorem 4.3, by taking an arbitrary point $x_0 \in A$, we can construct a sequence (x_n) where $x_n = Tx_{n-1}$ for all $n \in \mathbb{N}$ such that $x_n \in A$ if n is even and $x_n \in B$ if n is odd. Moreover, we assume that $x_{n-1} \neq x_n$ for all $n \in \mathbb{N}$.

From condition (4.2), we get

$$\begin{aligned} d(x_{2n-1}, x_{2n}) &= d(Tx_{2n-2}, Tx_{2n-1}) \\ &\leq \max \begin{Bmatrix} d(Tx_{2n-2}, Tx_{2n-1}), d(Tx_{2n-1}, Tx_{2n-2}), \\ d(Tx_{2n-2}, Tx_{2n-2}), d(Tx_{2n-1}, Tx_{2n-1}) \end{Bmatrix} \\ &\leq \gamma \min \begin{Bmatrix} d(x_{2n-2}, Tx_{2n-2}) + d(x_{2n-1}, Tx_{2n-1}), \\ d(Tx_{2n-2}, x_{2n-2}) + d(Tx_{2n-1}, x_{2n-1}) \end{Bmatrix} \\ &= \gamma \min \begin{Bmatrix} d(x_{2n-2}, x_{2n-1}) + d(x_{2n-1}, x_{2n}), \\ d(x_{2n-1}, x_{2n-2}) + d(x_{2n}, x_{2n-1}) \end{Bmatrix} \\ &\leq \gamma\{d(x_{2n-2}, x_{2n-1}) + d(x_{2n-1}, x_{2n})\}. \end{aligned}$$

This gives that,

$$d(x_{2n-1}, x_{2n}) \leq \frac{\gamma}{1-\gamma} d(x_{2n-2}, x_{2n-1}) = \alpha d(x_{2n-2}, x_{2n-1}), \tag{4.16}$$

where $\alpha = \frac{\gamma}{1-\gamma} \in [0, \frac{1}{s})$.

Again,

$$\begin{aligned} d(x_{2n}, x_{2n+1}) &= d(Tx_{2n-1}, Tx_{2n}) \\ &\leq \max \begin{Bmatrix} d(Tx_{2n-1}, Tx_{2n}), d(Tx_{2n}, Tx_{2n-1}), \\ d(Tx_{2n}, Tx_{2n}), d(Tx_{2n-1}, Tx_{2n-1}) \end{Bmatrix} \\ &\leq \gamma \min \begin{Bmatrix} d(x_{2n-1}, Tx_{2n-1}) + d(x_{2n}, Tx_{2n}), \\ d(Tx_{2n-1}, x_{2n-1}) + d(Tx_{2n}, x_{2n}) \end{Bmatrix} \\ &= \gamma \min \begin{Bmatrix} d(x_{2n-1}, x_{2n}) + d(x_{2n}, x_{2n+1}), \\ d(x_{2n}, x_{2n-1}) + d(x_{2n+1}, x_{2n}) \end{Bmatrix} \\ &\leq \gamma\{d(x_{2n-1}, x_{2n}) + d(x_{2n}, x_{2n+1})\} \end{aligned}$$

which gives that,

$$d(x_{2n}, x_{2n+1}) \leq \frac{\gamma}{1-\gamma} d(x_{2n-1}, x_{2n}) = \alpha d(x_{2n-1}, x_{2n}). \tag{4.17}$$

From conditions (4.16) and (4.17), it follows that

$$d(x_n, x_{n+1}) \leq \alpha d(x_{n-1}, x_n), \forall n \in \mathbb{N}. \tag{4.18}$$

By repeated use of condition (4.18), we get

$$d(x_n, x_{n+1}) \leq \alpha^n \, d(x_0, x_1), \forall n \in \mathbb{N}. \tag{4.19}$$

Similarly,

$$d(x_{n+1}, x_n) \leq \alpha^n d(x_1, x_0), \forall n \in \mathbb{N}.$$

Moreover,

$$\begin{aligned} d(x_n, x_n) &= d(Tx_{n-1}, Tx_{n-1}) \\ &\leq \max \begin{Bmatrix} d(Tx_{n-1}, Tx_n), d(Tx_n, Tx_{n-1}), \\ d(Tx_{n-1}, Tx_{n-1}), d(Tx_n, Tx_n) \end{Bmatrix} \\ &\leq \gamma \min \begin{Bmatrix} d(x_{n-1}, x_n) + d(x_n, x_{n+1}), \\ d(x_n, x_{n-1}) + d(x_{n+1}, x_n) \end{Bmatrix} \\ &\leq \gamma\{d(x_{n-1}, x_n) + d(x_n, x_{n+1})\} \\ &\leq \gamma\alpha^{n-1}(1+\alpha)d(x_0, x_1) \\ &= \gamma(\alpha^{n-1} + \alpha^n)d(x_0, x_1), \quad \forall n \in \mathbb{N}. \end{aligned}$$

By an argument similar to that used in Theorem 4.3, we have $x_m \neq x_n$ for all $m,\, n \in \mathbb{N}$ with $m \neq n$.

Proceeding similarly to that of Theorem 4.3, we can show that (x_n) is a 0-Cauchy sequence in (X, d). As (X, d) is 0-complete, there exists $u \in X$ such that

$$\lim_{n\to\infty} d(u, x_n) = \lim_{n\to\infty} d(x_n, u) = d(u, u) = 0.$$

Obviously, subsequences $(x_{2n}) \subseteq A$ and $(x_{2n-1}) \subseteq B$ are also converge to $u \in X$. As A and B are closed, by Lemma 2.12, it follows that $u \in A \cap B$.

