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Quasi-Sasakian 3-manifolds endowed with an η-Einstein Metrics

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Abstract: The aim of this research note is to discuss the characteristics of a 3-dimensional quasi-Sasakian manifold in terms of η-Einstein Solitons. We prove that an η-Einstein soliton on 3-dimensional quasi-Sasakian manifold is an η-Einstien manifold. Moreover, we consider η-Einstein solitons in a 3-dimensional quasi-Sasakian manifold with a Ricci tensor of Codazzi type and cyclic parallel Ricci tensor. Besides these, we discuss, conformally flat and φ-Ricci symmetric η-Einstein soliton in a 3-dimensional quasi-Sasakian manifold. Also, η-Einstein soliton on a 3-dimensional quasi-Sasakian manifold with the curvature condition $Q.R = 0$ and $Q.W_2 = 0$ have been discussed. Moreover, we furnish an example of η-Einstein solitons in a 3-dimensional quasi-Sasakian-manifold. Finally, we explore an application of η-Einstein solitons in a complete and compact 3-dimensional quasi-Sasakian-manifold to Number theory in terms of homotopy group.

1 Introduction

Geometric flows are important tools for understanding the topological and geometric structures in Riemannian geometry. In 1982, Hamilton [16] introduced that the Ricci solitons move under the Ricci flow simply by diffeomorphisms of the initial metric that is they are stationary points of the Ricci flow given by

$$\frac{\partial g}{\partial t} = -2Ric(g). \tag{1.1}$$

A Ricci soliton (g, V, λ) on a Riemannian manifold is defined by

$$\mathcal{L}_V g + 2S + 2\lambda = 0, \tag{1.2}$$

where S is the Ricci tensor, $\mathcal{L}_V$ is the Lie derivative along the vector field V on M and λ is a real scalar. Ricci soliton is said to be shrinking, steady, or expanding according to $\lambda < 0, \lambda = 0$ and $\lambda > 0$, respectively.

If the vector field V is the gradient of a potential function -ψ , where ψ is some smooth function $\psi : M \to \mathbb{R}$, then g is called a gradient Ricci soliton and equation (1.2) assumes the form

$$\nabla\nabla\psi = S + \lambda g. \tag{1.3}$$

It is well known that the quantity

$$a(g, \psi) := R + \left|\nabla\psi^2\right| - \psi$$

must be constant on M and it is often called the *auxiliary constant*. When ψ is constant the gradient Ricci soliton is simply an *Einstein manifold*.Thus Ricci solitons are natural extensions of Einstein metrics, an Einstein manifold with a constant potential function is called a trivial gradient Ricci soliton. Gradient Ricci solitons play an important role in Hamiltonian Ricci flow as they correspond to self-similar solutions, and often arise as singularity models. They are also related to smooth metric measure spaces, since equation (1.3) is equivalent to ∞-Bakry-Emery Ricci tensor $Ric\psi = 0$. In physics, a smooth metric space $(M, g, e^{\psi}, dvol)$ with $Ric\psi = \lambda g$ is called quasi-Einstein manifold. Therefore it is important to study the geometry and topology of gradient Ricci solitons and their classifications.

In general, one cannot expect potential function ψ to grow or decay linearly along all directions at infinity, because of the product property: the product of any two gradient steady Ricci solitons is also a gradient steady Ricci soliton. Consider for example (R, g, ψ), where g is the standard Euclidean metric, $\psi(x_1, x_2) = x_1$. ψ is constant along x_2 direction, so without additional conditions, ψ may not have linear growth at infinity.

In 2016, Catino and Mazzieri introduced the notion of Einstein solitons [8], which generate self-similar solutions to Einstein flow

$$\frac{\partial g}{\partial t} = -2\left(S - \frac{scal}{2}g\right). \tag{1.4}$$

Our interest in studying this equation from different points of view arises from concrete physical problems. On the other hand, gradient vector fields play a central role in Morse-Smale theory. In what follows, after characterizing the manifold of constant scalar curvature via the existence of η-Einstein solitons. The existence of an η-Einstein soliton implies that the manifold is quasi-Einstein. Remark that quasi-Einstein manifolds arose during the study of exact solutions of Einstein field equations.

