Palestine Journal of Mathematics

Vol. 12(4)(2023) , 215–229

Generalization of S-primary superideals over commutative super-rings

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Communicated by Ayman Badawi

MSC 2010 Classifications: Primary 13A15; Secondary 13C05.

Keywords and phrases: Prime superideal, S-prime ideal, δ-primary ideal, ϕ-δ-primary ideal.

The authors would like to thank the reviewers and the editor for their constructive comments and valuable suggestions that improved the quality of our paper.

Abstract Let R be a commutative super-ring with unity $(1 \neq 0)$ and let $\mathfrak{J}(R)$ be the set of all superideals of R. Let $\phi : \mathfrak{J}(R) \to \mathfrak{J}(R) \cup \{\emptyset\}$ be a reduction function of superideals of R and let $\delta : \mathfrak{J}(R) \to \mathfrak{J}(R)$ be an expansion function of superideals of R. We recall that a proper superideal I of R is called a ϕ-δ-primary superideal of R if whenever $a, b \in h(R)$ and $ab \in I - \phi(I)$, then $a \in I$ or $b \in \delta(I)$. In this paper, we introduce a new class of superideals that is a generalization to the class of ϕ-δ-primary superideals. Let S be a multiplicative subset of $h(R)$ such that $1 \in S$ and let I be a proper superideal of R with $S \cap I = \emptyset$, then I is called a ϕ-δ-S-primary superideal of R associated to $s \in S$ if whenever $a, b \in h(R)$ and $ab \in I - \phi(I)$, then $sa \in I$ or $sb \in \delta(I)$. In this paper, we have presented a range of different examples, properties, and characterizations of this new class of superideals.

1 Introduction

A supercase on a ring is a $\mathbb{Z}_2$-grading on that ring. In general the grading on a ring, or a module, usually leads computation by allowing one to focus on the homogeneous elements, which are simpler and easier than random elements. However, to do this work you need to know that the constructions being studied are graded. One approach to this issue is to redefine the constructions entirely in terms of graded modules and avoid any consideration of non-graded modules or non-homogeneous elements. Unfortunately, while such an approach helps to understand the graded modules, it will only help to understand the original construction, where the graded version of the concept coincides with the original one. Therefore, notably, the studying of the graded rings (or modules) is very important.

Let R be a commutative ring with unity $(1 \neq 0)$, then R is called a commutative super-ring if R is a $\mathbb{Z}_2$-graded ring such that if $a, b \in \mathbb{Z}_2$ then $R_a R_b \subseteq R_{a+b}$ where the subscripts are taken modulo 2. Let $h(R) = R_0 \cup R_1$. Then $h(R)$ is the set of homogeneous elements in R and $1 \in R_0$. Moreover, if $x \in R_i$ for some $i \in \mathbb{Z}_2$, then we say that x is of degree i ($|x| = i$). An ideal I of a super-ring R is called a superideal if $I = (I \cap R_0) \oplus (I \cap R_1) = I_0 \oplus I_1$. Moreover, if $x \in I$, where I is a superideal in a super-ring R, then x is written uniquely in the form $x = x_0 + x_1$, where $x_i \in I_i$ for $i = 0, 1$. A super radical of a superideal I is denoted by Srad(I) and it is defined as follows Srad$(I) = \{x = x_0 + x_1 \in R \ : \ \text{there exist } n_0, n_1 \in \mathbb{N} \text{ such that } x_0^{n_0}, x_1^{n_1} \in I\}$. Note that if I is a superideal of R, then Srad(I) is a superideal. Also, a superhomomorphism is nothing but a homomorphism of super-rings that is $\mathbb{Z}_2$-gradation preserving. Let $R = R_0 + R_1$ be a super-ring and let I be a superideal of R. Then the quotient ring R/I is also a super-ring. Indeed, $R/I = (R/I)_0 + (R/I)_1$ where $(R/I)_i = \{x + I : x \in R_i\}$, for $i = 0, 1$. Also, if A and B are two super-rings, then $A \times B$ is a super-ring with $(A \times B)_0 = (A_0 \times B_0)$ and $(A \times B)_1 = (A_1 \times B_1)$. Let R be a super-ring and let S be a multiplicative closed subset of $h(R)$. Then the ring of fractions $S^{-1}R$ is a super-ring which is called the super-ring of fractions.

Indeed, $S^{-1}R = \left(S^{-1}R\right)_0 \oplus \left(S^{-1}R\right)_1$ where

$$\left(S^{-1}R\right)_i = \left\{\frac{r}{s} : r \in h(R),\ s \in S \text{ and } i = |r| + |s|\right\}.$$

Let $\eta : R \to S^{-1}R$ be a superhomomorphism defined by $\eta(r) = \frac{r}{1}$. Then for any superideal I of R, the superideal of $S^{-1}R$ generated by $\eta(I)$ is denoted by $S^{-1}I$. Similar to non-graded cases one can prove that

$$S^{-1}I = \left\{\lambda \in S^{-1}R : \lambda = \frac{r}{s} \text{ for } r \in I \text{ and } s \in S\right\},$$

and that $S^{-1}I \neq S^{-1}R$ if and only if $S \cap I = \emptyset$. If $\mathcal{J}$ is a superideal in $S^{-1}R$, then $\mathcal{J} \cap R$ denotes the superideal $\eta^{-1}(\ \mathcal{J}\ )$ of R. Moreover, one can prove that $S^{-1}I \cap h(R) = \{x \in h(R) : xs \in I \text{ for some } s \in S\}$. Throughout, R will be a commutative super-ring with unity. Let S be a multiplicative subset of $h(R)$ such that $1 \in S$. In this paper, we call a proper superideal I of R, with $I \cap S = \emptyset$, a ϕ-δ-S-primary superideal of R associated to some $s \in S$ if whenever $a, b \in h(R)$ such that $ab \in I - \phi(I)$, then $sa \in I$ or $sb \in \delta(I)$, where ϕ and δ are reduction and expansion functions of superideals of R, respectively.
Among many results in this paper, it is shown in Proposition 2.18 that if I is an α-ϕ-δ-S-primary superideal of R associated to some $s \in S$ which is not α-δ-S-primary where $\alpha \in \mathbb{Z}_2$, then $I_\alpha^2 \subseteq \phi(I)$. Theorem 2.19 proves that a proper superideal I of R is an α-ϕ-δ-S-primary superideal of R associated to some $s \in S$ if and only if for each homogenous element $a \in R_\alpha - (\delta(I) : s)$ we have either $(I :_{R_\alpha} a) \subseteq (I :_{R_\alpha} s)$ or $(I :_{R_\alpha} a) = (\phi(I) :_{R_\alpha} a)$, where $\alpha \in \mathbb{Z}_2$. Similarly, in Theorem 2.20, we prove that a proper superideal I of R is an α-ϕ-δ-S-primary superideal of R associated to some $s \in S$ if and only if for each homogenous element $a \in R_\alpha - (I : s)$ we have either $(I :_{R_\alpha} a) \subseteq (\delta(I) :_{R_\alpha} s)$ or $(I :_{R_\alpha} a) = (\phi(I) :_{R_\alpha} a)$, where $\alpha \in \mathbb{Z}_2$. In Theorem 2.29, we show that if S satisfies the condition $\phi(I) = (\phi(I) : t)\ \forall t \in S$, then I is a ϕ-δ-S-primary superideal of R associated to some $s \in S$ if and only if $(I : s)$ is a ϕ-δ-primary superideal of R if and only if $S^{-1}I$ is a ϕ_S-δ_S-primary superideal of $S^{-1}R$ and $S^{-1}I \cap h(R) = h((I : s))$.
In the next section, let $f : X \to Y$ be a nonzero surjective $(\delta, \phi) - (\gamma, \psi)$-superhomomorphism. In Theorem 3.3, we prove that f induces one-to-one correspondence between ϕ-δ-S-primary superideals of X associated to some $s \in S$ consisting ker(f) and ψ-γ-$f(S)$-primary superideal of Y associated to $f(s) \in f(S)$. Also, in Theorem 3.6, we prove that if $a, b \in h(X)$, then (a, b) is a ϕ-δ-S-twin zero of I, where I is a ϕ-δ-S-primary superideal of X associated to some $s \in S$ consisting ker(f), if and only if $(f(a), f(b))$ is a ψ-γ-$f(S)$-twin zero of $f(I)$.
In the last section, we determine all ϕ-δ-S-primary superideals in direct product of super-rings and we prove some results concerning ϕ-δ-S-primary superideals in direct product of super-rings. (See Theorems 4.1- 4.4).

2 Properties of ϕ-δ-S-Primary superideals

Definition 2.1. Let R be a commutative super-ring with unity ($1 \neq 0$), and let $\mathfrak{J}(R)$ be the set of all superideals of R.
(1) Generalization of [10], a function $\delta : \mathfrak{J}(R) \to \mathfrak{J}(R)$ is called an expansion function of superideals of R if whenever I, J, K are superideals of R with $J \subseteq I$, then $\delta(J) \subseteq \delta(I)$ and $K \subseteq \delta(K)$.
(2) Generalization of [6], a function $\phi : \mathfrak{J}(R) \to \mathfrak{J}(R)$ is called a reduction function of superideals of R if $\phi(I) \subseteq I$ for all superideals I of R and if whenever $P \subseteq Q$, where P and Q are superideals of R, then $\phi(P) \subseteq \phi(Q)$.

Definition 2.2. Let R be a commutative super-ring with unity ($1 \neq 0$), and S a multiplicative subset of $h(R)$. Suppose δ, ϕ are expansion and reduction functions of superideals of R, respectively.
(1) A proper superideal I of R satisfying $I \cap S = \emptyset$ is said to be a δ-S-primary superideal of R associated to $s \in S$, if whenever $ab \in I$, then $sa \in I$ or $sb \in \delta(I)$ for all $a, b \in h(R)$.
(2) A proper superideal I of R satisfying $I \cap S = \emptyset$ is said to be a ϕ-δ-S-primary superideal of R associated to $s \in S$, if whenever $ab \in I - \phi(I)$, then $sa \in I$ or $sb \in \delta(I)$ for all $a, b \in h(R)$.

Throughout this section, R denotes a commutative super-ring with unity ($1 \neq 0$), S denotes a multiplicative subset of $h(R)$ such that $1 \in S$, $\delta, \gamma : \mathfrak{J}(R) \to \mathfrak{J}(R)$ denote expansion functions, and $\phi, \psi : \mathfrak{J}(R) \to \mathfrak{J}(R)$ denote reduction functions.

In the following example, we recall from [3] some examples of expansion functions and generalize them to the supercase of a given super-ring R.

Example 2.3.
(1) The identity function δ_0, where $\delta_0(I) = I$ for any $I \in \mathfrak{J}(R)$.
(2) For each superideal I of R define $\delta_1(I)$ by $\delta_1(I) = Srad(I)$ for every superideal I in $\mathfrak{J}(R)$.
(3) Let J be a given proper superideal of R, define $\delta(I)$ by $\delta(I) = I + J$ for every superideal I in $\mathfrak{J}(R)$.
(4) Let J be a given proper superideal of R, define $\delta(I)$ by $\delta(I) = (I : J)$ for every superideal I in $\mathfrak{J}(R)$.
(5) Assume that δ_1, δ_2 are expansion functions of superideals of R. Define $\delta(I)$ by $\delta(I) = \delta_1(I) + \delta_2(I)$ for every superideal I in $\mathfrak{J}(R)$.
(6) Assume that δ_1, δ_2 are expansion functions of superideals of R. Define $\delta(I)$ by $\delta(I) = \delta_1(I) \cap \delta_2(I)$ for every superideal I in $\mathfrak{J}(R)$.
(7) Assume that δ_1,...,δ_n are expansion functions of superideals of R. Define $\delta(I)$ by $\delta(I) = \cap_{i=1}^{n} \delta_i(I)$ for every superideal I in $\mathfrak{J}(R)$.
(8) Assume that δ_1, δ_2 are expansion functions of superideals of R. Define $\delta(I)$ by $\delta(I) = \delta_1(\delta_2(I))$ for every superideal I in $\mathfrak{J}(R)$.