We shall show that u is a fixed point of T. As in Theorem 4.3, we assume that $x_n,\, x_{n+1} \in X \setminus \{u,\, Tu\}$ for large $n \in \mathbb{N}$. Then by using conditions (4.2) and (4.19), we obtain that

$$\begin{aligned} d(u, Tu) &\leq s\,[d(u, x_n) + d(x_n, x_{n+1}) + d(x_{n+1}, Tu)] \\ &\leq s\,[d(u, x_n) + \alpha^n\, d(x_0, x_1) + d(Tx_n, Tu)] \\ &\leq s\,[d(u, x_n) + \alpha^n\, d(x_0, x_1)] \\ &\quad + s\, \max\{d(Tx_n, Tu), d(Tu, Tx_n), d(Tx_n, Tx_n), d(Tu, Tu)\} \\ &\leq s\,[d(u, x_n) + \alpha^n\, d(x_0, x_1)] \\ &\quad + s\,\gamma \min\{d(x_n, Tx_n) + d(u, Tu), d(Tx_n, x_n) + d(Tu, u)\} \\ &\leq s\,[d(u, x_n) + \alpha^n\, d(x_0, x_1)] + s\,\gamma\{d(x_n, x_{n+1}) + d(u, Tu)\} \\ &\leq s\,[d(u, x_n) + \alpha^n\, d(x_0, x_1)] + s\,\gamma\{\alpha^n\, d(x_0, x_1) + d(u, Tu)\}. \end{aligned}$$

Taking limit as $n \to \infty$, we get

$$d(u, Tu) \leq s\gamma\, d(u, Tu).$$

Since $0 \leq s\gamma < 1$, it follows that $d(u, Tu) = 0$.

Similarly, we can prove that $d(Tu, u) = 0$. Hence, we have $u = Tu$, i.e., u is a fixed point of T.

To prove the uniqueness, let v be another fixed point of T such that $v \in A \cap B$ and $d(v, v) = 0$. Then,

$$\begin{aligned} d(u, v) &= d(Tu, Tv) \\ &\leq \max\{d(Tu, Tv), d(Tv, Tu), d(Tu, Tu), d(Tv, Tv)\} \\ &\leq \gamma\, \min\{d(u, Tu) + d(v, Tv), d(Tu, u) + d(Tv, v)\} \\ &= \gamma\,[d(u, u) + d(v, v)] \\ &= 0. \end{aligned}$$

This proves that $d(u, v) = 0$. By similar arguments, we have $d(v, u) = 0$ and hence, $u = v$. Therefore, T has a unique fixed point $u \in A \cap B$ with $d(u, u) = 0$. □

Corollary 4.6. *Let (X,d) be a complete metric space and $T : X \to X$ be a mapping such that*

$$d(Tx,Ty) \leq \gamma[d(x,Tx) + d(y,Ty)]$$

for all x, $y \in X$, where $\gamma \in [0, \frac{1}{2})$ is a constant. Then T has a unique fixed point in X.

Proof. The proof follows from Theorem 4.5 by considering the metric d as a dislocated quasi rectangular b-metric with coefficient $s = 1$ and $A = B = X$. □

Definition 4.7. Let A and B be nonempty subsets of a dislocated quasi rectangular b-metric space (X,d) with coefficient s. A cyclic mapping $T : A \cup B \to A \cup B$ is called a dq-rectangular b-cyclic Fisher mapping if there exists $\gamma \in [0, \frac{1}{s+s^2})$ such that

$$\begin{aligned} \max\begin{Bmatrix} d(Tx,Ty), d(Ty,Tx), \\ d(Tx,Tx), d(Ty,Ty) \end{Bmatrix} \quad \leq \quad & \gamma\Bigg[\min\begin{Bmatrix} d(x,Ty)+d(y,Tx), \\ d(Ty,x)+d(Tx,y) \end{Bmatrix} \\ & -(1+s)\max\{d(x,x), d(y,y)\}\Bigg] \end{aligned} \tag{4.20}$$

for all $x \in A$, $y \in B$.
If $A = B = X$, then $T : X \to X$ satisfying condition (4.20) is called a dq-rectangular b-Fisher mapping.

Theorem 4.8. *Let (X,d) be a 0-complete dislocated quasi rectangular b-metric space with coefficient $s \geq 1$ and A, B be two nonempty closed subsets of X. If $T : A \cup B \to A \cup B$ is a dq-rectangular b-cyclic Fisher mapping, then T has a unique fixed point u (say) in $A \cap B$ with $d(u,u) = 0$.*

Proof. As in Theorem 4.3, if $x_0 \in A$ is arbitrary, then we can construct a sequence (x_n) where $x_n = Tx_{n-1}$ for all $n \in \mathbb{N}$ such that $x_n \in A$ if n is even and $x_n \in B$ if n is odd. Moreover, we assume that $x_{n-1} \neq x_n$ for all $n \in \mathbb{N}$.