Sharma [25] initiated the study of Ricci solitons in contact Riemannian geometry. After that, many geometers extensively studied Ricci soliton in almost contact metric manifolds (see [13, 32]). In 2009, Cho and Kimura [9] introduced the notion of η-Ricci solitons and gave a classification of real hypersurfaces in non-flat complex space forms admitting η-Ricci solitons. In [4], Blaga studied the notion of η-Einstein solitons. Siddiqi also studied some properties of η-Einstein solitons in [26, 27, 28, 29, 30, 31] which is closely related to this study.

On the other side, in 1967, Blair [5] introduced the study of quasi-Sasakian structures to unify Sasakian and cosymplectic structures. Quasi-Sasakian manifold can be viewed as an odd dimensional counter part of Kahler structure. In [33] Tanno also studied some facts about the quasi-Sasakian structurs. The properties of quasi-Sasakian manifolds have been studied by several authors, viz., Gonzalez and Chinea [15], Kanemaki [17] and Oubina [19]. In 1990, Kim [18] studied quasi-Sasakian manifolds and proved that fibred Riemannian spaces with invariant fibers normal to the structure vector field do not admit nearly Sasakian or contact structure but a quasi-Sasakian or cosymplectic structure.

Recently, quasi-Sasakian manifolds have been the subject of growing interest in view of finding significant applications to physics, in particular to supergravity and magnetic theory (see [1], [2]). Quasi-Sasakian structures have wide applications in the mathematical analysis of string theory (see[3], [14]). Motivated by the roles of curvature tensor and Ricci tensor of quasi-Sasakian manifolds in string theory [3]. On a 3-dimensional quasi-Sasakian manifold, the structure function β was defined by Olszak [20] and with the help of this function. He has obtained the necessary and sufficient conditions for the manifold to be conformally flat [21]. Next, he has proved that if the manifold is additionally conformally flat with $\beta = constant$, then
(a) the manifold is locally a product of R and a two-dimensional Käehlerian space of constant Gauss curvature (the cosymplectic case), or,
(b) the manifold is of constant positive curvature (the non-cosymplectic case, here the quasi-Sasakian structure is homothetic to a Sasakian structure)[33].

Therefore inspired by the above research literature in the present framework authors extensively studies the η-Einstein solitons on 3-dimensional quasi-Sasakian manifold and a Number theoretic approach to the same setting.

2 Preliminaries

Let M be a $(2n+1)$-dimensional connected differentiable manifold endowed with an almost contact metric structure (φ, ξ, η, g), where φ, ξ, η are tensor field on M of types $(1, 1)$, $(1, 0)$,

$(0,1)$ respectively, such that (see [6], [7])

$$\varphi^2 X = -X + \eta(X)\xi, \tag{2.1}$$

$$\eta(\xi) = 1, \tag{2.2}$$

$$\eta(X) = g(X,\xi), \quad \varphi\xi = 0, \quad \eta \circ \varphi = 0, \tag{2.3}$$

$$g(\varphi X, \varphi Y) = g(X,Y) - \eta(X)\eta(Y), \quad X, Y \in T(M), \tag{2.4}$$

where $T(M)$ is the Lie algebra of the vector fields of manifold M.

Let Φ be the fundamental 2-form of M defined by

$$\Phi(X,Y) = g(X,\varphi Y), \quad X, Y \in T(M). \tag{2.5}$$

Then $\Phi(X,\xi) = 0$, $X \in T(M)$. M is said to be quasi-Sasakian if the almost contact structure (φ, ξ, η) is normal and the fundamental 2-form Φ is closed, that is, for every $X, Y \in \Gamma(M)^{2n+1}$, where $\Gamma(M)^{2n+1}$ denotes the module of vector fields on M,

$$[\varphi,\varphi](X,Y) + d\eta(X,Y)\xi = 0, \tag{2.6}$$

$$d\Phi = 0, \quad \Phi(X,Y) = g(X,\varphi Y). \tag{2.7}$$

This was first introduced by Blair [5]. There are many types of quasi-Sasakian structures ranging from the cosymplectic case, $d\eta = 0 (rank\eta = 1)$, to the Sasakian case, $\eta(d\eta)n \neq 0 (rank\eta = 2n+1, \varphi = d\eta)$. The 1 form η has rank $r = 2p$ if $(d\eta)^p \neq 0$ and $\eta(d\eta)^p = 0$, and has rank $r = 2p+1$ if $(d\eta)^p = 0$ and $\eta(d\eta)^p \neq 0$. We also say that r is the rank of the quasi-Sasakian structure. Blair [5] also proved that there are no quasi-Sasakian structure of even rank. In order to study the properties of quasi-Sasakian manifolds Blair [5] proved some theorems regarding Kählerian manifolds and existence of quasi-Sasakian manifolds.