Recall that if $\psi_1, \psi_2 : \mathfrak{J}(R) \to \mathfrak{J}(R) \cup \{\emptyset\}$ are expansion (reduction) functions of superideals of R, we define $\psi_1 \leq \psi_2$ if $\psi_1(I) \subseteq \psi_2(I)$ for each $I \in \mathfrak{J}(R)$.

In the following example, we recall from [2] some examples of reduction functions and generalize them to the supercase of a given super-ring R.

Example 2.4.
(1) The function $\phi_\emptyset$, where $\phi_\emptyset(I) = \emptyset$ for any $I \in \mathfrak{J}(R)$.
(2) The function ϕ_0, where $\phi_0(I) = \{0\}$ for any $I \in \mathfrak{J}(R)$.
(3) The function ϕ_2, where $\phi_2(I) = I^2$ for any $I \in \mathfrak{J}(R)$.
(4) The function ϕ_n, where $\phi_n(I) = I^n$ for any $I \in \mathfrak{J}(R)$.
(5) The function ϕ_ω, where $\phi_\omega(I) = \cap_{n=1}^{\infty} I^n$ for any $I \in \mathfrak{J}(R)$.
(6) The function ϕ_1, where $\phi_1(I) = I$ for any $I \in \mathfrak{J}(R)$.
Observe that $\phi_\emptyset \leq \phi_0 \leq \phi_\omega \leq \cdots \leq \phi_{n+1} \leq \phi_n \leq \cdots \leq \phi_2 \leq \phi_1$.

Remark 2.5.
(1) If $\delta \leq \gamma$. Then every ϕ-δ-S-primary superideal of R is a ϕ-γ-S-primary superideal. In particular, every ϕ-S-prime superideal of R is a ϕ-δ-S-primary superideal. However, the converse is not true in general.
(2) If $\phi \leq \psi$. Then every ϕ-δ-S-primary superideal of R is a ψ-δ-S-primary superideal. In particular, every δ-S-primary superideal of R is a ϕ-δ-S-primary superideal. However, the converse is not true in general.

Example 2.6.
(1) Set $R = \mathbb{Z}_{12} + u\mathbb{Z}_{12}$, where $u^2 = 1$. Then R is a super-ring with $R_0 = \mathbb{Z}_{12}$, $R_1 = u\mathbb{Z}_{12}$. Let $I = 4\mathbb{Z}_{12} + u4\mathbb{Z}_{12}$. Then I is a superideal of R and $\delta_1(I) = Srad(I) = 2\mathbb{Z}_{12} + u2\mathbb{Z}_{12}$. Take $S = \{1\}$, $\phi = \phi_\emptyset$. Then it is easy to check that I is a δ_1-S-primary superideal of R. Moreover, I is not an S-prime superideal, since $(2)(2) = 4 \in I$ but $2 \notin I$.
(2) Let $R = \mathbb{Z}_{12} + u\mathbb{Z}_{12}$, where $u^2 = 1$. Then R is a super-ring with $R_0 = \mathbb{Z}_{12}$, $R_1 = u\mathbb{Z}_{12}$. Let $S = \{1, 5\}$. Then S is a multiplicative subset of $h(R)$. Let $I = \{0\}$. Then $\delta_1(I) = Srad(I) = 6\mathbb{Z}_{12} + u6\mathbb{Z}_{12}$, $\phi_2(I) = I^2 = (0)$. So, I is an almost-δ_1-S-primary superideal of R associated to $s = 5$. Moreover, $(3)(4) = 0 \in I$ but neither $(3)(5) = 4 \in \delta_1(I)$ nor $(4)(5) = 8 \in \delta_1(I)$. Thus, I is not a δ_1-S-primary superideal of R associated to $s = 5$.

Proposition 2.7. Let $\{J_i : i \in \triangle\}$ be a directed set of ϕ-δ-S-primary superideals of R associated to $s \in S$. Then the superideal $J = \cup_{i \in \triangle} J_i$ is a ϕ-δ-S-primary superideal of R associated to $s \in S$.

Proof. Let $ab \in J - \phi(J)$, where $a, b \in h(R)$. Suppose $sa \notin J$. We want to show that $sb \in \delta(J)$. Since $ab \notin \phi(J)$, we have $ab \notin \phi(J_i)$ for all $i \in \triangle$. Let $t \in \triangle$ such that $ab \in J_t - \phi(J_t)$, then $sa \in J_t$ or $sb \in \delta(J_t)$, since J_t is a ϕ-δ-S-primary superideal of R associated to $s \in S$. Since $sa \notin J$, we have $sa \notin J_t$ which implies that $sb \in \delta(J_t) \subseteq \delta(J)$. Hence J is a ϕ-δ-S-primary superideal of R associated to $s \in S$. □

Proposition 2.8. Let $\{Q_i : i \in \triangle\}$ be a directed set of ϕ-δ-S-primary superideals of R associated to $s \in S$. Suppose $\phi(Q_i) = \phi(Q_j)$ and $\delta(Q_i) = \delta(Q_j)$ for every $i, j \in \triangle$. If ϕ, δ have the intersection property, then the superideal $J = \cap_{i \in \triangle} Q_i$ is a ϕ-δ-S-primary superideal of R associated to $s \in S$.

Proof. Let $t \in \triangle$, since $\phi(Q_i) = \phi(Q_t)$ and $\delta(Q_i) = \delta(Q_t)$ for every $i \in \triangle$, and since ϕ, δ have the intersection property, then $\phi(J) = \phi(Q_t)$ and $\delta(J) = \delta(Q_t)$. Let $ab \in J - \phi(J)$, where $a, b \in h(R)$ such that $sb \notin \delta(J)$. Then $ab \in Q_t - \phi(Q_t)$. Since Q_t is a ϕ-δ-S-primary superideal of R associated to $s \in S$, we conclude that $sa \in Q_t$ or $sb \in \delta(Q_t)$. Since $sb \notin \delta(J)$, $sb \notin \delta(Q_t) = \delta(J)$. Hence we conclude that $sa \in Q_t$ for each $t \in \triangle$ which implies that $sa \in J$. Thus, J is a ϕ-δ-S-primary superideal of R associated to $s \in S$. □

Obviously, every ϕ-δ-primary superideal of R is a ϕ-δ-S-primary superideal. In particular, every weakly-δ-primary (δ-primary) superideal of R is a weakly-δ-S-primary (δ-S-primary). However, the next two examples show that the converse is not true in general.

Example 2.9. Let $R = \mathbb{Z}_{80} + u\mathbb{Z}_{80}$, where $u^2 = 1$. Then R is a super-ring with $R_0 = \mathbb{Z}_{80}$, $R_1 = u\mathbb{Z}_{80}$, then $I = 20\mathbb{Z}_{80} + u20\mathbb{Z}_{80}$ is a superideal of R. Let $S = \{1, 5, 25, 45, 65\}$. Then S is a multiplicative subset of $h(R)$ such that $I \cap S = \emptyset$. Let $\delta = \delta_1$ and $\phi = \phi_0$, then $\delta_1(I) = Srad(I) = 10\mathbb{Z}_{80} + u10\mathbb{Z}_{80}$ and $\phi_0(I) = (0)$. Let $a, b \in h(R)$ such that $0 \neq ab \in I$, then $2/ab$ which implies that $2/a$ or $2/b$. Thus, $5a \in \delta_1(I)$ or $5b \in \delta_1(I)$. Hence we conclude that I is a weakly-δ_1-S-primary superideal of R associated to $s = 5$. Moreover, $0 \neq (4)(5) = 20 \in I$ but neither $4 \in \delta_1(I)$ nor $5 \in \delta_1(I)$. Thus, I is not a weakly-δ_1-primary superideal.

Example 2.10. Let $R = \mathbb{Z}[x] + u\mathbb{Z}[x]$, where $u^2 = 1$. Then R is a super-ring with $R_0 = \mathbb{Z}[x]$, $R_1 = u\mathbb{Z}[x]$. Let $I = 4x\mathbb{Z}[x] + u4x\mathbb{Z}[x]$. Let $\phi_2(I) = I^2 = 16x^2\mathbb{Z}[x] + u16x^2\mathbb{Z}[x]$ and $\delta_1(I) = Srad(I) = 2x\mathbb{Z}[x] + u2x\mathbb{Z}[x]$. Let $S = \{2^k : k \geq 0\}$. Then S is a multiplicative subset of $h(R)$ such that $I \cap S = \emptyset$. Moreover, I is an almost-δ_1-S-primary superideal of R associated to $s = 2 \in S$, since if $f(x), g(x) \in h(R)$ with $f(x)g(x) \in I - I^2$, then $x/f(x)$ or $x/g(x)$ which implies that $2f(x) \in \delta_1(I)$ or $2g(x) \in \delta_1(I)$. Since $4x \in I - I^2$ and neither $4 \in \delta_1(I)$ nor $x \in \delta_1(I)$, then we get that I is not an almost-δ_1-primary superideal of R.

Proposition 2.11. Let I be a proper superideal of R such that I is a ϕ-S-primary superideal of R associated to $s \in S$ such that $Srad(\delta(I)) \subseteq \delta(Srad(I))$ and $Srad(\phi(I)) \subseteq \phi(Srad(I))$, then $Srad(I)$ is a ϕ-δ-S-primary superideal of R associated to s.

Proof. Let $a, b \in h(R)$ such that $ab \in Srad(I) - \phi(Srad(I))$. Then $ab \in Srad(I)$ which implies that $(ab)^n = a^n b^n \in I$ for some $n \geq 1$. If $a^n b^n \in \phi(I)$, then $ab \in Srad(\phi(I)) \subseteq \phi(Srad(I))$, a contradiction. Thus, $a^n b^n \in I - \phi(I)$ which implies that $sa^n \in I$ or $sb^n \in \delta(I)$. Thus, $sa \in Srad(I)$ or $sb \in Srad(\delta(I)) \subseteq \delta(Srad(I))$. Hence, $Srad(I)$ is a ϕ-δ-S-primary superideal of R associated to s. □

Corollary 2.12. Let I be a proper ϕ-S-primary superideal of R associated to $s \in S$. Suppose that $Srad(\phi(I)) \subseteq \phi(Srad(I))$. Then $Srad(I)$ is a ϕ-S-prime superideal of R associated to s.

Proof. Let $\delta(J) = Srad(J)$ for every superideal J of R. Then, by the above proposition, if I is a ϕ-S-primary superideal of R associated to s then $Srad(I)$ is a ϕ-S-prime superideal of R associated to s. □

Proposition 2.13. Let I be a proper ϕ-S-primary superideal of R associated to $s \in S$. Suppose that $Srad(\phi(I)) \subseteq \phi(Srad(I))$ and $(\phi(Srad(I)) : x) \subseteq (\phi(Srad(I)) : s)$ for each $x \in S$. If $a \in h(R) - (Srad(I) : s)$, then $S \cap (Srad(I) : a) = \emptyset$.

