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A NONLINEAR FRACTIONAL PANTOGRAPH EQUATION WITH A STATE-DEPENDENT NON-LOCAL CONDITION

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Abstract In a Banach space, we investigate the existence and uniqueness of mild solutions to a nonlinear fractional pantograph equation with a state-dependent non-local condition where the right-hand side of the equation is assumed to have singular coefficients. Moreover, we discuss the continuous dependence of mild solutions of the problem on the fractional order, associated parameter, and state-dependent non-local condition. Two examples are given to illustrate our theoretical findings.

1 Introduction

Pantograph equation is a delay differential equation arising in electrodynamics. There are hundreds of papers on pantograph equations with various boundary conditions or derivatives. In fact, Iserles and Liu [19] considered the classical pantograph integro-differential equations and obtained some results on well-posedness of the initial-value problem, monotonicity and oscillation of the solution, unboundedness of the solution, and asymptotic stability of the solution. In [5], the authors investigated the existence and uniqueness of solutions and their asymptotic behavior for the pantograph equation. Numerical methods for pantograph equations were also studied in [3, 18, 21] and references therein. Fractional pantograph differential equations have also been studied in recent years. In [1], the authors Investigated a class of pantograph differential equations involving Riemann-Liouville derivative with multi-point boundary conditions. They obtained some of the existence and Ulam stability results for the problem. Balachandran et al [2] considered nonlinear fractional pantograph equations with the initial and nonlocal condition, and obtained some of the existence by using the Banach and Krasnoselskii fixed point theorems. In [22], the authors investigated the Ulam-Hyers stability of the nonlinear pantograph fractional differential equation involving the Atangana-Baleanu derivative.

In the literature, usually, the existence and uniqueness results of solutions of differential equations derived from the continuity or boundedness of sources. Even so, in some cases, sources may have time-singular coefficients. We can find some papers that considered differential equations with these source types such as [4, 6, 7, 9, 10, 23]. But, there is not exist any paper that considers pantograph equations with these source types. Furthermore, the differential equations with state-dependent non-local conditions also study in some papers. In [15, 16, 20], authors considered various abstract delayed differential equations with state-dependent non-local conditions, and obtained the existence and uniqueness results for these problems. Herzallah and Radwan [17] investigated the existence and uniqueness of mild solutions to a hybrid fractional differential equation subject to state-dependent non-local conditions. However, we can not find any paper deals with pantograph equations with state-dependent non-local conditions or/and sources having time-singular coefficients.

It is well-known that involved quantities of physical processes are usually determined experimentally or from analytic mathematical models. In any case, these quantities are obtained only as approximate values. For this reason, studying the continuous dependence of solutions of fractional differential equations with respect to fractional orders and associated quantities play a significant role and is worth investigating. Related to this topic, in [6, 7, 9], we considered the continuity of solutions of Langevin equations regarding to fractional orders and associated parameters. In [11, 12, 13, 24, 25, 26], we proposed some suitable conditions such that solutions of parabolic equations depend continuously on fractional orders and associated parameters. To the best of our knowledge, the continuity of solutions of fractional pantograph problems concerning fractional orders, associated parameters, and state-dependent nonlocal conditions are still not considered.

Motivated by the above analyses, we consider the fractional pantograph equation with a state-dependent nonlocal condition. Precisely, let $T > 0$, $\alpha \in (0, 1]$ and $\lambda \in (0, 1)$, and let $(X, \|\cdot\|)$ be a Banach space. We consider the problem the following equation

$$D_t^\alpha u(t) = f(t, u(t), u(\lambda t)), \quad t \in (0, T] \tag{1.1}$$

subject to the state-dependent non-local condition in the following form

$$u(0) = H(\sigma(u), u) \tag{1.2}$$

where D_t^α is the Caputo fractional derivative defined by

$$D_t^\alpha u(t) = \begin{cases} \dfrac{1}{\Gamma(1-\alpha)} \displaystyle\int_0^t \dfrac{u'(s)}{(t-s)^\alpha} \, \mathrm{d}s & \text{as } 0 < \alpha < 1, \\ u'(t) & \text{as } \alpha = 1. \end{cases}$$

In this work, we aim to consider the cases of Lipschitz and non-Lipschitz sources with time-singular coefficients. Moreover, the continuity of solutions of the problem (1.1)-(1.2) with respect to inputs (fractional order, associated parameter, and state-dependent nonlocal condition) will be discussed. Here is the discrepancy of our work if compared with previous works.

The contributions of this work are:
(i) Give some sufficient conditions such that the problem has at least one solution/unique solution.
(ii) Propose some suitable conditions such that the solution of the problem is dependent continuously on the fractional order, associated parameter, and state-dependent nonlocal condition.

The rest of the paper divide into three sections. In section 2, we present some notations and lemmas that we will use in subsequent sections of the paper. In part 3, we present the main results: the existence, uniqueness of solutions, and dependence continuously of solutions on given data. In section 4, we construct some examples to illustrate the theoretical findings.

2 Preliminaries

This section is devoted to introduce some symbols and lemmas that we will use in the subsequent sections. Throughout the current paper, we also denote $\mathbb{R}_+ = [0, +\infty)$.