An almost contact metric manifold M is a 3-dimensional quasi-Sasakian manifold, if and only if its satisfies the following relations [6]

$$\nabla_X \xi = -\beta\varphi X, \tag{2.8}$$

for a certain function β on M, such that $\xi\beta = 0$, ∇ being the operator of the covariant differentiation with respect to the Levi-Civita connection of M. Clearly, such a quasi-Sasakian manifold is cosymplectic if and only if $\beta = 0$.

As a consequence of (2.8), we have [20]

$$(\nabla_X \varphi)Y = \beta[g(X,Y)\xi - \eta(Y)X] \tag{2.9}$$

$$(\nabla_X \eta)Y = -\beta g(\varphi X, Y) \tag{2.10}$$

for all $X, Y \in T(M)$.

In a 3-dimensional Riemannian manifold, the curvature tensor is given by:

$$R(X,Y)Z = [S(Y,Z)X - S(X,Z)Y + g(Y,Z)QX - g(X,Z)QY] \tag{2.11}$$

$$-\frac{r}{2}[g(Y,Z)X - g(X,Z)Y], \tag{2.12}$$

where r is the scalar curvature of M.

Throughout this paper, we consider β as a constant. Let 3-dimensional quasi-Sasakian manifold. Since β is constant The Ricci tensor S and the Ricci operator Q is given by [20]

$$S(X,Y) = \left(\frac{r}{2} - \beta^2\right) g(X,Y) + \left(3\beta^2 - \frac{r}{2}\right) \eta(X)\eta(Y), \tag{2.13}$$

$$QX = \left(\frac{r}{2} - \beta^2\right) X + \left(3\beta^2 - \frac{r}{2}\right) \eta(X)\xi \tag{2.14}$$

Moreover, the curvature tensor R, the Ricci tensor S and the Ricci operator Q in a 3-dimensional quasi-Sasakian manifold M also satisfy the following relations [20]

$$R(X,Y)\xi = \beta^2[\eta(Y)X - \eta(X)Y] \tag{2.15}$$

$$R(\xi,X)Y = \beta^2[g(X,Y) - \eta(X)Y] \tag{2.16}$$

$$S(X,\xi) = 2\beta^2\eta(X) \tag{2.17}$$

$$Q\xi = 2\beta^2\xi, \tag{2.18}$$

where $g(QX,Y) = S(X,Y)$. Also, from (2.13) we have

$$S(\varphi X, \varphi Y) = S(X,Y) - 2\beta^2\eta(X)\eta(Y). \tag{2.19}$$

3 η-Einstein solitons on (M,φ,ξ,η,g)

Let $(M,\varphi,\xi,\eta,g,\beta)$ be an almost contact metric manifold. Consider the equation

$$\mathcal{L}_\xi g + 2S + (2\lambda - scal) + 2\mu\eta\otimes\eta = 0, \tag{3.1}$$

where $\mathcal{L}_\xi$ is the Lie derivative operator along the vector field ξ, S is the Ricci curvature tensor field of the metric g, *scal* is the scalar curvature of the Riemannian metric g and λ and μ are real constants. For $\mu \neq 0$, the data (g,ξ,λ,μ) will be called η-Einstein soliton [4].

Remark that if the scalar curvature $scal$ of the manifold is constant, then the η-Einstein soliton $(g,\xi,\lambda - \frac{scal}{2},\mu)$ reduces to an η-Ricci soliton [9] and, moreover, if $\mu = 0$, to a Ricci soliton [25] $(g,\xi,\lambda - \frac{scal}{2})$. Therefore, the two concepts of η-Einstein soliton [4] and η-Ricci soliton are distinct on manifolds of non constant scalar curvature.

Writing $\mathcal{L}_\xi g$ in terms of the Levi-Civita connection ∇, we obtain [4]:

$$2S(X,Y) = -g(\nabla_X\xi,Y) - g(X,\nabla_Y\xi) - (2\lambda - scal)g(X,Y) - 2\mu\eta(X)\eta(Y), \tag{3.2}$$

for any $X,Y \in \chi(M)$.