Proof. It is easy to see that $Srad(I) \cap S = \emptyset$, since $I \cap S = \emptyset$. Also, by the above corollary, $Srad(I)$ is a ϕ-S-prime superideal of R associated to s. We show that $S \cap (Srad(I) : a) = \emptyset$. Let $t \in S$ such that $ta \in Srad(I)$. If $ta \in \phi(Srad(I))$, then $a \in (\phi(Srad(I)) : t) \subseteq (\phi(Srad(I)) : s)$ which implies that $sa \in \phi(Srad(I)) \subseteq Srad(I)$, a contradiction. Thus, $ta \in Srad(I) - \phi(Srad(I))$ implies that $sa \in Srad(I)$ or $st \in Srad(I)$, which is a contradiction again, since $a \notin (Srad(I) : s)$ and $S \cap Srad(I) = \emptyset$. Thus, $S \cap (Srad(I) : a) = \emptyset$. □

Corollary 2.14. Let I be a proper ϕ-δ-S-primary superideal of R associated to $s \in S$ with $\delta(I) \subseteq Srad(I)$. Suppose $(\phi(Srad(I)) : x) \subseteq (\phi(Srad(I)) : s)$ for each $x \in S$ and $(\delta(I) : s) = (Srad(I) : s)$. Then $(\delta(I) : s) = (\delta(I) : s^2)$ and if whenever $a \in h(R) - (\delta(I) : s)$, then $S \cap (\delta(I) : a) = \emptyset$.

Proof. Since I is a ϕ-δ-S-primary superideal of R associated to s and $\delta(I) \subseteq Srad(I)$, it is easy to see that I is a ϕ-S-primary superideal of R associated to s and $(Srad(I) : s) = (Srad(I) : s^2)$. Since $\delta(I) \subseteq Srad(I)$ and $(\delta(I) : s) = (Srad(I) : s)$, we have $(\delta(I) : s) = (\delta(I) : s^2)$. Moreover, if $a \in h(R) - (\delta(I) : s)$, then $sa \notin Srad(I)$. Thus, by the above proposition, $S \cap (Srad(I) : a) = \emptyset$. Hence $S \cap (\delta(I) : a) \subseteq S \cap (Srad(I) : a) = \emptyset$, since $\delta(I) \subseteq Srad(I)$. □

Recall that if I, J, K are ideals of R such that $K = I \cup J$, then $K = I$ or $K = J$.

Theorem 2.15. Let I be a proper ϕ-δ-S-primary superideal of R associated to $s \in S$. If $a \in h(R) - (\delta(I) : s^2)$, then $(I : sa) = (I : s)$ or $(I : sa) = (\phi(I) : sa)$.

Proof. It is enough to show that $(I : sa) = (I : s) \cup (\phi(I) : sa)$. It is easy to see that $(I : s)$ and $(\phi(I) : sa)$ are subsets of $(I : sa)$. Let $r \in h((I : sa))$, then $rsa \in I$. If $rsa \in \phi(I)$ then $r \in (\phi(I) : sa)$. So we may assume that $rsa \notin \phi(I)$. Thus, $rsa \in I - \phi(I)$ implies that $sr \in I$ since $s^2a \notin \delta(I)$. So, $r \in (I : s)$. Thus, $(I : sa) = (I : s) \cup (\phi(I) : sa)$. Hence $(I : sa) = (I : s)$ or $(I : sa) = (\phi(I) : sa)$. □

Corollary 2.16. Let I be a proper ϕ-S-primary superideal of R associated to $s \in S$. If $a \in h(R) - (Srad(I) : s)$, then $(I : sa) = (I : s)$ or $(I : sa) = (\phi(I) : sa)$.

Proof. It is easy to see that $(Srad(I) : s) = (Srad(I) : s^2)$. So, if $a \in h(R) - (Srad(I) : s)$, then $a \in h(R) - (Srad(I) : s^2)$. Hence, by Theorem 2.15, $(I : sa) = (I : s)$ or $(I : sa) = (\phi(I) : sa)$. □

Definition 2.17. Let I be a superideal of R such that $I_\alpha \neq R_\alpha$ for some $\alpha \in \mathbb{Z}_2$. Then I is called an α-ϕ-δ-S-primary ideal of R associate to $s \in S$, if whenever $a, b \in R_\alpha$ such that $ab \in I - \phi(I)$, then $sa \in I$ or $sb \in \delta(I)$.

Proposition 2.18. Let I be a superideal of R such that $I_\alpha \neq R_\alpha$ for some $\alpha \in \mathbb{Z}_2$, and I an α-ϕ-δ-S-primary ideal of R associate to $s \in S$. If I is not an α-δ-S-primary, then $I_\alpha^2 \subseteq \phi(I)$.

Proof. Suppose that $I_\alpha^2 \nsubseteq \phi(I)$. We claim that I is an α-δ-S-primary ideal of R associated to s. Let $a, b \in R_\alpha$ such that $ab \in I$. If $ab \in I - \phi(I)$, then $sa \in I$ or $sb \in \delta(I)$. So, we may assume that $ab \in \phi(I)$. Suppose that $aI_\alpha \nsubseteq \phi(I)$, then there exists $p \in I_\alpha$ such that $ap \notin \phi(I)$. So, $a(p + b) \in I - \phi(I)$ implies that $sa \in I$ or $s(p + b) \in \delta(I)$, and since $sp \in I \subseteq \delta(I)$, we have $sb \in \delta(I)$. Similarly, if $bI_\alpha \nsubseteq \phi(I)$, we obtain that $sa \in I$. Thus we may assume that $aI_\alpha \subseteq \phi(I)$ and $bI_\alpha \subseteq \phi(I)$. Since $I_\alpha^2 \nsubseteq \phi(I)$, there exist $p, q \in I_\alpha$ such that $pq \notin \phi(I)$. Thus, $(a + p)(b + q) \in I - \phi(I)$, since $ab + aq + pb \in \phi(I)$. Hence $s(a + p) \in I$ or $s(b + q) \in \delta(I)$. Consequently, we conclude that I is an α-δ-S-primary ideal of R associated to s. □

Theorem 2.19. Let I be a proper superideal of R. Suppose $I_\alpha \neq R_\alpha$ for some $\alpha \in \mathbb{Z}_2$. Then the following statements are equivalent.

(1) I is an α-ϕ-δ-S-primary ideal of R associated to $s \in S$.

(2) For each $a \in R_\alpha$ such that $a \notin (\delta(I) : s)$ we have either $(I :_{R_\alpha} a) \subseteq (I :_{R_\alpha} s)$ or $(I :_{R_\alpha} a) = (\phi(I) :_{R_\alpha} a)$.

(3) For all A and B superideals of R such that $A_\alpha \neq R_\alpha$, $B_\alpha \neq R_\alpha$, $A_\alpha B_\alpha \subseteq I$ but $A_\alpha B_\alpha \nsubseteq \phi(I)$, either $sA_\alpha \subseteq I$ or $sB_\alpha \subseteq \delta(I)$.

Proof. $(1 \to 2)$: Let $a \in R_\alpha$ such that $a \notin (\delta(I) : s)$, then $sa \notin \delta(I)$. Suppose that $(I :_{R_\alpha} a) \neq (\phi(I) :_{R_\alpha} a)$. We show that $(I :_{R_\alpha} a) \subseteq (I :_{R_\alpha} s)$. Let $r \in (I :_{R_\alpha} a)$. If $r \notin (\phi(I) :_{R_\alpha} a)$, then $ra \in I - \phi(I)$ implies that $sr \in I$, since $sa \notin \delta(I)$. Suppose $r \in (\phi(I) :_{R_\alpha} a)$. Since $(I :_{R_\alpha} a) \neq (\phi(I) :_{R_\alpha} a)$, let $r' \in (I :_{R_\alpha} a)$ such that $r' \notin (\phi(I) :_{R_\alpha} a)$. So, $ar' \in I - \phi(I)$ implies that $sr' \in I$, since $sa \notin \delta(I)$. Thus, $a(r + r') \in I - \phi(I)$ implies that $s(r + r') \in I$, since $sa \notin \delta(I)$. Because $s(r + r') \in I$ and $sr' \in I$ we get $sr \in I$. Consequently, we conclude that $(I :_{R_\alpha} a) \subseteq (I :_{R_\alpha} s)$.
$(2 \to 1)$: Let $a, b \in R_\alpha$ such that $ab \in I - \phi(I)$. Suppose that $sa \notin \delta(I)$, we show that $sb \in I$. Since $b \in (I :_{R_\alpha} a)$ and $b \notin (\phi(I) :_{R_\alpha} a)$, we get immediately, $(I :_{R_\alpha} a) \subseteq (I :_{R_\alpha} s)$. Thus, $b \in (I :_{R_\alpha} s)$ and $sb \in I$. Accordingly, I is an α-ϕ-δ-S-primary ideal of R associated to s.
$(2 \to 3)$: Let A, B be superideals of R such that $A_\alpha \neq R_\alpha$, $B_\alpha \neq R_\alpha$, $A_\alpha B_\alpha \subseteq I$. Suppose that $sA_\alpha \nsubseteq I$ and $sB_\alpha \nsubseteq \delta(I)$. We claim that $A_\alpha B_\alpha \subseteq \phi(I)$. Let $b \in B_\alpha - (\delta(I) :_{R_\alpha} s)$, then $(I :_{R_\alpha} b) \subseteq (I :_{R_\alpha} s)$ or $(I :_{R_\alpha} b) = (\phi(I) :_{R_\alpha} b)$. Since $A_\alpha \subseteq (I :_{R_\alpha} b)$ and $A_\alpha \nsubseteq (I :_{R_\alpha} s)$ we get immediately, $(I :_{R_\alpha} b) = (\phi(I) :_{R_\alpha} b)$. So, $A_\alpha b \subseteq \phi(I)$. For any $c \in B_\alpha \cap (\delta(I) :_{R_\alpha} s)$, then $b + c \in B_\alpha - (\delta(I) :_{R_\alpha} s)$ implies that $A_\alpha \subseteq (I :_{R_\alpha} b + c) = (\phi(I) :_{R_\alpha} b + c)$. Thus, $A_\alpha(b + c) \subseteq \phi(I)$ implies that $A_\alpha c \subseteq \phi(I)$, since $A_\alpha b \subseteq \phi(I)$. Consequently, we conclude that $A_\alpha B_\alpha \subseteq \phi(I)$.
$(3 \to 1)$: Let $a, b \in R_\alpha$ such that $ab \in I - \phi(I)$. Let $A = aR$, $B = bR$. Then A, B are superideals of R such that $A_\alpha = aR_0$, $B_\alpha = bR_0$. Moreover, $A_\alpha B_\alpha \subseteq I$ and $A_\alpha B_\alpha \nsubseteq \phi(I)$, since $ab \in I - \phi(I)$. Hence $sA_\alpha \subseteq I$ or $sB_\alpha \subseteq \delta(I)$ implies that $sa \in I$ or $sb \in \delta(I)$. Accordingly, I is an α-ϕ-δ-S-primary ideal of R associated to s. □

The following result can be proved similar to the previous theorem. Hence, we omit the proof.