From condition (4.20), we get

$$\begin{aligned} d(x_{2n-1},x_{2n}) &= d(Tx_{2n-2},Tx_{2n-1}) \\ &\leq \max\begin{Bmatrix} d(Tx_{2n-2},Tx_{2n-1}), d(Tx_{2n-1},Tx_{2n-2}), \\ d(Tx_{2n-2},Tx_{2n-2}), d(Tx_{2n-1},Tx_{2n-1}) \end{Bmatrix} \\ &\leq \gamma\Bigg[\min\begin{Bmatrix} d(x_{2n-2},Tx_{2n-1}) + d(x_{2n-1},Tx_{2n-2}), \\ d(Tx_{2n-1},x_{2n-2}) + d(Tx_{2n-2},x_{2n-1}) \end{Bmatrix} \\ &\quad -(1+s)\max\{d(x_{2n-2},x_{2n-2}), d(x_{2n-1},x_{2n-1})\}\Bigg] \\ &= \gamma\Bigg[\min\begin{Bmatrix} d(x_{2n-2},x_{2n}) + d(x_{2n-1},x_{2n-1}), \\ d(x_{2n},x_{2n-2}) + d(x_{2n-1},x_{2n-1}) \end{Bmatrix} \\ &\quad -(1+s)\max\{d(x_{2n-2},x_{2n-2}), d(x_{2n-1},x_{2n-1})\}\Bigg] \\ &\leq \gamma\Big[d(x_{2n-2},x_{2n}) + d(x_{2n-1},x_{2n-1}) - (1+s)d(x_{2n-1},x_{2n-1}) \Big] \\ &\leq \gamma\Big[s\{d(x_{2n-2},x_{2n-1}) + d(x_{2n-1},x_{2n-1}) + d(x_{2n-1},x_{2n})\} \\ &\quad -sd(x_{2n-1},x_{2n-1}) \Big] \\ &= \gamma s\Big[d(x_{2n-2},x_{2n-1}) + d(x_{2n-1},x_{2n}) \Big]. \end{aligned}$$

This gives that

$$d(x_{2n-1}, x_{2n}) \leq \frac{\gamma s}{1-\gamma s} d(x_{2n-2}, x_{2n-1}) = \alpha d(x_{2n-2}, x_{2n-1}), \tag{4.21}$$

where $\alpha = \frac{\gamma s}{1-\gamma s} \in [0, \frac{1}{s})$.

Again,

$$\begin{aligned}
d(x_{2n}, x_{2n+1}) &= d(Tx_{2n-1}, Tx_{2n}) \\
&\leq \max \left\{ \begin{array}{c} d(Tx_{2n}, Tx_{2n-1}), d(Tx_{2n-1}, Tx_{2n}), \\ d(Tx_{2n}, Tx_{2n}), d(Tx_{2n-1}, Tx_{2n-1}) \end{array} \right\} \\
&\leq \gamma \Bigg[\min \left\{ \begin{array}{c} d(x_{2n}, Tx_{2n-1}) + d(x_{2n-1}, Tx_{2n}), \\ d(Tx_{2n-1}, x_{2n}) + d(Tx_{2n}, x_{2n-1}) \end{array} \right\} \\
&\quad -(1+s) \max\{d(x_{2n}, x_{2n}), d(x_{2n-1}, x_{2n-1})\} \Bigg] \\
&= \gamma \Bigg[\min \left\{ \begin{array}{c} d(x_{2n}, x_{2n}) + d(x_{2n-1}, x_{2n+1}), \\ d(x_{2n}, x_{2n}) + d(x_{2n+1}, x_{2n-1}) \end{array} \right\} \\
&\quad -(1+s) \max\{d(x_{2n}, x_{2n}), d(x_{2n-1}, x_{2n-1})\} \Bigg] \\
&\leq \gamma \Big[d(x_{2n}, x_{2n}) + d(x_{2n-1}, x_{2n+1}) - (1+s) d(x_{2n}, x_{2n}) \Big] \\
&\leq \gamma \Big[s\{d(x_{2n-1}, x_{2n}) + d(x_{2n}, x_{2n}) + d(x_{2n}, x_{2n+1})\} - s d(x_{2n}, x_{2n}) \Big] \\
&= \gamma s \Big[d(x_{2n-1}, x_{2n}) + d(x_{2n}, x_{2n+1}) \Big].
\end{aligned}$$

This implies that

$$d(x_{2n}, x_{2n+1}) \leq \frac{\gamma s}{1-\gamma s} d(x_{2n-1}, x_{2n}) = \alpha d(x_{2n-1}, x_{2n}). \tag{4.22}$$

From conditions (4.21) and (4.22), we get

$$d(x_n, x_{n+1}) \leq \alpha d(x_{n-1}, x_n), \forall n \in \mathbb{N}. \tag{4.23}$$

By repeated use of condition (4.23), we obtain that

$$d(x_n, x_{n+1}) \leq \alpha^n d(x_0, x_1), \forall n \in \mathbb{N}. \tag{4.24}$$