Lemma 2.1 (see [8]). *Let $0 < \alpha \le 1$, and $\beta < \alpha$. For $0 \le s \le t \le T$, we define*

$$\phi_{\alpha,\beta}(s,t) = \int_s^t (t-\tau)^{\alpha-1} \tau^{-\beta} \, \mathrm{d}\tau, \quad \phi_{\alpha,\beta}(t) = \int_0^t (t-\tau)^{\alpha-1} \tau^{-\beta} \, \mathrm{d}\tau.$$

Then, one has

$$\phi_{\alpha,\beta}(s,t) \to 0 \ \text{ as } \ |s-t| \to 0, \ \phi_{\alpha,\beta}(t) = t^{\alpha-\beta} B(\alpha, 1-\beta).$$

As a consequence,

$$\phi_{\alpha,\beta}(t) \le \phi_{\alpha,\beta}(T)$$

for any $t \in [0, T]$.

Lemma 2.2 (The nonlinear Leray-Schauder alternatives fixed point theorem [14]). *Let X be a Banach space, and let V be a closed convex subset of X. Let U be a relatively open subset of V and $0 \in U$. Suppose that $\Psi : \overline{U} \to V$ is a continuous compact mapping. Then we have either (i) Ψ has a fixed point in $\overline{U}$ or (ii) there exist $\kappa \in (0, 1)$ and $u \in \partial U$ such that $u = \kappa \Psi u$.*

3 Main results

In this section, we present main results of the paper. First, it is easy to see that the problem (1.1) and (1.2) can transformed to the following integral equation

$$u(t) = H(\sigma(u), u) + \frac{1}{\Gamma(\alpha)} \int_0^t (t-\tau)^{\alpha-1} f(\tau, u(\tau), u(\lambda\tau))\, \mathrm{d}\tau. \tag{3.1}$$

Definition 3.1. A function $u \in C([0,T], X)$ satisfying the integral equation (3.1) is called mild solution of the problem (1.1) and (1.2).

In order to state the main results of the present paper, we list here some assumptions.

- *Assumption ($\mathcal{C}1$).* There exist $\gamma < \alpha$ and continuous, non-decreasing function $\varphi : \mathbb{R}_+ \to \mathbb{R}_+$ such that
$$\|f(t,u,v)\| \le t^{-\gamma}\varphi(\|u\|)$$
for any $u, v \in X$ satisfying $\|v\| \le \|u\|$, and
$$\|f(t,u_1,v_1) - f(t,u_2,v_2)\| \le t^{-\gamma}\vartheta(u_1,u_2,v_1,v_2),$$
where $\vartheta : X \times X \to X$ satisfying $\|\vartheta(u_1,u_2,v_1,v_2)\| \to 0$ as $u_1 \to u_2$ and $v_1 \to v_2$.
- *Assumption ($\mathcal{C}2$).* $H \in C([0,T] \times C([0,T],X), X)$, $\sigma \in C(C([0,T],X),[0,T])$, and there exists a continuous, non-decreasing function $\varrho : \mathbb{R}_+ \to \mathbb{R}_+$ such that
$$\|H(\sigma(u), u)\| \le \varrho(\|u\|)$$
for any $u \in C([0,T],X)$.
- *Assumption ($\mathcal{C}3$).* There exist $\gamma < \alpha$ and $L_f > 0$ such that
$$\|f(t,u_1,v_1) - f(t,u_2,v_2)\| \le L_f t^{-\gamma}(\|u_1-u_2\| + \|v_1-v_2\|)$$
$$\|f(t,0,0)\| \le L_f t^{-\gamma}$$
for any $0 < t \le T$ and $u_1,u_2,v_1,v_2 \in X$.
- *Assumption ($\mathcal{C}4$).* There exist $L_H, L_\sigma > 0$ such that
$$\begin{aligned} \|H(u_1,v_1) - H(u_2,v_2)\| &\le L_H(|u_1-u_2| + \|v_1-v_2\|), \\ \|H(u_1,v_1)\| &\le L_H(|u_1| + \|v_1\| + 1), \\ |\sigma(v_1) - \sigma(v_2)| &\le L_\sigma \|v_1-v_2\| \\ |\sigma(v_1)| &\le L_\sigma(\|v_1\| + 1) \end{aligned}$$
for any $u_1,u_2 \in [0,T]$ and $v_1,v_2 \in C([0,T],X)$.

To study the continuity of solutions to the problem with respect to the input: $(\alpha, \lambda, H, \sigma)$, we consider the following approximate problem

$$D_t^{\alpha_k} u_k(t) = f(t, u_k(t), u_k(\lambda_k t)), \quad t \in (0,T] \tag{3.2}$$

subject to the state-dependent non-local condition

$$u_k(0) = H_k(\sigma_k(u_k), u_k). \tag{3.3}$$

It is easy to see that the problem (3.2) and (3.3) can be transform to the following integral equation

$$u_k(t) = H_k(\sigma_k(u_k), u_k) + \frac{1}{\Gamma(\alpha_k)} \int_0^t (t-\tau)^{\alpha_k-1} f(\tau, u_k(\tau), u_k(\lambda_k\tau))\, \mathrm{d}\tau. \tag{3.4}$$

Moreover, the following assumptions will be posed.