Theorem 2.20. Let I be a proper superideal of R. Suppose $I_\alpha \neq R_\alpha$ for some $\alpha \in \mathbb{Z}_2$. Then the following statements are equivalent.
(1) I is an α-ϕ-δ-S-primary ideal of R associated to $s \in S$.
(2) For each $a \in R_\alpha$ such that $a \notin (I : s)$ we have either $(I :_{R_\alpha} a) \subseteq (\delta(I) :_{R_\alpha} s)$ or $(I :_{R_\alpha} a) = (\phi(I) :_{R_\alpha} a)$.
(3) For all A and B superideals of R such that $A_\alpha \neq R_\alpha$, $B_\alpha \neq R_\alpha$, $A_\alpha B_\alpha \subseteq I$ but $A_\alpha B_\alpha \nsubseteq \phi(I)$, either $sA_\alpha \subseteq I$ or $sB_\alpha \subseteq \delta(I)$.

Theorem 2.21. Let P be a ϕ-δ-S-primary superideal of R associated to $s \in S$. If $(\phi(P) : a) \subseteq \phi(P : a)$ for each $a \in h(R) - P$, then $(P : a)$ is also ϕ-δ-S-primary superideal of R associated to s.

Proof. If $(P : a) = R$, then $a \in P$, a contradiction. Thus, $(P : a) \neq R$. Let $x, y \in h(R)$ such that $xy \in (P : a) - \phi(P : a)$. So, $xya \in P - \phi(P)$ implies that $sxa \in P$ or $sy \in \delta(P)$. Hence, $sx \in (P : a)$ or $sy \in \delta(P) \subseteq (\delta(P) : a)$. Thus, $(P : a)$ is a ϕ-δ-S-primary superideal of R associated to s. □

Proposition 2.22. Let δ be an expansion function of superideals of R satisfies the intersection property, and let I be a ϕ-δ-S-primary superideal of R associated to $s \in S$ such that $\phi(J) = \phi(I)$ for each superideal $J \subseteq I$. If P is a superideal in R such that $P \cap S \neq \emptyset$, then $I \cap P$ and IP are ϕ-δ-S-primary superideals of R.

Proof. It is clear that $(P \cap I) \cap S = PI \cap S = \emptyset$. Pick $t \in P \cap S$. We show that $I \cap P$ is a ϕ-δ-S-primary superideal of R associated to ts. Let $a, b \in h(R)$ such that $ab \in I \cap P - \phi(I \cap P)$, then $ab \in I \cap P - \phi(I) \subseteq I - \phi(I)$. Thus, $sa \in I$ or $sb \in \delta(I)$ implies that $tsa \in I \cap P$ or $tsb \in \delta(I) \cap \delta(P) = \delta(I \cap P)$. Consequently, $I \cap P$ is a ϕ-δ-S-primary superideal of R associated to ts. We have the similar proof for IP. □

Let $\phi \neq \phi_\emptyset$ be a reduction function of ideals of R such that $P = \phi(P)$ for each ideal P of R. Then the following result holds.

Proposition 2.23. The following statements are equivalent.
(1) Every ϕ-δ-S-primary superideal of R is a δ-primary.
(2) If $I \in \mathfrak{J}(R)$, then $\phi(I)$ is a δ-primary superideal of R and every δ-S-primary superideal is a δ-primary.

Proof. $(1 \to 2)$: Let $I \in \mathfrak{J}(R)$. From the definition of ϕ-δ-S-primary superideals of R and $I = \phi(I)$, we have $\phi(I)$ is a ϕ-δ-S-primary superideal of R and every ϕ-δ-S-primary superideal of R is a δ-primary. Hence, $\phi(I)$ is a δ-primary superideal of R.
$(2 \to 1)$: Let I be a ϕ-δ-S-primary superideal of R associated to $s \in S$. It is enough to show that I is a δ-S-primary superideal of R associated to s. Let $a, b \in h(R)$ such that $ab \in I$. If $ab \notin \phi(I)$, then $ab \in I - \phi(I)$ implies that $sa \in I$ or $sb \in \delta(I)$. But, if $ab \in \phi(I)$, then $a \in \phi(I)$ implies $sa \in \phi(I) \subseteq I$ or $b \in \delta(\phi(I)) \subseteq \delta(I)$ implies $sb \in \delta(I)$, since $\phi(I)$ is a δ-primary superideal of R. Thus, I is a δ-S-primary superideal of R associated to s hence, by (2), I is a δ-primary. □

Corollary 2.24. The following assertions are equivalent:
(1) Every weakly S-prime superideal of R is a prime superideal.
(2) R is a superdomain and every S-prime superideal of R is a prime superideal.

Proof. It suffices to take $\phi = \phi_0$ and $\delta = \delta_0$ in Proposition 2.23. □

Corollary 2.25. The following assertions are equivalent:
(1) Every weakly S-primary superideal of R is a primary superideal.
(2) R is a domain and every S-primary superideal of R is a primary superideal.

Proof. It suffices to take $\phi = \phi_0$ and $\delta = \delta_1$ in Proposition 2.23. □

Remark 2.26. Let $S_1 \subseteq S_2$ be multiplicative subsets of $h(R)$ and I a superideal of R disjoint with S_2. Clearly, if I is a ϕ-δ-S_1-primary superideal of R associated to $s \in S_1$, then I is a ϕ-δ-S_2-primary superideal of R associated to $s \in S_2$. However, the converse is not true in general (take $\phi = \phi_0$, $\delta = \delta_0$, $R_1 = \{0\}$ in [1, Example 2.3]).

Proposition 2.27. Let $S_1 \subseteq S_2$ be multiplicative subsets of $h(R)$ such that for any $s \in S_2$, there exists $t \in S_2$ with $st \in S_1$. If I is a ϕ-δ-S_2-primary superideal of R associated to $s \in S_2$, then, for some $t \in S_2$, I is a ϕ-δ-S_1-primary superideal of R associated to $ts \in S_1$.

Proof. Let $t \in S_2$ such that $st \in S_1$. We show that I is a ϕ-δ-S_1-primary superideal of R associated to $st \in S_1$. Let $a, b \in h(R)$ such that $ab \in I - \phi(I)$, then $sa \in I$ implies that $sta \in I$ or $sb \in \delta(I)$ implies that $stb \in \delta(I)$. Consequently, I is a ϕ-δ-S_1-primary superideal of R associated to $st \in S_1$. □

Recall that if S is a multiplicative subset of $h(R)$ with $1 \in S$, then $S^* = \{r \in h(R) : \frac{r}{1} \in U(S^{-1}R)\}$ is said to be the saturation of S. One can easily see that S^* is a multiplicative subset of $h(R)$ containing S. If $S = S^*$, then S is called saturated.

Proposition 2.28. Let I be a proper superideal of R. Then I is a ϕ-δ-S-primary superideal of R if and only if I is a ϕ-δ-S^*-primary superideal of R.

Proof. First we show that $S^* \cap I = \emptyset$. Let $r \in S^* \cap I$, then $\frac{r}{1}$ is a unit in $S^{-1}R$, so there exist $a \in h(R)$, $s \in S$ such that $(\frac{r}{1})(\frac{a}{s}) = 1$. Hence, there exists $t \in S$ such that $tra = ts$ which implies that $tra \in I \cap S$, a contradiction. Therefore, $S^* \cap I = \emptyset$. Since $S \subseteq S^*$, I is a ϕ-δ-S-primary superideal of R associated to $s \in S$ implies that I is a ϕ-δ-S^*-primary superideal of R associated to $s \in S^*$. Conversely, suppose that I is a ϕ-δ-S^*-primary superideal of R associated to $s \in S^*$. Let $r \in S^*$, then $\frac{r}{1} \in U(S^{-1}R)$ implies that $(\frac{r}{1})(\frac{a}{x}) = 1$, where $a \in h(R)$, $x \in S$. Hence, there exists $t \in S$ such that $tra = tx \in S$. Take $r' = ta$, then $r' \in S^*$ with $r'r = tx \in S$. Let $S_1 = S$, $S_2 = S^*$, then, by Proposition 2.27, I is a ϕ-δ-S-primary superideal of R. □

Recall that $\delta_S(S^{-1}J) = S^{-1}\delta(J)$ and $\phi_S(S^{-1}J) = S^{-1}\phi(J)$ for each $J \in \mathfrak{J}(R)$. Let I be a proper superideal of R such that $\phi(I : a) = (\phi(I) : a)$, $\delta(I : a) = (\delta(I) : a)$ for each $a \in h(R)$. Moreover, assume that $\delta(S^{-1}I \cap R) = S^{-1}\delta(I) \cap R$. Then under the condition $\phi(I) = (\phi(I) : t)$ for each $t \in S$, the following result holds.

Theorem 2.29. Let I be a proper superideal of R such that $I \cap S = \emptyset$. Suppose that $\delta_S(S^{-1}I) \neq S^{-1}R$, if $S^{-1}I \neq S^{-1}R$. Then the following statements are equivalent:
(1) I is a ϕ-δ-S-primary superideal of R associated to some $s \in S$.
(2) $(I : s)$ is a ϕ-δ-primary superideal of R.
(3) $S^{-1}I$ is a ϕ_S-δ_S-primary superideal of $S^{-1}R$ and $(I : t) \subseteq (I : s)$ for each $t \in S$.
(4) $S^{-1}I$ is a ϕ_S-δ_S-primary superideal of $S^{-1}R$ and $S^{-1}I \cap h(R) = h((I : s))$.