Similarly,

$$d(x_{n+1}, x_n) \leq \alpha^n d(x_1, x_0), \forall n \in \mathbb{N}.$$

Moreover,

$$\begin{aligned} d(x_n, x_n) &= d(Tx_{n-1}, Tx_{n-1}) \\ &\leq \max \begin{Bmatrix} d(Tx_{n-1}, Tx_n), d(Tx_n, Tx_{n-1}), \\ d(Tx_{n-1}, Tx_{n-1}), d(Tx_n, Tx_n) \end{Bmatrix} \\ &\leq \gamma \Bigg[\min \begin{Bmatrix} d(x_{n-1}, x_{n+1}) + d(x_n, x_n), \\ d(x_{n+1}, x_{n-1}) + d(x_n, x_n) \end{Bmatrix} \\ &\quad -(1+s) \max\{d(x_{n-1}, x_{n-1}), d(x_n, x_n)\} \Bigg] \\ &\leq \gamma \Big[d(x_{n-1}, x_{n+1}) + d(x_n, x_n) - (1+s) d(x_n, x_n) \Big] \\ &\leq \gamma \Big[s\{d(x_{n-1}, x_n) + d(x_n, x_n) + d(x_n, x_{n+1})\} - s d(x_n, x_n) \Big] \\ &= \gamma s \Big[\{d(x_{n-1}, x_n) + d(x_n, x_{n+1})\} \Big] \\ &\leq \gamma s \alpha^{n-1} (1+\alpha) d(x_0, x_1) \\ &\leq \gamma s \alpha^{n-1} (1 + \frac{1}{s}) d(x_0, x_1) \\ &\leq \alpha^{n-1} d(x_0, x_1), \quad \forall n \in \mathbb{N} \end{aligned}$$

and

$$\begin{aligned} d(x_{n-1}, x_{n+1}) &\leq s[d(x_{n-1}, x_n) + d(x_n, x_n) + d(x_n, x_{n+1})] \\ &\leq s[\alpha^{n-1} d(x_0, x_1) + \alpha^{n-1} d(x_0, x_1) + \alpha^n d(x_0, x_1)] \\ &= s(2\alpha^{n-1} + \alpha^n) d(x_0, x_1). \end{aligned}$$

Proceeding similarly to that of Theorem 4.3, we can show that $x_m \neq x_n$ for all $m, n \in \mathbb{N}$ with $m \neq n$.

By the techniques that adapted in Theorem 4.3, it follows that (x_n) is 0-Cauchy in (X, d). As (X, d) is 0-complete, there exists $u \in X$ such that

$$\lim_{n\to\infty} d(u, x_n) = \lim_{n\to\infty} d(x_n, u) = d(u, u) = 0.$$

Obviously, subsequences $(x_{2n}) \subseteq A$ and $(x_{2n-1}) \subseteq B$ are also converge to $u \in X$. As A and B are closed, by Lemma 2.12, it follows that $u \in A \cap B$.

We shall show that u is a fixed point of T. If possible, suppose that $Tu \neq u$. As in Theorem 4.3, we assume that $x_n,\, x_{n+1} \in X \setminus \{u,\, Tu\}$ for large $n \in \mathbb{N}$. Then by using conditions (4.20) and (4.24), we obtain that

$$\begin{aligned}
d(u,Tu) &\leq s\left[d(u,x_n)+d(x_n,x_{n+1})+d(x_{n+1},Tu)\right]\\
&\leq s[d(u,x_n)+\alpha^n d(x_0,x_1)+d(Tx_n,Tu)]\\
&\leq s\Bigg[d(u,x_n)+\alpha^n d(x_0,x_1)\\
&\quad +\max\{d(Tx_n,Tu),d(Tu,Tx_n),d(Tx_n,Tx_n),d(Tu,Tu)\}\Bigg]\\
&\leq sd(u,x_n)+s\alpha^n d(x_0,x_1)\\
&\quad +s\gamma\Bigg[\min\{d(x_n,Tu)+d(u,Tx_n),d(Tu,x_n)+d(Tx_n,u)\}\\
&\quad -(1+s)\max\{d(x_n,x_n),d(u,u)\}\Bigg]\\
&\leq sd(u,x_n)+s\alpha^n d(x_0,x_1)\\
&\quad +s\gamma\Big[d(x_n,Tu)+d(u,x_{n+1})-(1+s)d(x_n,x_n)\Big]\\
&\leq sd(u,x_n)+s\alpha^n d(x_0,x_1)\\
&\quad +s\gamma\Big[s\{d(x_n,x_{n+1})+d(x_{n+1},u)+d(u,Tu)\}+d(u,x_{n+1})\Big]\\
&\leq sd(u,x_n)+s\alpha^n d(x_0,x_1)\\
&\quad +s\gamma\Big[s\{\alpha^n d(x_0,x_1)+d(x_{n+1},u)+d(u,Tu)\}+d(u,x_{n+1})\Big].
\end{aligned}$$

Taking limit as $n\to\infty$, we get

$$d(u,Tu)\leq s^2\gamma d(u,Tu).$$

Since $0\leq s^2\gamma<1$, it follows that $d(u,Tu)=0$. Similarly, $d(Tu,u)=0$. Therefore, $Tu=u$, a contradiction. Hence, we have $u=Tu$, i.e., u is a fixed point of T.