- *Assumption* $(\mathcal{C}5)$. For each k, there exist $L_{H,k}, L_{\sigma,k} > 0$ such that
$$\|H_k(u_1, v_1) - H_k(u_2, v_2)\| \le L_{H,k}\left(|u_1 - u_2| + \|v_1 - v_2\|\right),$$
$$\|H_k(u_1, v_1)\| \le L_{H,k}(|u_1| + \|v_1\| + 1),$$
$$|\sigma_k(v_1) - \sigma_k(v_2)| \le L_{\sigma,k}\|v_1 - v_2\|$$
$$|\sigma_k(v_1)| \le L_{\sigma,k}\left(\|v_1\| + 1\right)$$
for any $u_1, u_2 \in [0, T]$ and $v_1, v_2 \in C([0, T], X)$.
- *Assumption* $(\mathcal{C}6)$. $\alpha_k \to \alpha$, $\lambda_k \to \lambda$, $L_{H,k} \to L_H$, $L_{\sigma,k} \to L_\sigma$, $\sup_{\|u\|\le M} |\sigma_k(u) - \sigma(u)| \to 0$, and $\sup_{\|u\|\le M} \|H_k(\sigma(u), u) - H(\sigma(u), u)\| \to 0$ as $k \to +\infty$.

We are now in a position to state and prove the main results of the paper. The first result is devoted to the existence of solutions to the problem.

Theorem 3.2. *Assume that Assumptions* $(\mathcal{C}1)$ *and* $(\mathcal{C}2)$ *are satisfied. Then the problem (1.1) and (1.2) has at least one mild solution belongs to* $C([0, T], X)$ *whenever there exists* $E > 0$ *such that*
$$E > \varrho(E) + \frac{T^{\alpha-\gamma}\Gamma(\gamma)}{\Gamma(\alpha + 1 - \gamma)}\varphi(E). \tag{3.5}$$

Remark 3.3. If
$$\lim_{x\to+\infty}\left(\varrho(x) + \frac{T^{\alpha-\gamma}\Gamma(\gamma)}{\Gamma(\alpha + 1 - \gamma)}\varphi(x)\right)/x < 1$$
then (3.5) holds for E large enough.

Proof. We begin the proof by considering the operator $\Psi : C([0, T], X) \to C([0, T], X)$ given by
$$\Psi u(t) = H(\sigma(u), u) + \frac{1}{\Gamma(\alpha)}\int_0^t (t - \tau)^{\alpha-1} f(\tau, u(\tau), u(\lambda\tau))\, \mathrm{d}\tau. \tag{3.6}$$
We first verify that Ψ is a compact operator. To this aim, for each $M > 0$, let us define
$$\Omega_M = \{u \in C([0, T], X) : \|u\| \le M\}.$$

Claim 1 We prove that Ψ is a continuous operator. For convenience in writing, we denote $u_\lambda(t) = u(\lambda t)$. It is easy to see that $\|u_\lambda\| = \sup_{0\le t\le T} \|u(\lambda t)\| = \sup_{0\le t\le \lambda T<T} \|u(t)\| \le \|u\|$, and if $u \to v$ then $u_\lambda \to v_\lambda$ in $C([0, T], X)$. Using this fact and in view of Lemma 2.1 together with Assumptions $(\mathcal{C}1)$-$(\mathcal{C}2)$, one has
$$\begin{aligned}
\|\Psi u(t) - \Psi v(t)\| &\le \|H(\sigma(u), u) - H(\sigma(v), v)\| \\
&\quad + \frac{1}{\Gamma(\alpha)}\int_0^t (t - \tau)^{\alpha-1}\|f(\tau, u(\tau), u(\lambda\tau)) - f(\tau, v(\tau), v(\lambda\tau))\|\, \mathrm{d}\tau \\
&\le \|H(\sigma(u), u) - H(\sigma(v), v)\| + \frac{1}{\Gamma(\alpha)}\|\vartheta(u(\cdot), u_\lambda(\cdot), v(\cdot), v_\lambda(\cdot))\|\phi_{\alpha,\gamma}(t) \\
&\le \|H(\sigma(u), u) - H(\sigma(v), v)\| + \frac{1}{\Gamma(\alpha)}\|\vartheta(u(\cdot), u_\lambda(\cdot), v(\cdot), v_\lambda(\cdot))\|\phi_{\alpha,\gamma}(T) \\
&\to 0 \ \text{ as } \ u \to v.
\end{aligned}$$