Proof. Let I be a proper superideal of R such that $I \cap S = \emptyset$. Then $S^{-1}I \neq S^{-1}R$ implies that $\delta_S(S^{-1}I) \neq S^{-1}R$, since $I \cap S = \emptyset$. Moreover, it is easy to check that $\delta(I) \cap S = \emptyset$.
$(1 \to 2)$: Since $I \cap S = \emptyset$, $(I : s) \neq R$. Let $a, b \in h(R)$ such that $ab \in (I : s) - (\phi(I) : s)$, then $sab \in I - \phi(I)$ which implies that $s^2a \in I - \phi(I)$ or $sb \in \delta(I)$. Thus, $sa \in I$, since $s^3 \notin \delta(I)$, or $sb \in \delta(I)$. Hence, $a \in (I : s)$ or $b \in (\delta(I) : s)$. Consequently, we conclude that $(I : s)$ is a ϕ-δ-primary superideal of R.
$(2 \to 3)$: $S^{-1}I \neq S^{-1}R$ since $I \cap S = \emptyset$. Let $\frac{a}{s_1}, \frac{b}{s_2}$ be homogeneous elements in $S^{-1}R$ such that $\frac{a}{s_1}\frac{b}{s_2} \in S^{-1}I - \phi_S(S^{-1}I)$. Then $\frac{ab}{s_1s_2} = \frac{u}{s_3} \in S^{-1}I - \phi_S(S^{-1}I)$ for some $u \in h(I)$. So there exists $t \in S$ such that $tabs_3 = ts_1s_2u \in h(I)$. If $ts_1s_2u \in \phi(I)$, then $\frac{u}{s_3} \in \phi_S(S^{-1}I)$, a contradiction. Hence, $tabs_3 \in I - \phi(I)$ which implies that $tabs_3 \in (I : s) - (\phi(I) : s)$, since $\phi(I) = (\phi(I) : s)$. Thus, $a \in (I : s)$ or $tbs_3 \in (\delta(I) : s)$. Therefore, $sa \in I$ implies that $\frac{a}{s_1} \in S^{-1}I$ or $stbs_3 \in \delta(I)$ implies that $\frac{b}{s_2} \in S^{-1}\delta(I)$. So, we conclude that $S^{-1}I$ is a ϕ_S-δ_S-primary superideal of $S^{-1}R$. Let $t \in S$ and let $a \in (I : t) \cap h(R)$. If $a \in (\phi(I) : t)$, then $a \in (\phi(I) : s) \subseteq (I : s)$. Therefore, we may assume that $a \notin (\phi(I) : t)$. So, $ta \in I \subseteq (I : s)$ and $ta \notin \phi(I) = (\phi(I) : s)$. Thus, $ta \in (I : s) - (\phi(I) : s)$ which implies that $a \in (I : s)$, since $t \notin (\delta(I) : s)$. Consequently, we conclude that $h((I : t)) \subseteq h((I : s))$ implies that $(I : t) \subseteq (I : s)$.
$(3 \to 4)$: By using part(3) we have $S^{-1}I$ is a ϕ_S-δ_S-primary superideal of $S^{-1}R$. Let $s \in S$ such that $(I : t) \subseteq (I : s)$ for each $t \in S$. Then it is easy to check that $h((I : s)) \subseteq S^{-1}I \cap h(R)$. Conversely, let $a \in S^{-1}I \cap h(R)$, then there exists $t \in S$ such that $ta \in I$. Thus, $a \in h((I : t)) \subseteq h((I : s))$ implies that $S^{-1}I \cap h(R) \subseteq h((I : s))$. Hence we conclude that $S^{-1}I \cap h(R) = h((I : s))$.
$(4 \to 1)$: Suppose that $S^{-1}I$ is a ϕ_S-δ_S-primary superideal of $S^{-1}R$ and $S^{-1}I \cap h(R) = h((I : s))$ for some $s \in S$. We show that I is a ϕ-δ-S-primary superideal of R associated to s. Let $a, b \in h(R)$ such that $ab \in I - \phi(I)$. Then $\frac{a}{1}, \frac{b}{1}$ are homogeneous elements in $S^{-1}R$ such that $\frac{a}{1}\frac{b}{1} \in S^{-1}I - \phi_S(S^{-1}I)$. Thus, $\frac{a}{1} \in S^{-1}I$ or $\frac{b}{1} \in \delta_S(S^{-1}I) = S^{-1}\delta(I)$, since $S^{-1}I$ is a ϕ_S-δ_S-primary superideal of $S^{-1}R$. If $\frac{a}{1} \in S^{-1}I$ then there exists $t \in S$ such that $ta \in h(I)$ which implies that $a = \frac{ta}{t} \in S^{-1}I \cap h(R) = h((I : s))$. So, $sa \in I$. Also, if $S^{-1}I \cap h(R) = h((I : s))$, then $S^{-1}I \cap R = (I : s)$ which implies that $\delta(S^{-1}I \cap R) = \delta((I : s)) = (\delta(I) : s)$, and since $\delta(S^{-1}I \cap R) = S^{-1}\delta(I) \cap R$, we have that $S^{-1}\delta(I) \cap R = (\delta(I) : s)$. Now, if $\frac{b}{1} \in S^{-1}\delta(I)$ then there exists $t' \in S$ such that $t'b \in h(\delta(I))$ which implies that $b = \frac{t'b}{t'} \in S^{-1}\delta(I) \cap R = (\delta(I) : s)$. So, $sb \in \delta(I)$. Hence we conclude that I is a ϕ-δ-S-primary superideal of R associated to s. □

Let R be a super-ring, $S \subseteq R$ be a multiplicative subset of $h(R)$. Next we give an example of a proper superideal P of R with $h(P) \cap S = \emptyset$ such that if $(\phi(P) : s) \neq \phi(P)$ for some $s \in S$, then P is a ϕ-δ-S-primary superideal of R associated to s but $(P : s)$ is not a ϕ-δ-primary superideal.

Example 2.30. Let $R = \mathbb{Z}[x]$, with $R_0 = \mathbb{Z}[x]$ and $R_1 = \{0\}$, and let $P =< 4x >= 4x\mathbb{Z}[x]$, then $P_0 = 4x\mathbb{Z}[x]$ and $P_1 = \{0\}$. Let $\phi(P) = p^2 =< 16x^2 >$ and $\delta(P) = Srad(P) =< 2x >$. Let $S = \{2^k : k \geq 0\}$. Then it is easy to check that $P \cap S = \emptyset$ and P is an almost-δ-S-primary superideal of R associated to $s = 2 \in S$. Also, it is easy to check that $(P^2 : s) = (< 16x^2 >: 2) =< 8x^2 >\neq P^2$. Moreover, $(P : s) = (< 4x >: 2) =< 2x >$ is not an almost-δ-primary superideal of R, since $2x \in< 2x > - < 4x^2 >$ but neither $2 \in< 2x >= Srad(< 2x >)$ nor $x \in< 2x >= Srad(< 2x >)$.

3 $(\phi, \delta) - (\psi, \gamma)$-Superhomomorphisms

A generalization of [9], let X, Y be commutative super-rings with unities and let $f : X \to Y$ be a superhomomorphism. Suppose δ, ϕ are expansion and reduction functions of superideals of X and γ, ψ are expansion and reduction functions of superideals of Y, respectively. Then f is said to be (δ, ϕ)-(γ, ψ)-superhomomorphism if $\delta(f^{-1}(J)) = f^{-1}(\gamma(J))$ and $\phi(f^{-1}(J)) = f^{-1}(\psi(J))$ for all $J \in \mathfrak{J}(Y)$.

Remark 3.1.
(1) If $f : X \to Y$ is a nonzero surjective superhomomorphism and 1 is the unity of X, then $f(1)$ is the unity of Y.

(2) Suppose $f : X \to Y$ is a nonzero surjective (δ, ϕ)-(γ, ψ)-superhomomorphism and let I be a proper superideal of X containing ker(f). Then, by doing a generalization of ([9, Remark 2.11]) to the supercase, it is easy to see that $\gamma(f(I)) = f(\delta(I))$ and $\psi(f(I)) = f(\phi(I))$.
(3) If S is a multiplicative subset of $h(X)$ containing 1, then $f(S)$ is a multiplicative subset of $h(Y)$ containing $f(1)$.

Theorem 3.2. Let $f : X \to Y$ be a nonzero surjective (δ, ϕ)-(γ, ψ)-superhomomorphism. Then the following statements are satisfied.
(1) If J is a ψ-γ-$f(S)$-primary superideal of Y associated to $f(s) \in f(S)$, then $f^{-1}(J)$ is a ϕ-δ-S-primary superideal of X associated to $s \in S$.
(2) If I is a ϕ-δ-S-primary superideal of X associated to $s \in S$ containing ker(f) and f is surjective, then $f(I)$ is a ψ-γ-$f(S)$-primary superideal of Y associated to $f(s) \in f(S)$.

Proof. (1) If S is a multiplicative subset of $h(X)$ with $1 \in S$, then $f(S)$ is a multiplicative subset of $h(Y)$ with $1 = f(1) \in f(S)$, since f is a nonzero surjective superhomomorphism. Let J be a ψ-γ-$f(S)$-primary superideal of Y associated to $f(s) \in f(S)$. Choose $a, b \in h(X)$ such that $ab \in f^{-1}(J) - \phi(f^{-1}(J))$. Since f respect the grading, $f(a)$, $f(b)$ are homogeneous elements in Y such that $f(a)f(b) \in J - \psi(J)$. Since J is a ψ-γ-$f(S)$-primary superideal of Y associated to $f(s) \in f(S)$, we conclude that $f(s)f(a) \in J$ or $f(s)f(b) \in \gamma(J)$ which implies that $sa \in f^{-1}(J)$ or $sb \in f^{-1}(\gamma(J)) = \delta(f^{-}(J))$. Hence $f^{-1}(J)$ is a ϕ-δ-S-primary superideal of X associated to s.
(2) Let I be a ϕ-δ-S-primary superideal of X associated to s containing ker(f), then the unity in Y is $f(1) \in f(S)$, since f is a nonzero surjective (δ, ϕ)-(γ, ψ)-superhomomorphism. Choose $x, y \in h(Y)$ such that $xy \in f(I) - \psi(f(I))$. Since f is onto map and it is respect the grading, we can choose $a, b \in h(I)$ such that $f(a) = x$, $f(b) = y$. This implies that $f(a)f(b) = f(ab) \in f(I)$. Since ker$(f) \subseteq I$, we conclude that $ab \in I$. If $ab \in \phi(I)$, then $xy = f(ab) \in f(\phi(I)) = \psi(f(I))$, which is a contradiction. So, $ab \in I - \phi(I)$. As I is a ϕ-δ-S-primary superideal of X associated to s, we have $sa \in I$ or $sb \in \delta(I)$. Thus, $f(s)x \in f(I)$ or $f(s)y \in f(\delta(I)) = \gamma(f(I))$. Therefore, $f(I)$ is a ψ-γ-$f(S)$-primary superideal of Y associated to $f(s)$. □

From the above theorem we obtain the following result.

Theorem 3.3. [Correspondence Theorem] Let $f : X \to Y$ be a nonzero surjective (δ, ϕ)-(γ, ψ)-superhomomorphism. Then f induces a one-to-one correspondence between the ϕ-δ-S-primary superideals of X associated to $s \in S$ containing ker(f) and the ψ-γ-$f(S)$-primary superideals of Y associated to $f(s) \in f(S)$ in such a way that if I is a ϕ-δ-S-primary superideal of X associated to $s \in S$ containing ker(f), then $f(I)$ is the corresponding ψ-γ-$f(S)$-primary superideal of Y associated to $f(s) \in f(S)$, and if J is a ψ-γ-$f(S)$-primary superideal of Y associated to $f(s) \in f(S)$, then $f^{-1}(J)$ is the corresponding ϕ-δ-S-primary superideal of X associated to $s \in S$ containing ker(f).

Assume that δ, ϕ are expansion and reduction functions of superideals of R, respectively. Let J be a proper superideal of R such that $J = \phi(J)$. Then $\gamma : \mathfrak{J}(R/J) \to \mathfrak{J}(R/J)$, defined by $\gamma(I/J) = \delta(I)/J$, and $\psi : \mathfrak{J}(R/J) \to \mathfrak{J}(R/J)$, defined by $\psi(I/J) = \phi(I)/J$ if $\phi \neq \phi_\emptyset$ and $\psi(I/J) = \emptyset$ if $\phi = \phi_\emptyset$, are expansion and reduction functions of superideals of R/J, respectively. Moreover, if S is a multiplicative subset of $h(R)$, then S/J is a multiplicative subset of R/J, where $S/J = \{s + J \in R/J \ : \ s \in S\}$.

Let Q be a proper superideal of R, and let S be a multiplicative subset of $h(R)$. Recall that Q is said to be a weakly δ-S-primary superideal of R associated to $s \in S$, if whenever $0 \neq ab \in Q$ for some $a, b \in h(R)$ then $sa \in Q$ or $sb \in \delta(Q)$.

Theorem 3.4. Let δ, ϕ be expansion and reduction functions of superideals of R such that $\phi \neq \phi_\emptyset$, and let J be a proper superideal of R with $J = \phi(J)$. For every $L \in \mathfrak{J}(R)$ let $\gamma : \mathfrak{J}(R/J) \to \mathfrak{J}(R/J)$ be an expansion function of superideals of R/J defined by $\gamma(L + J/J) = \delta(L + J)/J$ and $\psi : \mathfrak{J}(R/J) \to \mathfrak{J}(R/J)$ be a reduction function of superideals of R/J defined by $\psi(L + J/J) = \phi(L + J)/J$. Then the followings statements hold.
(1) A map $f : R \to R/J$ defined by $f(r) = r + J$ for every $r \in R$ is a surjective (δ, ϕ)-(γ, ψ)-superhomomorphism.