The data (g,ξ,λ,μ) which satisfy the equation (3.1) is said to be an η-*Einstein soliton* on M [10]. In particular if $\mu = 0$ then (g,ξ,λ) is called *Ricci soliton* [32] and it is called *shrinking*, *steady* or *expanding*, according as λ is negative, zero or positive respectively [9].

Here is an example of η-Einstein soliton on 3-dimensional quasi-Sasakian manifold.

Example 3.1. Let $M = \{(x,y,z) \in R^3 : z \neq 0\}$ where (x,y,z) are the standard coordinates of $\mathbb{R}^3$.

The vector fields are

$$e_1 = \frac{\partial}{\partial z} - y\frac{\partial}{\partial x}, \qquad e_2 = \frac{\partial}{\partial y}, \qquad e_3 = 2\frac{\partial}{\partial x}$$

Let g be the Riemannian metric defined by

$$g(e_1,e_1) = g(e_2,e_2) = g(e_3,e_3) = 1, \qquad g(e_1,e_3) = g(e_2,e_3) = g(e_1,e_2) = 0$$

that is, the form of the metric becomes Let η be the 1-form defiend by $\eta(Z) = g(Z,e_3)$ for any $Z \in \chi(M)$.

Also, let φ be the $(1,1)$ tensor field defined by

$$\varphi(e_1) = -e_2, \quad \varphi(e_2) = e_1, \quad \varphi(e_3) = 0.$$

Thus using the linearity of φ and g, we have

$$\eta(e_3) = 0, \quad ,\eta(e_1) = 0, \quad \eta(e_2) = 0,$$

$$[e_1, e_2] = \frac{1}{2}e_3, \quad [e_2, e_3] = 0, \quad [e_1, e_3] = 0,$$

$$\varphi^2 Z = -Z + \eta(Z)e_3$$

,

$$g(\varphi Z, \varphi W) = g(Z, W) - \eta(Z)\eta(W)$$

for any $Z, W \in \chi(M)$.

Then for $e_3 = \xi$, the structure $\varphi, \xi, \eta, g)$ defines an almost contact metric structure on M. Let ∇ be the Levi-Civita connection with respect to the metric g, then we have

$$2g(\nabla_X Y, Z) = Xg(Y, Z) + Yg(Z, X) - Zg(X, Y) - g(X, [Y, Z])$$
$$-g(Y, [X, Z]) + g(Z, [X, Y]),$$

which is known as Koszul's formula.
Using Koszul's formula we have

$$\nabla_{e_1} e_1 = 0, \quad \nabla_{e_1} e_2 = -\frac{1}{4}e_3, \quad \nabla_{e_1} e_3 = \frac{1}{4}e_3,$$

$$\nabla_{e_2} e_1 = \frac{1}{4}e_3, \quad \nabla_{e_2} e_2 = 0, \quad \nabla_{e_2} e_3 = -\frac{1}{4}e_1,$$

$$\nabla_{e_3} e_1 = \frac{1}{4}e_2, \quad \nabla_{e_3} e_2 = -\frac{1}{4}e_1, \quad \nabla_{e_3} e_0 = 0. \tag{3.3}$$

From (3.3) we find that the structure (φ, ξ, η, g) satisfies the formula (2.8) for $\beta = \frac{1}{4}$ and $\xi = e_3$. Hence the manifold is a 3-dimensional quasi-Sasakian manifold with the constant structure function $\beta = \frac{1}{4}$ [13].

Then the Riemannian and Ricci curvature tensor fields are given by:

$$R(e_1, e_2)e_3 = 0, \quad R(e_2, e_3)e_3 = \frac{1}{16}e_2, \quad R(e_1, e_3)e_3 = \frac{1}{16}e_1,$$

$$R(e_1, e_2)e_2 = -\frac{3}{16}e_1, \quad R(e_2, e_3)e_2 = -\frac{1}{16}e_3, \quad R(e_1, e_3)e_2 = 0,$$

$$R(e_1, e_2)e_1 = \frac{3}{16}e_2, \quad R(e_2, e_3)e_1 = 0, \quad R(e_1, e_3)e = -\frac{1}{16}e_3.$$

From the above expressions of the curvature tensor we obtain

$$S(e_1, e_1) = g(R(e_1, e_2)e_2, e_1) + g(R(e_1, e_3)e_3, e_1) = -\frac{1}{8}$$

similarly we have

$$S(e_1, e_1) = S(e_2, e_2) = -\frac{1}{8}, \quad \text{and} \quad S(e_3, e_3) = \frac{1}{8}$$