Claim 2 We show that Ψ maps bounded sets into bounded sets in $C([0, T], X)$. For $u \in \Omega_M$, using Lemma 2.1 and Assumptions $(\mathcal{C}1)$-$(\mathcal{C}2)$, we have
$$\begin{aligned}
\|\Psi u(t)\| &\le \|H(\sigma(u), u)\| + \frac{1}{\Gamma(\alpha)}\int_0^t (t - \tau)^{\alpha-1}\|f(\tau, u(\tau), u(\lambda\tau))\|\, \mathrm{d}\tau \\
&\le \varrho(\|u\|) + \frac{1}{\Gamma(\alpha)}\varphi(\|u\|)\phi_{\alpha,\gamma}(t) \\
&\le \varrho(M) + \frac{1}{\Gamma(\alpha)}\varphi(M)\phi_{\alpha,\gamma}(T).
\end{aligned}$$

Claim 3 Ψ maps bounded sets into equicontinuous set of $C([0,T];X)$. Without lost of general, we assume $0 \le s \le t \le T$ and have $(s-\tau)^{\alpha-1} - (t-\tau)^{\alpha-1} \ge 0$ for any $0 \le \tau < s$. Then, for any $u \in \Omega_M$, we obtain from Assumption ($\mathcal{C}1$) that

$$\begin{aligned}
\|\Psi u(s) - \Psi u(t)\| &\le \frac{1}{\Gamma(\alpha)} \int_0^s \left((s-\tau)^{\alpha-1} - (t-\tau)^{\alpha-1}\right) \|f(\tau, u(\tau), u(\lambda\tau))\| \,\mathrm{d}\tau \\
&\quad - \frac{1}{\Gamma(\alpha)} \int_s^t (t-\tau)^{\alpha-1} \|f(\tau, u(\tau), u(\lambda\tau))\| \,\mathrm{d}\tau \\
&\le \frac{\varphi(\|u\|)}{\Gamma(\alpha)} \left(\int_0^s \left((s-\tau)^{\alpha-1} - (t-\tau)^{\alpha-1}\right) \tau^{-\gamma} \,\mathrm{d}\tau - \int_s^t (t-\tau)^{\alpha-1} \tau^{-\gamma} \,\mathrm{d}\tau \right) \\
&\le \frac{\varphi(\|u\|)}{\Gamma(\alpha)} \left(\int_0^s (s-\tau)^{\alpha-1} \tau^{-\gamma} \,\mathrm{d}\tau - \int_0^t (t-\tau)^{\alpha-1} \tau^{-\gamma} \,\mathrm{d}\tau \right) \\
&\le \frac{\varphi(M) B(\alpha; 1-\gamma)}{\Gamma(\alpha)} \left(s^{\alpha-\gamma} - t^{\alpha-\gamma} \right) \to 0 \ \text{ as } \ |t-s| \to 0.
\end{aligned}$$

Using the compactness of the operator Ψ, we continue the proof of the result of Theorem by defining the set

$$S = \{u \in C([0,T], X) : \|u\| \le E\}.$$

Suppose that there are $\kappa \in (0,1)$ and $u \in \partial S$ such that $u = \kappa \Psi u$, then, similar to the way that we have used in Claim 2 to obtain

$$\|u(t)\| = \|\kappa \Psi u(t)\| \le \|\Psi u(t)\| \le \varrho(E) + \frac{1}{\Gamma(\alpha)} \varphi(E) \phi_{\alpha,\gamma}(T).$$

The latter inequality leads to

$$E \le \varrho(E) + \frac{T^{\alpha-\gamma}\Gamma(\gamma)}{\Gamma(\alpha+1-\gamma)} \varphi(E)$$

due to $\frac{1}{\Gamma(\alpha)} \phi_{\alpha,\gamma}(T) = \frac{T^{\alpha-\gamma}\Gamma(\gamma)}{\Gamma(\alpha+1-\gamma)}$. This contradicts the hypothesis of Theorem. Thus, we conclude from Lemma 2.2 that Ψ has a unique fixed point in $C([0,T], X)$, which is a (unique) mild solution of the problem (1.1) and (1.2). This finishes the proof of Theorem. □

In the next theorem, we present the result on the uniqueness of solutions of the problem.

Theorem 3.4. *Assume that Assumptions ($\mathcal{C}3$) and ($\mathcal{C}4$) hold. Then the problem (1.1) and (1.2) has a unique mild solution provided that*

$$c = L_H(L_\sigma + 1) + 2L_f \frac{T^{\alpha-\gamma}\Gamma(\gamma)}{\Gamma(\alpha+1-\gamma)} < 1.$$

Furthermore, we have the following estimate

$$\|u\| \le \frac{1}{1-c} \left(L_H(L_\sigma + 1) + L_f \frac{T^{\alpha-\gamma}\Gamma(\gamma)}{\Gamma(\alpha+1-\gamma)} \right).$$