(2) Let I be a proper superideal of R such that $J \subseteq I$, S a multiplicative subset of $h(R)$. Then I is a ϕ-δ-S-primary superideal of R associated to $s \in S$ if and only if I/J is a γ-ψ-S/J-primary superideal of R/J associated to $s + J \in S/J$.
(3) Let I be a nonzero proper superideal of R such that $\phi^2(I) = \phi(I)$. Then I is a ϕ-δ-S-primary superideal of R associated to $s \in S$ if and only if $I/\phi(I)$ is a weakly γ-$S/\phi(I)$-primary superideal of $R/\phi(I)$ associated to $s + \phi(I) \in S/\phi(I)$.

Proof. (1) It is easy to see that f is a surjective superhomomorphism with $\ker(f) = J$. Let K be a superideal in R/J, then $K = L + J/J$ for some superideal $L \in \mathfrak{J}(R)$. Therefore,

$$f^{-1}(\gamma(K)) = f^{-1}(\delta(L + J)/J) = \delta(L + J) = \delta(f^{-1}(K)),$$

$$f^{-1}(\psi(K)) = f^{-1}(\phi(L + J)/J) = \phi(L + J) = \phi(f^{-1}(K)),$$

since f is onto. Thus, f is a surjective (δ, ϕ)-(γ, ψ)-superhomomorphism.
(2) Let I be a proper superideal of R such that $J \subseteq I$, S a multiplicative subset of $h(R)$. Since the map f defined in (1) is a surjective (δ, ϕ)-(γ, ψ)-superhomomorphism with $\ker(f) = J$ and $f(I) = I/J$. Then, by the correspondence theorem, I is a ϕ-δ-S-primary superideal of R associated to $s \in S$ if and only if I/J is a γ-ψ-S/J-primary superideal of R/J associated to $s + J \in S/J$.
(3) Let $J = \phi(I)$, then $J = \phi(J)$. Moreover, $f(I) = I/\phi(I)$ and $\psi(I/\phi(I)) = \phi(I)/\phi(I) = \{0\} \in R/\phi(I)$. Hence, by the correspondence theorem, I is a ϕ-δ-S-primary superideal of R associated to $s \in S$ if and only if $I/\phi(I)$ is a weakly γ-$S/\phi(I)$-primary superideal of $R/\phi(I)$ associated to $s + \phi(I) \in S/\phi(I)$. □

Definition 3.5. Let I be a ϕ-δ-S-primary superideal of R. If there exist $a, b \in R_\alpha$, for some $\alpha \in \mathbb{Z}_2$, such that $ab \in \phi(I)$ with $sa \notin I$ and $sb \notin \delta(I)$. Then (a, b) is called an α-ϕ-δ-S-twin zero of I.

Lemma 3.1. Suppose that I is a ϕ-δ-S-primary ideal of R associated to $s \in S$. If there exist $a, b \in R_\alpha$, for some $\alpha \in \mathbb{Z}_2$, such that (a, b) is an α-ϕ-δ-S-twin zero of I. Then $I_\alpha^2 \subseteq \phi(I)$.

Proof. Let $\alpha \in \mathbb{Z}_2$, and let $a, b \in R_\alpha$ such that (a, b) is an α-ϕ-δ-S-twin zero of I. Then I is not an α-δ-S-primary ideal of R. Hence, by Proposition 2.18, $I_\alpha^2 \subseteq \phi(I)$. □

Theorem 3.6. Let $f : X \to Y$ be a nonzero surjective (δ, ϕ)-(γ, ψ)-superhomomorphism and let I a ϕ-δ-S-primary superideal of X associated to $s \in S$ such that $\ker(f) \subseteq I$. Let $a, b \in X_\alpha$, where $\alpha \in \mathbb{Z}_2$, then (a, b) is an α-ϕ-δ-S-twin zero of I if and only if $(f(a), f(b))$ is an α-ψ-γ-$f(S)$-twin zero of $f(I)$.

Proof. By Theorem 3.2, $f(I)$ is a ψ-γ-$f(S)$-primary superideal of Y associated to $f(s) \in f(S)$. For some $\alpha \in \mathbb{Z}_2$, let $a, b \in X_\alpha$ such that (a, b) is an α-ϕ-δ-S-twin zero of I. Then $ab \in \phi(I)$ with $sa \notin I$ and $sb \notin \delta(I)$. Hence $f(a), f(b) \in Y_\alpha$, since f respect the grading. So, $f(a)f(b) = f(ab) \in \psi(f(I))$ with $f(s)f(a) \notin f(I)$, since $\ker(f) \subseteq I$ and $sa \notin I$. Similarly, $f(s)f(b) \notin \gamma(f(I))$. Thus, $(f(a), f(b))$ is an α-ψ-γ-$f(S)$-twin zero of $f(I)$. Conversely, for some $\alpha \in \mathbb{Z}_2$, let $a, b \in X_\alpha$ such that $(f(a), f(b))$ is an α-ψ-γ-$f(S)$-twin zero of $f(I)$. Then $f(a)f(b) = f(ab) \in \psi(f(I)) = f(\phi(I))$ with $f(s)f(a) = f(sa) \notin f(I)$ and $f(s)f(b) \notin \gamma(f(I)) = f(\delta(I))$. Thus, $ab \in f^{-1}(\psi(f(I))) = \phi(f^{-1}(f(I))) = \phi(I)$, since $\ker(f) \subseteq I$. Moreover, $sa \notin f^{-1}(f(I)) = I$ and $sb \notin f^{-1}(\gamma(f(I))) = \delta(I)$. Consequently, we conclude that (a, b) is an α-ϕ-δ-S-twin zero of I. □

Corollary 3.7. Let δ, ϕ be expansion and reduction functions of superideals of R such that $\phi \neq \phi_\emptyset$ and let J be a proper superideal of R such that $J = \phi(J)$. For every $L \in \mathfrak{J}(R)$ let $\gamma : \mathfrak{J}(R/J) \to \mathfrak{J}(R/J)$ be an expansion function of superideals of R/J defined by $\gamma(L+J/J) = \delta(L + J)/J$ and $\psi : \mathfrak{J}(R/J) \to \mathfrak{J}(R/J)$ be a reduction function of superideals of R/J defined by $\psi(L + J/J) = \phi(L + J)/J$. For some $\alpha \in \mathbb{Z}_2$, let $a, b \in R_\alpha$. Then the followings statements hold.
(1) (a, b) is an α-ϕ-δ-S-twin zero of I if and only if $(a + J, b + J)$ is an α-ψ-γ-S/J-twin zero of I/J.
(2) (a, b) is an α-ϕ-δ-S-twin zero of I if and only if $(a + \phi(I), b + \phi(I))$ is an α-γ-S/J-twin zero of $I/\phi(I)$.

Proof. (1) It follows from Theorem 3.4(2) and Theorem 3.6.
(2) It follows from Theorem 3.4(3) and Theorem 3.6. □

Theorem 3.8. Suppose that I is a ϕ-δ-S-primary superideal of R associated to $s \in S$. For some $\alpha \in \mathbb{Z}_2$, if there exist $a, b \in R_\alpha$ such that (a, b) is an α-ϕ-δ-S-twin zero of I. Then $aI_\alpha \subseteq \phi(I)$, $bI_\alpha \subseteq \phi(I)$.

Proof. Since (a, b) is an α-ϕ-δ-S-twin zero of I, we have $ab \in \phi(I)$ such that $sa \notin I$ and $sb \notin \delta(I)$. So, I is not an α-δ-S-primary ideal of R associated to s. So, by Proposition 2.18, $I_\alpha^2 \subseteq \phi(I)$. Now, we show that $aI_\alpha \subseteq \phi(I)$ and $bI_\alpha \subseteq \phi(I)$ case by case.
Since $sb \notin \delta(I)$, by Theorem 2.19, $(I :_{R_\alpha} b) \subseteq (I :_{R_\alpha} s)$ or $(I :_{R_\alpha} b) = (\phi(I) :_{R_\alpha} b)$. But, since $a \in (I :_{R_\alpha} b)$ and $a \notin (I :_{R_\alpha} s)$, we get that $(I :_{R_\alpha} b) = (\phi(I) :_{R_\alpha} b)$. Hence we conclude that $bI \subseteq \phi(I)$.
Similarly, by Theorem 2.20, $sa \notin I$ implies that $(I :_{R_\alpha} a) \subseteq (\delta(I) :_{R_\alpha} s)$ or $(I :_{R_\alpha} a) = (\phi(I) :_{R_\alpha} a)$. Since $b \in (I :_{R_\alpha} a)$ and $b \notin (\delta(I) :_{R_\alpha} s)$, we get that $(I :_{R_\alpha} a) = (\phi(I) :_{R_\alpha} a)$. Hence we conclude that $aI \subseteq \phi(I)$. □

Definition 3.9. Suppose I is a ϕ-δ-S-primary superideal of R such that $A_\alpha B_\alpha \subseteq I$ and $A_\alpha B_\alpha \nsubseteq \phi(I)$ for some $\alpha \in \mathbb{Z}_2$, where A, B are proper superideals of R. Then I is said to be an α-ϕ-δ-S-free twin zero with respect to $A_\alpha B_\alpha$ if (a, b) is not an α-ϕ-δ-S-twin zero of I for every $a \in A_\alpha$ and $b \in B_\alpha$. In particular, I is said to be an α-ϕ-δ-S-free twin zero, if whenever $A_\alpha B_\alpha \subseteq I$ with $A_\alpha B_\alpha \nsubseteq \phi(I)$, for any superideals A, B of R, then I is an α-ϕ-δ-S-free twin zero with respect to $A_\alpha B_\alpha$.

Theorem 3.10. Let I be a ϕ-δ-S-primary superideal of R associated to $s \in S$. Then I is an α-ϕ-δ-S-free twin zero if and only if for superideals A, B of R with $A_\alpha B_\alpha \subseteq I$ and $A_\alpha B_\alpha \nsubseteq \phi(I)$, either $sA_\alpha \subseteq I$ or $sB_\alpha \subseteq \delta(I)$.