. In case of η-Einstein soliton, from the relation (3.9) it is sufficient to verify that

$$S(e_i, e_i) = -\left(\lambda - \frac{scal}{2}\right) g(e_i, e_i) - \mu\eta(e_i)\eta(e_i) \tag{3.4}$$

for all $i = 1, 2, 3$, and $scal = -\frac{1}{8}$, we get

$$S(e_1, e_1) = -\left(\lambda - \frac{scal}{2}\right) g(e_1, e_1)$$

which implies

$$S(e_1, e_1) = -\left(\lambda - \frac{scal}{2}\right) \Rightarrow \lambda = \frac{1}{16}.$$

Also,

$$S(e_3, e_3) = -\left(\lambda - \frac{scal}{2}\right) g(e_3, e_3) - \mu\eta(e_3)\eta(e_3) \tag{3.5}$$

By using $\lambda = \frac{1}{16}$ and $scal = -\frac{1}{8}$ in (3.5) we obtain $\mu = -\frac{1}{4}$.
Therefore, the data $(g, \xi, \lambda - \frac{scal}{2}, \mu)$ is an η-Einstein soliton in 3-dimensional quasi-Sasakian manifold.

For this example we have $\lambda = \frac{1}{16}$, i.e. $\lambda > 0$ so that the η-Einstein soliton in 3-dimensional quasi-Sasakian manifold is expanding.

Proposition 3.2. *An η-Einstein soliton on a 3-dimensional quasi-Sasakian manifold is an η-Einstien manifold.*

Proof. Let us consider that a 3-dimensional quasi-Sasakian manifold admits a proper η-Einstein soliton $(g, \xi, \lambda - \frac{scal}{2}, \mu)$. Then from the relation (3.1) yields

$$(\mathcal{L}_\xi g)(X, Y) + 2S(X, Y) + (2\lambda - scal)g(X, Y) + 2\mu\eta(X)\eta(Y) = 0. \tag{3.6}$$

It follows that

$$2S(X, Y) = -(\mathcal{L}_\xi g)(X, Y) - (2\lambda - scal)g(X, Y) - 2\mu\eta(X)\eta(Y) \tag{3.7}$$

for all smooth vector fileds $X, Y(M)$. Since ξ is Killing and its integral curves are geodesics in 3-dimensional quasi-Sasakian manifold [20], that is

$$(\mathcal{L}_\xi g)(X, Y) = 0. \tag{3.8}$$

Now, using (3.8), we obtain

$$S(X, Y) = -\left(\lambda - \frac{scal}{2}\right) g(X, Y) - \mu\eta(X)\eta(Y). \tag{3.9}$$

Therefore, we conclude that η-Einstein soliton $(g, \xi, \lambda - \frac{scal}{2}, \mu)$ on a 3-dimensional quasi-Sasakian manifold is an η-Einstein manifold. This complete the proof. □

Proposition 3.3. *In a 3-dimensional quasi-Sasakian manifold for an η-Einstein soliton we have $\lambda + \mu = 2\beta^2 - \frac{r}{2}$.*

Proof. The Ricci tensor of a 3-dimensional quasi-Sasakian manifold is given by

$$S(X, Y) = \left(\frac{r}{2} - \beta^2\right) g(X, Y) + \left(3\beta^2 - \frac{r}{2}\right) \eta(X)\eta(Y), \tag{3.10}$$

where r is the scalr curvature. Comparing the equation (3.10) with equation (3.9), we obtain $-\left(\lambda - \frac{scal}{2}\right) = \frac{1}{2}(r - 2\beta^2)$ and $\mu = -3\beta^2 + \frac{r}{2}$. Since here $scal = r$ is the scalar curvature. From which it follows that $\lambda + \mu = 2\beta^2 - \frac{r}{2}$. This complete the proof. □

4 η-Einstein Soliton on 3-dimensional quasi-Sasakian manifolds with Ricci tensor of Coddazi type

In this section, we consider proper η-Einstein soliton on 3-dimensional quasi-Sasakian manifolds with Ricci tensor of Coddazi type. Therefore taking the covariant differentiation of (3.8) with respect to Z we obtain

$$(\nabla_Z S)(X, Y) = -\mu[(\nabla_X \eta)(X)\eta(Y) + \eta(X)(\nabla_Z \eta)(Y)]. \tag{4.1}$$