Proof. Regarding the operator Ψ given by (3.6), we will prove that Ψ is contraction. Indeed, by virtue of Lemma 2.1 together with Assumptions ($\mathcal{C}3$) and ($\mathcal{C}4$), we have

$$\begin{aligned}
\|\Psi u(t) - \Psi v(t)\| &\le \|H(\sigma(u), u) - H(\sigma(v), v)\| \\
&\quad + \frac{1}{\Gamma(\alpha)} \int_0^t (t-\tau)^{\alpha-1} \|f(\tau, u(\tau), u(\lambda\tau)) - f(\tau, v(\tau), v(\lambda\tau))\| \,\mathrm{d}\tau \\
&\le L_H(L_\sigma + 1)\|u - v\| \\
&\quad + \frac{L_f}{\Gamma(\alpha)} \int_0^t (t-\tau)^{\alpha-1} t^{-\gamma} \left(\|u(\tau) - v(\tau))\| + \|u(\lambda\tau) - v(\lambda\tau))\| \right) \,\mathrm{d}\tau \\
&\le L_H(L_\sigma + 1)\|u - v\| + \frac{2L_f \phi_{\alpha,\gamma}(t)}{\Gamma(\alpha)} \|u - v\| \\
&\le c\|u - v\|,
\end{aligned}$$

where $c = L_H(L_\sigma+1)+2L_f\frac{T^{\alpha-\gamma}\Gamma(\gamma)}{\Gamma(\alpha+1-\gamma)} < 1$. Here we have used the facts that $\frac{\phi_{\alpha,\gamma}(T)}{\Gamma(\alpha)} = \frac{T^{\alpha-\gamma}\Gamma(\gamma)}{\Gamma(\alpha+1-\gamma)}$ and

$$\sup_{0\leq t\leq T} \|u(\lambda t) - v(\lambda t)\| = \sup_{0\leq t\leq \lambda T<T} \|u(t) - v(t)\| \leq \sup_{0\leq t\leq T} \|u(t) - v(t)\| = \|u - v\|.$$

Thus, Ψ is a contraction and admits a unique fixed point in $C([0, T], X)$, which is a (unique) mild solution of the problem (1.1) and (1.2).

To end the proof, we prove the upper bounded estimate. One has from Assumption ($\mathcal{C}3$) that

$$\begin{aligned}\|f(t, u(t), u(\lambda t))\| &\leq \|f(t, u(t), u(\lambda t)) - f(t, 0, 0)\| + \|f(t, 0, 0)\| \\ &\leq L_f t^{-\gamma}(\|u(t)\| + \|u(\lambda t)\|) + L_f t^{-\gamma} \\ &\leq L_f t^{-\gamma}(2\|u\| + 1).\end{aligned} \tag{3.7}$$

Therefore, we have

$$\begin{aligned}\|\Psi u(t)\| &\leq \|H(\sigma(u), u)\| + \frac{1}{\Gamma(\alpha)}\int_0^t (t-\tau)^{\alpha-1}\|f(\tau, u(\tau), u(\lambda\tau))\| \,\mathrm{d}\tau \\ &\leq L_H((L_\sigma + 1)\|u\| + L_\sigma + 1) + \frac{L_f(2\|u\| + 1)}{\Gamma(\alpha)}\phi_{\alpha,\gamma}(t) \\ &\leq c\|u\| + L_H(L_\sigma + 1) + L_f\frac{T^{\alpha-\gamma}\Gamma(\gamma)}{\Gamma(\alpha + 1 - \gamma)},\end{aligned}$$

where $c = L_H(L_\sigma + 1) + 2L_f\frac{T^{\alpha-\gamma}\Gamma(\gamma)}{\Gamma(\alpha+1-\gamma)}$. This gives

$$\|u\| \leq c\|u\| + L_H(L_\sigma + 1) + L_f\frac{T^{\alpha-\gamma}\Gamma(\gamma)}{\Gamma(\alpha + 1 - \gamma)}.$$

Since $c < 1$, then the latter inequality deduces

$$\|u\| \leq \frac{1}{1-c}\left(L_H(L_\sigma + 1) + L_f\frac{T^{\alpha-\gamma}\Gamma(\gamma)}{\Gamma(\alpha + 1 - \gamma)}\right).$$

The proof is done. □

In the last result of the paper, we introduce a result on the continuity of solutions to the problem with respect to fractional order, associated parameter, and state-dependent nonlocal condition.

Theorem 3.5. *Assume that Assumptions ($\mathcal{C}3$)-($\mathcal{C}6$) hold, and*

$$c = L_H(L_\sigma + 1) + 2L_f\frac{T^{\alpha-\gamma}\Gamma(\gamma)}{\Gamma(\alpha + 1 - \gamma)} < 1.$$

Then the problem (1.1) and (1.2) has a unique mild solution, and there exist $N \in \mathbb{N}^*$ such that the approximate problem (3.2) and (3.3) has a unique mild solution for each $k \geq N$. Moreover, we have

$$u_k \to u \;\; in \;\; C([0, T], X). \tag{3.8}$$

Proof. Put

$$c_k = L_{H,k}(L_{\sigma,k} + 1) + 2L_f\frac{T^{\alpha_k-\gamma}\Gamma(\gamma)}{\Gamma(\alpha_k + 1 - \gamma)}. \tag{3.9}$$

Using Assumption ($\mathcal{C}6$), we can see that $c_k \to c < 1$ and $\alpha_k \to \alpha > \gamma$ as $k \to +\infty$. This implies that there is a $N \in \mathbb{N}^*$ such that $c_k < 1$ and $\alpha_k > \gamma$ for all $k \geq N$. Using (3.4) and Theorem 3.4, we conclude that the approximate problem (3.2) and (3.3) has a unique mild solution each $k \geq N$.