Proof. Suppose that I is an α-ϕ-δ-S-free twin zero for some $\alpha \in \mathbb{Z}_2$, and let A, B be superideals of R such that $A_\alpha B_\alpha \subseteq I$ and $A_\alpha B_\alpha \nsubseteq \phi(I)$. Then I is an α-ϕ-δ-S-free twin zero with respect to $A_\alpha B_\alpha$. We show that either $sA_\alpha \subseteq I$ or $sB_\alpha \subseteq \delta(I)$. Suppose $sB_\alpha \nsubseteq \delta(I)$. Then there exists $b \in B_\alpha$ such that $sb \notin \delta(I)$. Let $a \in A_\alpha$, then (a, b) is not an α-ϕ-δ-S-twin zero of I. If $ab \notin \phi(I)$, then $ab \in I - \phi(I)$ implies that $sa \in I$, since $sb \notin \delta(I)$. If $ab \in \phi(I)$, then $sa \in I$, since (a, b) is not an α-ϕ-δ-S-twin zero of I and $sb \notin \delta(I)$. Accordingly, we conclude that $sA_\alpha \subseteq I$. Conversely, suppose that if whenever A, B are superideals R with $A_\alpha B_\alpha \subseteq I$ and $A_\alpha B_\alpha \nsubseteq \phi(I)$, then either $sA_\alpha \subseteq I$ or $sB_\alpha \subseteq \delta(I)$. We show that I is an α-ϕ-δ-S-free twin zero. Let P, Q be superideals of R with $P_\alpha Q_\alpha \subseteq I$ and $P_\alpha Q_\alpha \nsubseteq \phi(I)$. Then, by assumption, either $sP_\alpha \subseteq I$ or $sQ_\alpha \subseteq \delta(I)$. Let $p \in P_\alpha$, $q \in Q_\alpha$. If (p, q) is an α-ϕ-δ-S-twin zero of I, then $pq \in \phi(I)$ with $sp \notin I$ and $sq \notin \delta(I)$, a contradiction, since $sP \subseteq I$ or $sQ \subseteq \delta(I)$. Thus, (p, q) is not an α-ϕ-δ-S-twin zero of I for every $p \in P_\alpha$ and $q \in Q_\alpha$. Hence we conclude that I is an α-ϕ-δ-S-free twin zero with respect to $P_\alpha Q_\alpha$ which implies that I is an α-ϕ-δ-S-free twin zero. □

4 ϕ-δ-S-Primary in direct product of super-rings

Let R_i be commutative super-rings with unity for each $i = 1, 2$ and $R = R_1 \times R_2$ denote the direct product of super-rings R_1, R_2. Also, let S_1, S_2 be multiplicative subsets of $h(R_1)$, $h(R_2)$ respectively, then $S = S_1 \times S_2$ is a multiplicative subset of $h(R)$. Suppose that ϕ_i, δ_i are reduction and expansion functions of superideals of R_i for each $i = 1, 2$ respectively. Generalizing of [9], we define the following two functions:

$$\hat{\delta}(I_1 \times I_2) = \delta_1(I_1) \times \delta_2(I_2),$$

$$\hat{\phi}(I_1 \times I_2) = \phi_1(I_1) \times \phi_2(I_2).$$

Then it is easy to see that $\hat{\delta}$, $\hat{\phi}$ are expansion and reduction functions of superideals of R, respectively.
Recall that if X and Y are commutative super-rings with unities, then $R = X \times Y$ a direct product super-ring with unity ($(1, 1) \neq 0$) such that $R_0 = (X_0 \times Y_0)$, $R_1 = (X_1 \times Y_1)$.

Theorem 4.1. Let X and Y be commutative super-rings with $(1 \neq 0)$, $R = X \times Y$ a direct product super-ring, and $S = S_1 \times S_2$ a multiplicative subset of $h(R)$. Suppose that δ_1, δ_2 are expansion functions of superideals of X, Y and ϕ_1, ϕ_2 are reduction functions of superideals of X, Y (respectively) such that $\phi_2(Y) \neq Y$. Then the following statements are equivalent
(1) $I_1 \times Y$ is a $\hat{\phi}$-$\hat{\delta}$-S-primary superideal of R associated to $(s_1, s_2) \in S$.
(2) I_1 is a δ_1-S_1-primary superideal of X associated to $s_1 \in S_1$ and $I_1 \times Y$ is a $\hat{\delta}$-S-primary superideal of R associated to (s_1, s_2), for some $s_2 \in S_2$.

Proof. $(1 \to 2)$: Suppose that $I_1 \times Y$ is a $\hat{\phi}$-$\hat{\delta}$-S-primary superideal of R associated to $(s_1, s_2) \in S$ and let $a, b \in h(X)$ such that $ab \in I_1$. Then we have $(a, 1)(b, 1) = (ab, 1) \in I_1 \times Y - \hat{\phi}(I_1 \times Y)$, since $\phi_2(Y) \neq Y$. This implies that $(s_1, s_2)(a, 1) \in I_1 \times Y$ or $(s_1, s_2)(b, 1) \in \hat{\delta}(I_1 \times Y)$. Hence we conclude that $s_1 a \in I_1$ or $s_1 b \in \delta_1(I_1)$ and thus, I_1 is a δ_1-S_1-primary superideal of X associated to s_1. Consequently, by the correspondence theorem, $I_1 \times Y$ is a $\hat{\delta}$-S-primary superideal of R associated to (s_1, s_2).
$(2 \to 1)$: It is clear, since every $\hat{\delta}$-S-primary superideal of R associated to (s_1, s_2) is a $\hat{\phi}$-$\hat{\delta}$-S-primary superideal. □

Theorem 4.2. Let X and Y be commutative super-rings with $(1 \neq 0)$, $R = X \times Y$ a direct product super-ring, and $S = S_1 \times S_2$ a multiplicative subset of $h(R)$. Suppose that δ_1, δ_2 are expansion functions of superideals of X, Y and ϕ_1, ϕ_2 are reduction functions of superideals of X, Y (respectively). If $I_1 \times Y$ is a $\hat{\phi}$-$\hat{\delta}$-S-primary superideal of R associated to $(s_1, s_2) \in S$ that is not α-$\hat{\delta}$-S-primary for some $\alpha \in \mathbb{Z}_2$. Then $\hat{\phi}(I_1 \times Y) \neq \emptyset$, $\phi_2(Y) = Y$ and I_1 is a ϕ_1-δ_1-S_1-primary superideal of X associated to s_1 that is not δ_1-S_1-primary.

Proof. Suppose that $I_1 \times Y$ is a $\hat{\phi}$-$\hat{\delta}$-S-primary superideal of R associated to (s_1, s_2) that is not α-$\hat{\delta}$-S-primary, then by Proposition 2.18, we have $(I_1 \times Y)^2_\alpha \subseteq \hat{\phi}(I_1 \times Y)$ which implies that $\hat{\phi}(I_1 \times Y) \neq \emptyset$. If $\phi_2(Y) \neq Y$, then by Theorem 4.1, $I_1 \times Y$ is a $\hat{\delta}$-S-primary ideal of R associated to (s_1, s_2) which implies that $I_1 \times Y$ is an α-$\hat{\delta}$-S-primary superideal of R associated to (s_1, s_2) which is a contradiction. Thus, $\phi_2(Y) = Y$. Moreover, if $I_1 \times Y$ is a $\hat{\phi}$-$\hat{\delta}$-S-primary superideal of R associated to (s_1, s_2) then, by the correspondence theorem, I_1 is a ϕ_1-δ_1-S_1-primary superideal of X associated to s_1. Also, if I_1 is a δ_1-S_1-primary superideal of X associated to s_1 then, by the correspondence theorem, $I_1 \times Y$ is a $\hat{\delta}$-S-primary superideal of R associated to (s_1, s_2) which is a contradiction. Hence I_1 is a ϕ_1-δ_1-S_1-primary superideal of X associated to s_1 that is not δ_1-S_1-primary. □

Now suppose that for each $i = 1, 2$, if $I_i \neq \phi_i(I_i)$, then $S_i \cap \phi_i(I_i) = \emptyset$ and if $S_i \cap \delta_i(I_i) \neq \emptyset$, then $S_i \cap I_i = S_i \cap \delta_i(I_i)$. Then we obtain the following result.

Theorem 4.3. Let X and Y be commutative super-rings with $(1 \neq 0)$, $R = X \times Y$ a direct product super-ring, and $S = S_1 \times S_2$ a multiplicative subset of $h(R)$. Suppose that δ_1, δ_2 are expansion functions of superideals of X, Y and ϕ_1, ϕ_2 are reduction functions of superideals of X, Y (respectively). Let $I = I_1 \times I_2$ be a proper superideal of R, for some superideals $I_1 \neq \phi_1(I_1)$, $I_2 \neq \phi_2(I_2)$ of X, Y, respectively, such that if $I_1 \neq X$ and $I_2 \neq Y$, then $\delta_1(I_1) \neq X$ and $\delta_2(I_2) \neq Y$. Then the following statements are equivalent
(1) I is a $\hat{\phi}$-$\hat{\delta}$-S-primary superideal of R associated to $(s_1, s_2) \in S$.
(2) $I_1 = X$ and I_2 is a δ_2-S_2-Primary superideal of Y associated to s_2 or $I_2 = Y$ and I_1 is a δ_1-S_1-Primary superideal of X associated to s_1 or $s_2 \in I_2 \cap S_2$ and I_1 is a δ_1-S_1-Primary superideal of X associated to s_1 or $s_1 \in I_1 \cap S_1$ and I_2 is a δ_2-S_2-Primary superideal of Y associated to s_2.
(3) I is a $\hat{\delta}$-S-primary superideal of R associated to $(s_1, s_2) \in S$.

Proof. $(1 \to 2)$: If $I_1 = X$, then by Theorem 4.1, I_2 is a δ_2-S_2-Primary superideal of Y associated to s_2. Similarly, if $I_2 = Y$, then by Theorem 4.1, I_1 is a δ_1-S_1-Primary superideal of X associated to s_1. Assume that I_1, I_2 are proper superideals of X, Y, respectively. Let $a \in h(X)$ such that $a \in I_1$, choose $b \in h(Y)$ such that $b \in I_2 - \phi_2(I_2)$. Then $(a, 1)(1, b) = (a, b) \in I - \hat{\phi}(I)$. As I is a $\hat{\phi}$-$\hat{\delta}$-S-primary superideal of R associated to (s_1, s_2), we have $(s_1, s_2)(a, 1) = (s_1 a, s_2) \in I = I_1 \times I_2$ or $(s_1, s_2)(1, b) = (s_1, s_2 b) \in \hat{\delta}(I) = \delta_1(I_1) \times \delta_2(I_2)$. Thus, $s_2 \in S_2 \cap I_2$ or $s_1 \in S_1 \cap \delta_1(I_1) = S_1 \cap I_1$. Assume that $s_2 \in S_2 \cap I_2$. Since $S \cap I = \emptyset$, we have $S_1 \cap I_1 = \emptyset$. We show that I_1 is a δ_1-S_1-Primary superideal of X associated to s_1. Let $a, b \in h(X)$

such that $ab \in I_1$, then $(a, s_2)(b, 1) \in I - \hat{\phi}(I)$, since $s_2 \in S_2 \cap I_2$ and $s_2 \notin \phi_2(I_2)$. As I is a $\hat{\phi}$-$\hat{\delta}$-S-primary superideal of R associated to $(s_1, s_2) \in S$, we have $(s_1, s_2)(a, s_2) = (s_1 a, (s_2)^2) \in I$ or $(s_1, s_2)(b, 1) = (s_1 b, s_2) \in \hat{\delta}(I)$ which implies that $s_1 a \in I_1$ or $s_1 b \in \delta_1(I_1)$. Thus, I_1 is a δ_1-S_1-Primary superideal of X associated to s_1. Similarly, if we assume that $s_1 \in S_1 \cap I_1$, then I_2 is a δ_2-S_2-Primary superideal of Y associated to s_2.
$(2 \to 3)$: If $I_1 = X$ and I_2 is a δ_2-S_2-Primary superideal of Y associated to s_2, then by the correspondence theorem, I is a $\hat{\delta}$-S-primary superideal of R associated to (s_1, s_2). Similarly, if $I_2 = Y$ and I_1 is a δ_1-S_1-Primary superideal of X associated to s_1, then I is a $\hat{\delta}$-S-primary superideal of R associated to (s_1, s_2). Now, suppose that $s_1 \in I_1 \cap S_1$ and I_2 is a δ_2-S_2-Primary superideal of Y associated to s_2, we show that I is a $\hat{\delta}$-S-primary superideal of R associated to (s_1, s_2). Let (a, c), (b, d) be homogeneous elements in R such that $(a, c)(b, d) = (ab, cd) \in I$, then $cd \in I_2$ which implies that $s_2 c \in I_2$ or $s_2 d \in \delta_2(I_2)$. Since $s_1 \in S_1 \cap I_1$, we have $(s_1, s_2)(a, c) = (s_1 a, s_2 c) \in I_1 \times I_2$ or $(s_1, s_2)(b, d) = (s_1 b, s_2 d) \in I_1 \times \delta_2(I_2) \subseteq \delta_1(I_1) \times \delta_2(I_2)$. Thus, I is a $\hat{\delta}$-S-primary superideal of R associated to (s_1, s_2). Similarly, if we assume that $s_2 \in S_2 \cap I_2$ and I_1 is a δ_1-S_1-Primary superideal of X associated to s_1, then I is a $\hat{\delta}$-S-primary superideal of R associated to (s_1, s_2).
$(3 \to 1)$: Clear. □

Suppose that for each $i = 1, 2$, if $I_i \neq \phi_i(I_i)$, then $S_i \cap \phi_i(I_i) = \emptyset$ and if $S_i \cap \delta_i(I_i) \neq \emptyset$, then $S_i \cap I_i = S_i \cap \delta_i(I_i)$. Then we obtain the following result.