To prove the uniqueness, let v be another fixed point of T such that $v\in A\cap B$ and $d(v,v)=0$. Then,

$$\begin{aligned}
d(u,v)=d(Tu,Tv) &\leq \max\{d(Tu,Tv),d(Tv,Tu),d(Tu,Tu),d(Tv,Tv)\}\\
&\leq \gamma\min\{d(u,Tv)+d(v,Tu),d(Tv,u)+d(Tu,v)\}\\
&= \gamma[d(u,v)+d(v,u)].
\end{aligned}$$

This implies that

$$d(u,v)\leq\frac{\gamma}{1-\gamma}d(v,u).$$

Similarly, $d(v,u)\leq\frac{\gamma}{1-\gamma}d(u,v)$. Thus,

$$d(u,v)\leq\frac{\gamma}{1-\gamma}d(v,u)\leq\left(\frac{\gamma}{1-\gamma}\right)^2 d(u,v).$$

Since $0\leq\frac{\gamma}{1-\gamma}<1$, it follows that $d(u,v)=0$. Similarly, we can show that $d(v,u)=0$. Therefore, $u=v$. Hence, T has a unique fixed point u in $A\cap B$ with $d(u,u)=0$. □

Corollary 4.9. *Let (X,d) be a complete metric space and $T:X\to X$ be a mapping such that*

$$d(Tx,Ty)\leq\gamma[d(x,Ty)+d(y,Tx)]$$

for all x, $y\in X$, where $\gamma\in[0,\frac{1}{2})$ is a constant. Then T has a unique fixed point in X.

Proof. The proof follows from Theorem 4.8 by considering the metric d as a dislocated quasi rectangular b-metric with coefficient $s = 1$ and $A = B = X$. □

Remark 4.10. We note that Theorem 4.3 is a proper generalization(see Example 4.13) of the famous Banach contraction theorem [5].

Remark 4.11. It is worthy to note that several fixed point results in quasi b-metric spaces and dislocated metric spaces can be derived from Theorems 4.3, 4.5 and 4.8.

Remark 4.12. The results of this study are obtained under the weaker assumption that the underlying dislocated quasi rectangular b-metric space is 0-complete. However, they also valid if the space is complete.

We now present an example to justify the validity of our first main result. It should be noticed that the well known Banach contraction theorem can not explain the existence of fixed point in the following example.

Example 4.13. Let $A = \{0, \frac{1}{2}, 2\}$, $B = \{\frac{1}{n} : n \in \mathbb{N}, n \geq 2\}$ and $X = A \cup B$. Let $d : X \times X \to [0, \infty)$ be defined by

$$d(x,y) = \begin{cases} \frac{1}{2}, & \text{if } x, y \in A \setminus \{\frac{1}{2}\}, \text{ except } x = y = 0; \\ \frac{1}{3}, & \text{if } x, y \in B \setminus \{\frac{1}{2}\}; \\ \frac{1}{4}, & \text{if } x \in A \setminus \{\frac{1}{2}\} \text{ and } y \in B \setminus \{\frac{1}{2}\}; \\ \frac{2}{9}, & \text{if } x \in B \setminus \{\frac{1}{2}\} \text{ and } y \in A \setminus \{\frac{1}{2}\}; \\ \frac{2}{9}, & \text{if } (x = \frac{1}{2}, y \in B \setminus \{\frac{1}{2}\}) \text{ or } (x \in B \setminus \{\frac{1}{2}\}, y = \frac{1}{2}); \\ \frac{1}{9}, & \text{if } (x = 0, y = \frac{1}{2}) \text{ or } (x = \frac{1}{2}, y = 0); \\ \frac{1}{2}, & \text{if } (x = 2, y = \frac{1}{2}) \text{ or } (x = \frac{1}{2}, y = 2); \\ 0, & \text{otherwise.} \end{cases}$$

Then (X, d) is a dislocated quasi rectangular b-metric space with coefficient $s = \frac{9}{5}$. Let $T : X \to X$ be defined by

$$Tx = \begin{cases} \frac{1}{2}, & \text{if } x \in A, \\ 0, & \text{if } x \in B \setminus \{\frac{1}{2}\}. \end{cases}$$

Obviously, $T(A) \subseteq B$ and $T(B) \subseteq A$ and hence T is cyclic.

If we consider the usual metric $\rho(x,y) =| x - y |$ for all $x, y \in X$, then (X, ρ) is a complete metric space. For $x = 0,\ y = \frac{1}{5}$, we have $Tx = \frac{1}{2},\ Ty = 0$. Therefore,

$$\rho(Tx, Ty) = \frac{1}{2} > \alpha\, \rho(x, y)$$

for any $\alpha \in [0, 1)$ and hence T is not a contraction operator on (X, ρ). So, Banach contraction theorem can not explain the existence of fixed point of T. However, our first main result can explain it.

We find that the constant sequence (x_n) where $x_n = 0$ for all $n \in \mathbb{N}$ and the constant sequence (x_n) where $x_n = \frac{1}{2}$ for all $n \in \mathbb{N}$ are the only 0-Cauchy sequences in (X, d). In fact, for $x_n = 0$ for all $n \in \mathbb{N}$, we have

$$\lim_{n\to\infty} d(x_n, x_{n+p}) = \lim_{n\to\infty} d(x_{n+p}, x_n) = 0,\ \forall p \in \mathbb{N}$$

and

$$\lim_{n\to\infty} d(x_n, 0) = \lim_{n\to\infty} d(0, x_n) = d(0,0) = 0.$$

Moreover, for $x_n = \frac{1}{2}$ for all $n \in \mathbb{N}$, we have

$$\lim_{n\to\infty} d(x_n, x_{n+p}) = \lim_{n\to\infty} d(x_{n+p}, x_n) = 0, \ \forall p \in \mathbb{N}$$

and

$$\lim_{n\to\infty} d(x_n, \frac{1}{2}) = \lim_{n\to\infty} d(\frac{1}{2}, x_n) = d(\frac{1}{2}, \frac{1}{2}) = 0.$$

Thus, every 0-Cauchy sequence in (X, d) converges to a point $x \in X$ such that $d(x, x) = 0$ and hence (X, d) is 0-complete.