Finally, we prove (3.8). We define

$$\Psi_k w(t) = H_k(\sigma_k(w), w) + \frac{1}{\Gamma(\alpha_k)} \int_0^t (t-\tau)^{\alpha_k - 1} f(\tau, w(\tau), w(\lambda_k \tau))\, \mathrm{d}\tau, \tag{3.10}$$

and denote u_k being a (unique) mild solution of equation $\Psi_k w = w$ with k large enough (which is mild solution of the problem (3.2) and (3.3)).

Using the upper bound estimate in Theorem 3.4 and Assumption ($\mathcal{C}6$), we can find a positive number M such that

$$\|u\| \le M$$

for all $k \ge N$. Using (3.7) and the latter inequality, we have

$$\|f(t, u(t), u(\lambda t))\| \le L_f t^{-\gamma}(2\|u\| + 1) \le L_f(2M+1)t^{-\gamma}. \tag{3.11}$$

Step 1 Using Assumption ($\mathcal{C}5$), one have

$$\begin{aligned}\|H_k(\sigma_k(u_k), u_k) - H_k(\sigma(u), u)\| &\le L_{H,k}\left(|\sigma_k(u_k) - \sigma(u)| + \|u_k - u\|\right)\\ &\le L_{H,k}\left(|\sigma_k(u_k) - \sigma_k(u)| + \|u_k - u\|\right) + L_{H,k}|\sigma_k(u) - \sigma(u)|\\ &\le L_{H,k}(L_{\sigma,k}+1)\|u_k - u\| + L_{H,k}|\sigma_k(u) - \sigma(u)|\end{aligned}$$

Using the latter inequality and Assumptions ($\mathcal{C}5$)-($\mathcal{C}6$), we have

$$\begin{aligned}J_k^{(1)} &= \|H_k(\sigma_k(u_k), u_k) - H(\sigma(u), u)\|\\ &\le \|H_k(\sigma_k(u_k), u_k) - H_k(\sigma(u), u)\| + \|H_k(\sigma(u), u) - H(\sigma(u), u)\|\\ &\le L_{H,k}(L_{\sigma,k}+1)\|u_k - u\| + \delta_k,\end{aligned}$$

where

$$\begin{aligned}\delta_k &= \|H_k(\sigma(u), u) - H(\sigma(u), u)\| + L_{H,k}|\sigma_k(u) - \sigma(u)|\\ &\le \sup_{\|u\|\le M} \|H_k(\sigma(u), u) - H(\sigma(u), u)\| + L_{H,k} \sup_{\|u\|\le M} |\sigma_k(u) - \sigma(u)| \to 0\end{aligned}$$

as $k \to +\infty$.

Step 2 Using (3.11) and Assumption ($\mathcal{C}6$), one has

$$\begin{aligned}J_k^{(2)}(t) &= \left|\frac{1}{\Gamma(\alpha)} - \frac{1}{\Gamma(\alpha_k)}\right| \int_0^t (t-\tau)^{\alpha-1}\|f(\tau, u(\tau), u(\lambda\tau))\|\, \mathrm{d}\tau\\ &\le \left|\frac{1}{\Gamma(\alpha)} - \frac{1}{\Gamma(\alpha_k)}\right| L_f(2M+1)\phi_{\alpha,\gamma}(t) \le \left|\frac{1}{\Gamma(\alpha)} - \frac{1}{\Gamma(\alpha_k)}\right| L_f(2M+1)\phi_{\alpha,\gamma}(T) \to 0\end{aligned}$$

uniformly for $t \in [0, T]$ as $k \to +\infty$.

Step 3 Since $g(\tau) = (t-\tau)^{\alpha-1} - (t-\tau)^{\alpha_k-1}$ does not change sign on the interval $[0, t]$ for each α_k. Thus, using (3.11) and Assumption ($\mathcal{C}6$), we have

$$\begin{aligned}J_k^{(3)}(t) &= \int_0^t \left|(t-\tau)^{\alpha-1} - (t-\tau)^{\alpha_k-1}\right| \|f(\tau, u(\tau), u(\lambda\tau))\|\, \mathrm{d}\tau\\ &\le L_f(2M+1)\int_0^t \left|(t-\tau)^{\alpha-1} - (t-\tau)^{\alpha_k-1}\right| \tau^{-\gamma}\, \mathrm{d}\tau\\ &= L_f(2M+1)\left|\int_0^t \left((t-\tau)^{\alpha-1} - (t-\tau)^{\alpha_k-1}\right)\tau^{-\gamma}\, \mathrm{d}\tau\right|\\ &\le \left|t^{\alpha-\gamma}B(\alpha, 1-\gamma) - t^{\alpha_k-\gamma}B(\alpha_k, 1-\gamma)\right| \to 0\end{aligned}$$

uniformly for $t \in [0, T]$ as $k \to +\infty$.