Theorem 4.4. Let X and Y be commutative super-rings with $(1 \neq 0)$, $R = X \times Y$ a direct product super-ring, and $S = S_1 \times S_2$ a multiplicative subset of $h(R)$. Suppose that δ_1, δ_2 are expansion functions of superideals of X, Y and ϕ_1, ϕ_2 are reduction functions of superideals of X, Y (respectively). Let $I = I_1 \times I_2$ be a proper superideal of R, such that $I \neq \hat{\phi}(I)$ for some superideals I_1, I_2 of X, Y, respectively, such that if $I_1 \neq X$, $I_2 \neq Y$ then $\delta_1(I_1) \neq X$, $\delta_2(I_2) \neq Y$ (respectively). Then I is a $\hat{\phi}$-$\hat{\delta}$-S-primary superideal of R associated to $(s_1, s_2) \in S$ that is not $\hat{\delta}$-S-primary if and only if one of the following conditions satisfies
(1) $I = I_1 \times I_2$, where $\phi_1(I_1) \subsetneqq I_1 \subsetneqq X$, such that I_1 is a ϕ_1-δ_1-S_1-primary superideal of X associated to s_1 that is not δ_1-S_1-primary and $I_2 = \phi_2(I_2)$ with $s_2 \in S_2 \cap \phi_2(I_2)$.
(2) $I = I_1 \times I_2$, where $\phi_2(I_2) \subsetneqq I_2 \subsetneqq Y$, such that I_2 is a ϕ_2-δ_2-S_2-primary superideal of Y associated to s_2 that is not δ_2-S_2-primary and $I_1 = \phi_1(I_1)$ with $s_1 \in S_1 \cap \phi_1(I_1)$.

Proof. Suppose that I is a $\hat{\phi}$-$\hat{\delta}$-S-primary superideal of R associated to (s_1, s_2) such that it is not $\hat{\delta}$-S-primary. Assume that $I_1 \neq \phi_1(I_1)$ and $I_2 \neq \phi_2(I_2)$, then by Theorem 4.3, I is a $\hat{\delta}$-S-primary superideal of R associated to (s_1, s_2), a contradiction. Therefore $I_1 = \phi_1(I_1)$ or $I_2 = \phi_2(I_2)$. Without loss of generality we may assume that $I_2 = \phi_2(I_2)$. We show that $s_2 \in S_2 \cap I_2$ or $s_1 \in S_1 \cap I_1$. Choose $x \in h(X)$ such that $x \in I_1 - \phi_1(I_1)$ then for $b \in h(Y)$ with $b \in I_2$ we have $(x, 1)(1, b) = (x, b) \in I - \hat{\phi}(I)$. Since I is a $\hat{\phi}$-$\hat{\delta}$-S-primary superideal of R associated to $(s_1, s_2) \in S$, we get $(s_1, s_2)(x, 1) \in I$ or $(s_1, s_2)(1, b) \in \hat{\delta}(I)$. Therefore, $(s_1 x, s_2) \in I = I_1 \times I_2$ or $(s_1, s_2 b) \in \hat{\delta}(I) = \delta_1(I_1) \times \delta_2(I_2)$ and hence $s_2 \in S_2 \cap I_2 = S_2 \cap \phi_2(I_2)$ or $s_1 \in S_1 \cap \delta_1(I_1) = S_1 \cap I_1$.
Case(1) Suppose that $s_2 \in S_2 \cap I_2 = S_2 \cap \phi_2(I_2)$. Then $S_1 \cap I_1 = \emptyset$, since $S \cap I = \emptyset$. Next, we show that I_1 is a ϕ_1-δ_1-S_1-primary superideal of X associated to s_1. Observe that $I_1 \neq X$. For if $I_1 = X$, then $S \cap I = (S_1 \times S_2) \cap (R_1 \times \phi_2(I_2)) = S_1 \times (S_2 \cap \phi_2(I_2)) \neq \emptyset$, a contradiction. Thus, $I_1 \neq X$. Let $a, b \in h(X)$ such that $ab \in I_1 - \phi_1(I_1)$. Then $(a, 1)(b, 0) = (ab, 0) \in I - \hat{\phi}(I)$ implies that $(s_1, s_2)(a, 1) = (s_1 a, s_2) \in I_1 \times I_2$ or $(s_1, s_2)(b, 0) = (s_1 b, 0) \in \delta_1(I_1) \times \delta_2(I_2)$. So, $s_1 a \in I_1$ or $s_1 b \in \delta_1(I_1)$. Therefore I_1 is a ϕ_1-δ_1-S_1-primary superideal of X associated to s_1. We show that I_1 is not a δ_1-S_1-primary superideal of X associated to s_1. Suppose that I_1 is a δ_1-S_1-primary superideal of X associated to s_1 and let $(a, c), (b, d) \in h(R)$ such that $(a, c)(b, d) = (ab, cd) \in I$. Then a, b are homogeneous elements in X with $ab \in I_1$ which implies that $s_1 a \in I_1$ or $s_1 b \in \delta_1(I_1)$. Since $s_2 \in S_2 \cap \phi_2(I_2)$ then $(s_1, s_2)(a, c) \in I$ or $(s_1, s_2)(b, d) \in \hat{\delta}(I)$. So, I is a $\hat{\delta}$-S-primary superideal of R associated to (s_1, s_2), a contradiction. Thus, I_1 is a ϕ_1-δ_1-S_1-primary superideal of X associated to s_1 that is not δ_1-S_1-primary.
Case(2) Suppose $s_1 \in S_1 \cap I_1$. Then $S_2 \cap I_2 = \emptyset$, since $S \cap I = \emptyset$. We show that $I_2 = \phi_2(I_2)$ is a δ_2-S_2-primary superideal of Y associated to s_2. Let $a, b \in h(Y)$ such that $ab \in I_2 = \phi_2(I_2)$. Choose $x \in h(X)$ such that $x \in I_1 - \phi_1(I_1)$, then $(x, a)(1, b) = (x, ab) \in I - \hat{\phi}(I)$, where (x, a), $(1, b)$ are homogeneous elements in R. Since I is a $\hat{\phi}$-$\hat{\delta}$-S-primary superideal of R associated

to (s_1, s_2), we have $(s_1, s_2)(x, a) = (s_1 x, s_2 a) \in I = I_1 \times I_2$ or $(s_1, s_2)(1, b) = (s_1, s_2 b) \in \hat{\delta}(I) = \delta_1(I_1) \times \delta_2(I_2)$ which implies that $s_2 a \in I_2$ or $s_2 b \in \delta_2(I_2)$. Thus, $I_2 = \phi_2(I_2)$ is a δ_2-S_2-primary superideal of Y associated to s_2. Now, we show that case(2) can't be happened by proving that I will be a $\hat{\delta}$-S-primary superideal of R associated to (s_1, s_2) which contradicts the assumption. Let $(a, c), (b, d) \in h(R)$ such that $(a, c)(b, d) = (ab, cd) \in I$, then $cd \in I_2$ implies that $s_2 c \in I_2$ or $s_2 d \in \delta_2(I_2)$. Since $s_1 \in S_1 \cap I_1$ we have $(s_1, s_2)(a, c) = (s_1 a, s_2 c) \in I_1 \times I_2 = I$ or $(s_1, s_2)(b, d) = (s_1 b, s_2 d) \in I_1 \times \delta_2(I_2) \subseteq \hat{\delta}(I)$. Thus, I is a $\hat{\delta}$-S-primary superideal of R associated to (s_1, s_2) which is a contradiction.
Conversely, suppose that (1) satisfies. Let $(a, c), (b, d) \in h(R)$ such that $(a, c)(b, d) = (ab, cd) \in I - \hat{\phi}(I)$. Then $ab \in I_1 - \phi_1(I_1)$ implies that $s_1 a \in I_1$ or $s_1 b \in \delta_1(I_1)$. Thus, $(s_1, s_2)(a, c) = (s_1 a, s_2 c) \in I_1 \times I_2 = I$ or $(s_1, s_2)(b, d) = (s_1 b, s_2 d) \in \delta_1(I_1) \times I_2 \subseteq \hat{\delta}(I)$. Thus, I is a $\hat{\phi}$-$\hat{\delta}$-S-primary superideal of R associated to (s_1, s_2). Finally, we show that I can not be a $\hat{\delta}$-S-primary superideal of R associated to (s_1, s_2). Suppose that I is a $\hat{\delta}$-S-primary superideal of R associated to (s_1, s_2), and let $a, b \in h(X)$ such that $ab \in I_1$, then $(a, s_2)(b, 1) = (ab, s_2) \in I$. So, $(s_1, s_2)(a, s_2) = (s_1 a, (s_2)^2) \in I$ or $(s_1, s_2)(b, 1) = (s_1 b, s_2) \in \hat{\delta}(I)$ which implies that $s_1 a \in I_1$ or $s_1 b \in \delta_1(I_1)$. Thus, I_1 is a δ_1-S_1-primary superideal of X associated to s_1, a contradiction. Hence I is a $\hat{\phi}$-$\hat{\delta}$-S-primary superideal of R associated to (s_1, s_2) that is not $\hat{\delta}$-S-primary. □

Corollary 4.5. Let X and Y be commutative super-rings with $(1 \neq 0)$, $R = X \times Y$. Let δ_1, δ_2 be expansion functions of superideals of X, Y, respectively. Let $I = I_1 \times I_2$ be a proper superideal of R for some superideals I_1, I_2 of X, Y, respectively, such that if $I_1 \neq X$, $I_2 \neq Y$, then $\delta_1(I_1) \neq X$, $\delta_2(I_2) \neq Y$. Then I is a nonzero weakly-$\hat{\delta}$-S-primary superideal of R associated to $(s_1, s_2) \in S$ that is not $\hat{\delta}$-S-primary if and only if one of the following conditions satisfies
(1) $I = I_1 \times I_2$, where I_1 is a nonzero proper superideal of X such that I_1 is a weakly-δ_1-S_1-primary superideal of X associated to $s_1 \in S_1$ that is not δ_1-S_1-primary and $I_2 = 0$, $s_2 = 0$.
(2) $I = I_1 \times I_2$, where I_2 is a nonzero proper superideal of Y such that I_2 is a weakly-δ_2-S_2-primary superideal of Y associated to $s_2 \in S_2$ that is not δ_2-S_2-primary and $I_1 = 0$, $s_1 = 0$.

Proof. In Theorem 4.4, let $\hat{\phi}(I) = \phi_1(I_1) \times \phi_2(I_2) = (0, 0)$ for each proper superideal $I = I_1 \times I_2$ of R. □

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Received: 2022-08-29
Accepted: 2023-09-16