Furthermore, it is easy to compute that $T : A \cup B \to A \cup B$ is a dq-rectangular b-cyclic Banach mapping for $\alpha = \frac{1}{2} \in [0, \frac{1}{s})$ and A, B are closed subsets of (X, d). Thus, all the hypotheses of Theorem 4.3 hold true and $\frac{1}{2} \in A \cap B$ is a unique fixed point of T with $d(\frac{1}{2}, \frac{1}{2}) = 0$.

We now cite an example to support our Theorem 4.5.

Example 4.14. Let us take $X = \{2\} \cup \{2 + \frac{1}{3n} : n \in \mathbb{N}\} = \{2\} \cup \{2 + \frac{1}{9q} : q \in \mathbb{N}\} \cup \{2 + \frac{1}{3r} : r \in \mathbb{N},\ r \ is\ not\ divisible\ by\ 3\}$. For r, q, r_1, $q_1 \in \mathbb{N}$ with $r \neq r_1$, $q \neq q_1$ and r, r_1 are not divisible by 3, we define $d : X \times X \to [0, \infty)$ by

$$d(2,2) = 0,\ d(2 + \frac{1}{3r}, 2 + \frac{1}{3r}) = 10,\ d(2 + \frac{1}{9q}, 2 + \frac{1}{9q}) = 3,$$

$$d(2, 2 + \frac{1}{3r}) = 6,\ d(2 + \frac{1}{3r}, 2) = 7,\ d(2 + \frac{1}{9q}, 2) = 2,$$

$$d(2, 2 + \frac{1}{9q}) = 1,\ d(2 + \frac{1}{3r}, 2 + \frac{1}{9q}) = 15,\ d(2 + \frac{1}{9q}, 2 + \frac{1}{3r}) = 16,$$

$$d(2 + \frac{1}{3r}, 2 + \frac{1}{3r_1}) = 11,\ d(2 + \frac{1}{9q}, 2 + \frac{1}{9q_1}) = 4.$$

Then (X, d) is a dislocated quasi rectangular b-metric space with coefficient $s = 2$.

Let $T : X \to X$ be defined by

$$Tx = \begin{cases} \frac{x+4}{3}, & \text{if } x \in \{2 + \frac{1}{3r} : r \in \mathbb{N},\ r\ is\ not\ divisible\ by\ 3\}, \\ 2, & otherwise. \end{cases}$$

If we consider the usual metric $\rho(x, y) =\mid x - y \mid$ for all x, $y \in X$, then (X, ρ) is a complete metric space. For $x = 2$ and $y = \frac{7}{3}$, we have
$\rho(Tx, Ty) = \rho(2, \frac{19}{9}) = \frac{1}{9}$ and $\rho(x, Tx) + \rho(y, Ty) = \rho(2, 2) + \rho(\frac{7}{3}, \frac{19}{9}) = \frac{2}{9}$. Therefore,

$$\rho(Tx, Ty) > \alpha[\rho(x, Tx) + \rho(y, Ty)]$$

for any $\alpha \in [0, \frac{1}{2})$ and hence T is not a Kannan operator on (X, ρ). So, Kannan fixed point theorem cannot be used to get fixed point of T. However, we can apply our second main result.

We note that (X, d) is 0-complete as the constant sequence (x_n) where $x_n = 2$ for all $n \in \mathbb{N}$ is the only 0-Cauchy sequence in (X, d) and

$$\lim_{n\to\infty} d(x_n, 2) = \lim_{n\to\infty} d(2, x_n) = d(2, 2) = 0.$$

We take two closed subsets A, B of X as follows:

$$A = \{2\} \cup \{2 + \frac{1}{9q} : q \in \mathbb{N}\},\ B = \{2\} \cup \{2 + \frac{1}{3r} : r \in \mathbb{N},\ r\ is\ not\ divisible\ by\ 3\}.$$

Then, $X = A \cup B$, $T(A) \subseteq B$ and

$$T(B) = \{2\} \cup \{2 + \frac{1}{9r} : r \in \mathbb{N},\ r \text{ is not divisible by } 3\} \subseteq A.$$

This proves that T is cyclic.

It is easy to compute that condition (4.2) holds true for $\gamma = \frac{4}{15} \in [0, \frac{1}{1+s})$. Hence, all the conditions of Theorem 4.5 are satisfied and T has a unique fixed point $2 \in A \cap B$ with $d(2, 2) = 0$.

The following example supports our Theorem 4.8.

Example 4.15. Let us take $X = [0, 1]$. We define $d : X \times X \to [0, \infty)$ by

$$d(x, y) = \begin{cases} 0, \text{ if } x = y = \frac{1}{2}; \\ \frac{1}{3}, \text{ if } x = y \neq \frac{1}{2}; \\ 2, \text{ if } (x = \frac{1}{2} \text{ and } y \in [0, \frac{1}{2})) \text{ or } (x \in [0, \frac{1}{2}) \text{ and } y = \frac{1}{2}); \\ 8, \text{ if } x,\ y \in [0, \frac{1}{2}) \text{ and } x \neq y; \\ 16, \text{ if } x \in (\frac{1}{2}, 1] \text{ and } y = \frac{1}{2}; \\ 18, \text{ if } x = \frac{1}{2} \text{ and } y \in (\frac{1}{2}, 1]; \\ 20, \text{ otherwise.} \end{cases}$$

Then (X, d) is a dislocated quasi rectangular b-metric space with coefficient $s = 2$.