Step 4 Using the fact that

$$\begin{aligned}\|u(\lambda t) - u_k(\lambda_k t)\| &\le \|u(\lambda t) - u(\lambda_k t)\| + \|u(\lambda_k t) - u_k(\lambda_k t)\| \\ &\le \sup_{0\le t\le T} \|u(\lambda t) - u(\lambda_k t)\| + \|u - u_k\|\end{aligned}$$

and Assumption ($\mathcal{C}3$), we have

$$\begin{aligned}J_k^{(4)}(t) &= \int_0^t (t-\tau)^{\alpha_k - 1} \|f(\tau, u(\tau), u(\lambda\tau)) - f(\tau, u_k(\tau), u_k(\lambda_k\tau))\| \,\mathrm{d}\tau \\ &\le L_f \int_0^t (t-\tau)^{\alpha_k-1}\tau^{-\gamma} \left(\|u(\tau) - u_k(\tau)\| + \|u(\lambda\tau) - u_k(\lambda_k\tau)\|\right) \mathrm{d}\tau \\ &\le 2L_f\phi_{\alpha_k,\gamma}(t)\|u_k - u\| + L_f\phi_{\alpha_k,\gamma}(t) \sup_{0\le t\le T}\|u(\lambda t) - u(\lambda_k t)\| \\ &\le 2L_f\phi_{\alpha_k,\gamma}(T)\|u_k - u\| + I_k,\end{aligned}$$

where $I_k = L_f\phi_{\alpha_k,\gamma}(T)\sup_{0\le t\le T}\|u(\lambda t) - u(\lambda_k t)\| \to 0$ as $k \to +\infty$.

Step 5 Regarding the operators Ψ and Ψ_k given by (3.6) and (3.10), respectively, combining Step 1 – Step 4, and a direct computation, we obtain

$$\begin{aligned}\|u_k(t) - u(t)\| &= \|\Psi_k u_k(t) - \Psi u(t)\| \\ &\le J_k^{(1)} + J_k^{(2)}(t) + \frac{1}{\Gamma(\alpha_k)}\left(J_k^{(3)}(t) + J_k^{(4)}(t)\right) \\ &\le c_k\|u_k - u\| + \theta_k(t),\end{aligned}$$

where $\theta_k(t) = \delta_k + J_k^{(2)}(t) + \left(J_k^{(3)}(t) + I_k\right)/\Gamma(\alpha_k) \to 0$ uniformly for $t \in [0,T]$ as $k \to +\infty$. The latter inequality implies that

$$\|u_k - u\| \le \frac{1}{1-c_k}\sup_{0\le t\le T}\theta_k(t) \to 0$$

as $k \to +\infty$. This completes the proof of Theorem. □

4 Examples

As applications of the obtained results, two illustrative examples are given as follows.

Example 4.1. Consider the fractional pantograph problem in $X = \mathbb{R}$

$$\begin{cases} D^{0.75}u(t) = t^{-0.25}u^{0.2} + u^{0.4}(0.8t) + t^{-0.5}\sin t, & 0 < t \le T = 1, \\ u(0) = 0.1u(0.5) + 0.2u(1), \end{cases} \tag{4.1}$$

where $\alpha = 0.75$, $\lambda = 0.8$, $f(t,u,v) = t^{-0.25}u^{0.2} + v^{0.4} + t^{-0.5}\sin t$, $H(\sigma(u),u) = H(u) = 0.1u(0.5) + 0.2u(1)$. We can directly check that $\|f(t,u,v)\| \le t^{-\gamma}\varphi(\|u\|)$ for all $\|u\| \ge \|v\|$ with $\gamma = 0.5$, $\varphi(x) = x^{0.2} + x^{0.4} + 1$, and $\|f(t,u_1,v_1) - f(t,u_2,v_2)\| \le t^{-\gamma}\vartheta(u_1,u_2,v_1,v_2)$ with $\vartheta(u_1,u_2,v_1,v_2) = u_1^{0.2} + v_1^{0.4} - u_2^{0.2} - v_2^{0.4}$. We also have $H \in C(C([0,1],\mathbb{R}),\mathbb{R})$ and $\|H(u)\| \le \varrho(\|u\|)$ with $\varrho(x) = 0.3x$. It is clear that φ, ϱ are continuous, non-decreasing functions and $\|\vartheta(u_1,u_2,v_1,v_2)\| \to 0$ as $u_1 \to u_2$ and $v_1 \to v_2$. Thus, Assumptions ($\mathcal{C}1$) and ($\mathcal{C}2$) are satisfied. Since $\lim_{x\to+\infty}\left(\varrho(x) + \frac{T^{\alpha-\gamma}\Gamma(\gamma)}{\Gamma(\alpha+1-\gamma)}\varphi(x)\right)/x = 0.3 < 1$, we find from Remark 3.3 that the condition (3.5) holds. So, we conclude from Theorem 3.2 that the problem (4.1) has at least one mild solution.