We define $T : X \to X$ by

$$Tx = \begin{cases} x - \frac{1}{2}, \text{ if } x \in (\frac{1}{2}, 1], \\ \frac{1}{2}, \text{ otherwise.} \end{cases}$$

We now verify that T does not satisfy Fisher fixed point theorem with respect to the usual metric defined on X. For $x = \frac{1}{2}$ and $y = \frac{3}{4}$, we have
$d(Tx, Ty) = d(\frac{1}{2}, \frac{1}{4}) = \frac{1}{4}$ and $d(x, Ty) + d(y, Tx) = d(\frac{1}{2}, \frac{1}{4}) + d(\frac{3}{4}, \frac{1}{2}) = \frac{1}{2}$. So,

$$d(Tx, Ty) > \alpha[d(x, Ty) + d(y, Tx)],$$

for any $\alpha \in [0, \frac{1}{2})$. Hence, Fisher fixed point theorem with respect to usual metric cannot be used to get fixed point of T. However, we can apply our third main result.

We note that (X, d) is 0-complete as the constant sequence (x_n) where $x_n = \frac{1}{2}$ for all $n \in \mathbb{N}$ is the only 0-Cauchy sequence in (X, d) and

$$\lim_{n\to\infty} d(x_n, \frac{1}{2}) = \lim_{n\to\infty} d(\frac{1}{2}, x_n) = d(\frac{1}{2}, \frac{1}{2}) = 0.$$

We take $A = [0, \frac{1}{2}],\ B = [\frac{1}{2}, 1]$. Then $A,\ B$ are closed subsets of X with $T(A) \subseteq B,\ T(B) = (0, \frac{1}{2}] \subseteq A$ and so T is cyclic. Finally, condition (4.20) holds true for $\gamma = \frac{4}{25} \in [0, \frac{1}{s+s^2})$. Thus, all the hypotheses of Theorem 4.8 are fulfilled and T has a unique fixed point $\frac{1}{2} \in A \cap B$ with $d(\frac{1}{2}, \frac{1}{2}) = 0$.

5 Some Coincidence Point Results

Definition 5.1. [1] Let f and g be self mappings of a set X. If $y = fx = gx$ for some x in X, then x is called a coincidence point of f and g and y is called a point of coincidence of f and g.

Definition 5.2. [15] The mappings $f,\ g : X \to X$ are weakly compatible, if for every $x \in X$, the following holds:

$$g(fx) = f(gx) \ whenever \ fx = gx.$$

Proposition 5.3. *[1] Let f and g be weakly compatible self maps of a nonempty set X. If f and g have a unique point of coincidence $y = fx = gx$, then y is the unique common fixed point of f and g.*

We state the following lemma which is a key result in this section.

Lemma 5.4. *[12] Let X be a nonempty set and $f : X \to X$ a function. Then there exists a subset $G \subseteq X$ such that $f(G) = f(X)$ and $f : G \to X$ is one-to-one.*

As an application of Theorem 4.3, we obtain the following result.

Theorem 5.5. *Let (X, d) be a 0-complete dislocated quasi rectangular b-metric space with coefficient $s \geq 1$. If $T : X \to X$ is a dq-rectangular b-Banach mapping, then T has a unique fixed point u in X with $d(u, u) = 0$.*

Proof. The proof follows from Theorem 4.3 by taking $A = B = X$. □

Theorem 5.6. *Let (X, d) be a dislocated quasi rectangular b-metric space with coefficient $s \geq 1$ and the mappings $T,\ g : X \to X$ satisfy the following condition:*

$$\max\{d(Tx, Ty), d(Ty, Tx), d(Tx, Tx), d(Ty, Ty)\} \leq \alpha \min\{d(gx, gy), d(gy, gx)\} \tag{5.1}$$

for all $x,\ y \in X$, where $\alpha \in [0, \frac{1}{s})$ is a constant. If $T(X) \subseteq g(X)$ and $g(X)$ is a 0-complete subspace of X, then T and g have a unique point of coincidence u(say) in $g(X)$ with $d(u, u) = 0$. Moreover, if T and g are weakly compatible, then T and g have a unique common fixed point in $g(X)$.

Proof. By Lemma 5.4, there exists $G \subseteq X$ such that $g(G) = g(X)$ and $g : G \to X$ is one-to-one. Define $h : g(G) \to g(G)$ by $h(gx) = Tx$. This is possible as $T(X) \subseteq g(X)$. Then h is well defined, as g is one-to-one on G.