Example 4.2. Consider the fractional pantograph problem in $X = \mathbb{R}$

$$\begin{cases} D^{0.9}u(t) = 0.1t^{-0.5}/(|u|+1) + 0.15u(0.9t), & 0 < t \le T = 1, \\ u(0) = 0.15\int_0^{\|u\|/(\|u\|+1)} 1/(|u(\tau)|^2+1)\,\mathrm{d}\tau, \end{cases} \tag{4.2}$$

where $\alpha = \lambda = 0.9$, $f(t,u,v) = 0.1t^{-0.5}/(|u|+1)+0.2v$, $\sigma(u) = \|u\|/(\|u\|+1)$, $H(\sigma(u),u) = 0.15\int_0^{\sigma(u)} 1/(|u(\tau)|^2+1)\,\mathrm{d}\tau$. We can easily check that $f(t,0,0) = 0$, and $\|f(t,u_1,v_1) - f(t,u_2,v_2)\| \le L_f t^{-\gamma}(\|u_1-u_2\| + \|v_1-v_2\|)$ with $L_f = 0.15, \gamma = 0.5$, and

$$\begin{aligned}
&|H(u_1,v_1) - H(u_2,v_2)| \\
&\le 0.15\left|\int_0^{u_1} \frac{1}{|v_1(\tau)|^2+1}\,\mathrm{d}\tau - \int_0^{u_2} \frac{1}{|v_2(\tau)|^2+1)\,\mathrm{d}\tau}\right| \\
&\le 0.15\left|\int_0^{u_1} \left(\frac{1}{|v_1(\tau)|^2+1} - \frac{1}{|v_2(\tau)|^2+1}\right)\mathrm{d}\tau\right| + 0.15\left|\int_{u_1}^{u_2} \frac{1}{|v_2(\tau)|^2+1}\,\mathrm{d}\tau\right| \\
&\le 0.15\left|\int_0^{1} \frac{(|v_2(\tau)-v_1(\tau))(v_2(\tau)+v_1(\tau))}{(|v_1(\tau)|^2+1))(|v_2(\tau)|^2+1))}\,\mathrm{d}\tau\right| + 0.15\left|\int_{u_1}^{u_2} \frac{1}{|v_2(\tau)|^2+1}\,\mathrm{d}\tau\right| \\
&\le L_H\left(|u_1-u_2| + \|v_1-v_2\|\right)
\end{aligned}$$

for all $u1, u_2 \in [0,1]$, $v_1, v_2 \in C([0,1],\mathbb{R})$ with $L_H = 0.15$. We also have

$$\begin{aligned}
|\sigma(v_1) - \sigma(v_1)| &= \left|\frac{\|v_1\|}{\|v_1\|+1} - \frac{\|v_2\|}{\|v_2\|+1}\right| \\
&\le L_\sigma \|v_1 - v_2\|
\end{aligned}$$

for all $v_1, v_2 \in C([0,1],\mathbb{R})$ with $L_\sigma = 1$. On the other hand, one has $\Gamma(\alpha+1-\gamma) = \Gamma(1.4) \approx 0.8872638, \Gamma(\gamma) = \Gamma(0.5) \approx 1.772454$. Thus, we obtain

$$c = L_H(L_\sigma+1) + 2L_f\frac{T^{\alpha-\gamma}\Gamma(\gamma)}{\Gamma(\alpha+1-\gamma)} \approx 0.899 < 1.$$

Using Theorem 3.4, we deduce that the problem (4.2) has a unique mild solution.

Now we consider the approximate problem of the problem (4.2) as follows

$$\begin{cases} D^{0.9-0.2/k}u_k(t) = 0.1t^{-0.5}/(|u_k|+1) + 0.15u_k((0.9-0.2/k)t), \quad 0 < t \le T = 1, \\ u_k(0) = (0.15+0.1/k^2)\int_0^{(1-1/(2k))\|u\|/(\|u\|+1)} u_k(\tau)\,\mathrm{d}\tau, \end{cases} \tag{4.3}$$

where $\alpha_k = 0.9 - 0.2/k \to 0.9 = \alpha$, $\lambda_k = 0.9 - 0.2/k \to 0.9 = \lambda$. We can also check that $L_{H,k} = (0.15+0.1/k^2) \to 0.15 = L_H$, $L_{\sigma,k} = 1 - 1/(2k) \to 1 = L_\sigma$, and

$$\sup_{\|u\|\le M} |H_k(\sigma(u),u) - H(\sigma(u),u)| = 1/(2k)\int_0^{\sigma(u)} u(\tau)\,\mathrm{d}\tau \le 1/(2k)\|u\| \le M/(2k) \to 0$$

as $k \to +\infty$. Using Theorem 3.5, we conclude that the problem (4.3) has a unique mild solution when k large enough. Moreover, we have $u_k \to u$ in $C([0,1],\mathbb{R})$ as $k \to +\infty$.

5 Conclusions

We have considered a nonlinear pantograph equation involving Caputo fractional derivative with the state-dependent non-local conditions. We obtained two results on the existence and uniqueness of solutions to the problem where the source function has time-singular coefficients. We have also proposed some suitable conditions so that solutions to the problem is dependent continuously on fractional order, associated parameter, and state-dependent non-local condition. In future works, we would like to investigate nonlinear pantograph equations involving generalized Caputo fractional derivative or generalized Hiffer fractional derivative with delays.

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