For all $gx,\ gy \in g(G)$, we obtain from condition (5.1) that

$$\begin{aligned} \max\begin{Bmatrix} d(h(gx), h(gy)), d(h(gy), h(gx)), \\ d(h(gx), h(gx)), d(h(gy), h(gy)) \end{Bmatrix} &= \max\begin{Bmatrix} d(Tx, Ty), d(Ty, Tx), \\ d(Tx, Tx), d(Ty, Ty) \end{Bmatrix} \\ &\leq \alpha \min\{d(gx, gy), d(gy, gx)\}. \end{aligned}$$

This proves that $h : g(G) \to g(G)$ is a dq-rectangular b-Banach mapping. Since $g(G) = g(X)$ is 0-complete, by Theorem 5.5, there exists a unique $gx_0 \in g(X)$ such that $h(gx_0) = gx_0 = u$, say with $d(u, u) = 0$. That is, $Tx_0 = gx_0 = u$. Hence, T and g have a unique point of coincidence u in $g(X)$.
If T and g are weakly compatible, then by Proposition 5.3 it follows that T and g have a unique common fixed point in $g(X)$. □

Corollary 5.7. *Let (X, d) be a complete metric space and let $g : X \to X$ be an onto mapping satisfying*

$$d(gx, gy) \geq k\, d(x, y)$$

for all $x,\ y \in X$, where $k > 1$ is a constant. Then g has a unique fixed point in X.

Proof. The proof follows from Theorem 5.6 by taking $T = I$, the identity map on X, $s = 1$ and $\alpha = \frac{1}{k}$. □

The following result is an application of Theorem 4.5.

Theorem 5.8. *Let (X,d) be a 0-complete dislocated quasi rectangular b-metric space with coefficient $s \geq 1$. If $T : X \to X$ is a dq-rectangular b-Kannan mapping, then T has a unique fixed point u in X with $d(u,u) = 0$.*

Proof. The proof follows from Theorem 4.5 by taking $A = B = X$. □

The following theorem is a consequence of Theorem 5.8 and Lemma 5.4.

Theorem 5.9. *Let (X,d) be a dislocated quasi rectangular b-metric space with coefficient $s \geq 1$ and the mappings $T,\ g : X \to X$ satisfy the following condition:*

$$\max \begin{Bmatrix} d(Tx,Ty), d(Ty,Tx), \\ d(Tx,Tx), d(Ty,Ty) \end{Bmatrix} \leq \gamma \min \begin{Bmatrix} d(gx,Tx) + d(gy,Ty), \\ d(Tx,gx) + d(Ty,gy) \end{Bmatrix} \tag{5.2}$$

for all $x,\ y \in X$, where $\gamma \in [0, \frac{1}{1+s})$ is a constant. If $T(X) \subseteq g(X)$ and $g(X)$ is a 0-complete subspace of X, then T and g have a unique point of coincidence u(say) in $g(X)$ with $d(u,u) = 0$. Moreover, if T and g are weakly compatible, then T and g have a unique common fixed point in $g(X)$.

Proof. By an argument similar to that used in Theorem 5.6 and by using condition (5.2), we can show that $h : g(G) \to g(G)$ is a dq-rectangular b-Kannan mapping. Then, by applying Theorem 5.8, there exists a unique $gx_0 \in g(X)$ such that $h(gx_0) = gx_0 = u$, say with $d(u,u) = 0$. That is, $Tx_0 = gx_0 = u$. Hence, T and g have a unique point of coincidence u in $g(X)$.
If T and g are weakly compatible, then by Proposition 5.3 it follows that T and g have a unique common fixed point in $g(X)$. □

We now apply Theorem 4.8 to obtain the following result.

Theorem 5.10. *Let (X,d) be a 0-complete dislocated quasi rectangular b-metric space with coefficient $s \geq 1$. If $T : X \to X$ is a dq-rectangular b-Fisher mapping, then T has a unique fixed point u in X with $d(u,u) = 0$.*

Proof. The proof follows from Theorem 4.8 by taking $A = B = X$. □

The following theorem is a consequence of Theorem 5.10 and Lemma 5.4.

Theorem 5.11. *Let (X,d) be a dislocated quasi rectangular b-metric space with coefficient $s \geq 1$ and the mappings $T,\ g : X \to X$ satisfy the following condition:*

$$\begin{aligned} \max \begin{Bmatrix} d(Tx,Ty), d(Ty,Tx), \\ d(Tx,Tx), d(Ty,Ty) \end{Bmatrix} \quad \leq \quad \gamma \Bigg[& \min \begin{Bmatrix} d(gx,Ty) + d(gy,Tx), \\ d(Ty,gx) + d(Tx,gy) \end{Bmatrix} \\ & -(1+s) \max\{d(gx,gx), d(gy,gy)\} \Bigg] \end{aligned} \tag{5.3}$$

for all $x,\ y \in X$, where $\gamma \in [0, \frac{1}{s+s^2})$ is a constant. If $T(X) \subseteq g(X)$ and $g(X)$ is a 0-complete subspace of X, then T and g have a unique point of coincidence u(say) in $g(X)$ with $d(u,u) = 0$. Moreover, if T and g are weakly compatible, then T and g have a unique common fixed point in $g(X)$.

Proof. By an argument similar to that used in Theorem 5.6 and by using condition (5.3), we can show that $h : g(G) \to g(G)$ is a dq-rectangular b-Fisher mapping. Then, by applying Theorem 5.10, there exists a unique $gx_0 \in g(X)$ such that $h(gx_0) = gx_0 = u$, say with $d(u,u) = 0$. That is, $Tx_0 = gx_0 = u$. Hence, T and g have a unique point of coincidence u in $g(X)$.
If T and g are weakly compatible, then by Proposition 5.3 it follows that T and g have a unique common fixed point in $g(X)$. □

6 Conclusion remarks

In this study, we introduced some new generalized cyclic mappings in dislocated quasi rectangular b-metric spaces and discussed their fixed points. Moreover, we have investigated some coincidence point results as applications of our main results. The results of this work extend several important results in the existing literature.

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