Using (2.10), we get

$$(\nabla_Z S)(X,Y) = \beta\mu[g(\varphi Z, X)\eta(Y) + \eta(X)g(\varphi Z, Y)]. \tag{4.2}$$

Now, using the fact that the Ricci tensor S of Coddazi type. Then

$$(\nabla_Z S)(X,Y) = (\nabla_Y S)(Z,X). \tag{4.3}$$

Using equations (4.2) and (4.3), we have

$$\beta\mu[g(\varphi Z, X)\eta(Y) + g(\varphi Z, Y)\eta(X)] = \beta\mu[g(\varphi Y, Z)\eta(X) + g(\varphi Y, X)\eta(Z)]. \tag{4.4}$$

Putting $Z = \xi$ in (4.4), we get $\beta\mu = 0$, which is a contradiction. Hence a 3-dimensional quasi-Sasakian manifold with Ricci tensor of Coddazi type does not admit proper η-Einstein soliton. Thus we conclude the following:

Theorem 4.1. *A 3-dimensional quasi-Sasakian manifold with Ricci tensor of Coddazi type does not admit a proper η-Einstein soliton.*

5 η-Einstein soliton on 3-dimensional quasi-Sasakian manifold with cyclic parallel Ricci tensor

This section deals with a proper η-Einstein soliton on a 3-dimensional quasi-Sasakian manifold with cyclic parallel Ricci tensor. Therefore, we have

$$(\nabla_X S)(Y,Z) + (\nabla_Y S)(Z,X) + (\nabla_Z S)(X,Y) = 0, \tag{5.1}$$

for all smooth vector fields $X, Y, Z \in \Gamma(TM)$. Using (3.9) in (5.1), we have

$$\beta\mu[g(Y, \varphi X)\eta(Z) + g(Z, \varphi X)\eta(Y) + g(Z, \varphi Y)\eta(X) \tag{5.2}$$

$$+g(X, \varphi Y)\eta(Z) + g(X, \varphi Z)\eta(Y) + g(Y, \varphi Z)\eta(X)] = 0.$$

Putting $X = \xi$ in (refs6), we get

$$\beta\mu g(\varphi Y, Z) = 0. \tag{5.3}$$

It follow that $\beta\mu = 0$, since $(\beta \neq 0)$ or $\mu = 0$, which is a contradiction. Thus we can state the following theorem:

Theorem 5.1. *A 3-dimensional quasi-Sasakian manifold with cyclic parallel Ricci tensor does not admit a proper η-Eisntein soliton.*

6 φ-Ricci Symmetric η-Einstein soliton on 3-dimensional quasi-Sasakian manifold

In this section, we study φ-Ricci symmetric proper η-Einstein soliton on a 3-dimensional quasi-Sasakian manifold. A quasi-Sasakian manifold is said to be φ-Ricci symmetric [13] if

$$\varphi^2(\nabla_X Q)Y = 0 \tag{6.1}$$

for all smooth vector fields X, Y on M.

The Ricci tensor for an η-Einstein soliton on a 3-dimensional quasi-Sasakian manifold is given by

$$S(X,Y) = -\left(\lambda - \frac{scal}{2}\right) g(X,Y) - \mu\eta(X)\eta(Y). \tag{6.2}$$

Then it follows that

$$QX = -\left(\lambda - \frac{scal}{2}\right) X - \mu\eta(X)\xi \tag{6.3}$$

for all smooth vector filed x on M. Now, taking covariant derivative of (s10), we obtain

$$(\nabla_X Q)Y = \beta\mu g(\varphi X, Y) + \beta\mu\eta(Y)\varphi X. \tag{6.4}$$

Applying φ^2 on both side of the equation (6.4), we get

$$\varphi^2(\nabla_X Q)Y = \beta\mu\eta(Y)\varphi^2 X. \tag{6.5}$$

From (6.1) and (6.5) we have

$$\beta\mu = 0. \tag{6.6}$$

Since $\beta \neq 0$. Therefore $\mu = 0$, which is a contradiction. Therefore a φ-Ricci symmetric 3-dimensional quasi-Sasakian manifold does not admit a proper η-Einstein soliton. Thus we can state the following theorem:

Theorem 6.1. *A φ-Ricci symmetric 3-dimensional quasi-Sasakian manifold does not admit a proper η-Einstein soliton.*

7 Conformally flat η-Einstein soliton on 3-dimensional quasi-Sasakian manifold

In this section we study conformally flat proper η-Einstein soliton in a 3-dimensional quasi-Sasakian manifold. Therefore [**?**]

$$(\nabla_X S)(Y, Z) - (\nabla_Y S)(X, Z) = \frac{1}{4}[g(Y, Z)dr(X) - g(X, Z)dr(Y)]. \tag{7.1}$$

Using (3.9) in (7.1)

$$\beta\mu[g(Y, \varphi X)\eta(Z) + g(Z, \varphi X)\eta(Y)?g(X, \varphi Y)\eta(Z)?g(Z, \varphi Y)\eta(X)] \tag{7.2}$$

$$= \frac{1}{4}[g(Y, Z)dr(X) - g(X, Z)dr(Y)].$$

Putting $X = \xi$ in (7.2), we get

$$4\beta\mu g(\varphi Y, Z) = -\eta(Z)dr. \tag{7.3}$$

It follow that $4\beta\mu\varphi Y = -dr(Y)\xi$. This implies that $\beta\mu\varphi^2 Y = 0$. Hence $\beta\mu = 0$. Since $\beta \neq 0$ or $\mu = 0$, which is a contradiction. Therefore a conformally flat 3-dimensional quasi-Sasakian manifold does not admit a proper η-Einstein soliton. Thus we can state the following theorem:

Theorem 7.1. *A conformally flat 3-dimensional quasi-Sasakian manifold does not admit a proper η-Einstein soliton.*

8 η-Einstein soliton on 3-dimensional quasi-Sasakian manifold satisfying the curvature condition $Q.R = 0$

In this section, we study the proper η-Einstein soliton in a 3-dimensional quasi-Sasakian manifold satisfying the curvature condition $Q.R = 0$. Therefore

$$(Q.R)(X, Y)Z = 0 \tag{8.1}$$

for all smooth vector fileds X, Y, Z on M. The explicit form of the above equation is

$$Q(R(X, Y)Z) - R(QX, Y)Z - R(X, QY)Z - R(X, Y)QZ = 0. \tag{8.2}$$

Using (3.9) in (8.2) we have

$$-\lambda_1 R(X, Y)Z - \mu\eta(R(X, Y)Z)\xi + \lambda_1 R(X, Y)Z + \mu\eta(X)R(\xi, Y)Z \tag{8.3}$$

$$+\lambda_1 R(X,Y)Z + \mu\eta(Y)R(X,\xi)Z + \lambda_1 R(X,Y)Z + \mu\eta(Z)R(X,Y)\xi = 0,$$

where $\lambda_1 = \left(\lambda - \frac{scal}{2}\right)$. Using (2.11) in (8.3) we have

$$-\mu g(Y,Z)\eta(QX)\xi + \mu g(X,Z)\eta(QY)\xi - \mu\eta(X)\eta(QZ)Y \tag{8.4}$$

$$+mu\eta(X)g(Y,Z)Q\xi - 2\mu\eta(X)\eta(Z)QY + r\mu\eta(X)\eta(Z)Y$$

$$+\mu\eta(Y)\eta(QZ)X + 2\mu\eta(Y)\eta(Z)QX - \mu\eta(Y)g(X,Z)Q\xi$$

$$-r\mu\eta(Y)\eta(Z)X + \mu\eta(Z)\eta(QY)X - \mu\eta(Z)\eta(QX)Y$$

$$+2\lambda_1[S(Y,Z)X - S(X,Z)Y + g(Y,Z)QX - g(X,Z)QY]$$

$$-\frac{r}{2}[g(Y,Z)X - g(X,Z)Y] = 0.$$

Again using (3.9) in (8.4) we obtain

$$\lambda_1\mu g(Y,Z)\eta(X)\xi + \mu^2 g(Y,Z)\eta(X)\xi - \mu g(Y,Z)\eta(-\lambda_1 X - \mu\eta(X)\xi)\xi \tag{8.5}$$

$$+\mu g(X,Z)\eta(-\lambda_1 X - \mu\eta(Y)\xi)\xi - \eta(X)\eta(-\lambda_1 Z - \mu\eta(Z)\xi)Y$$

$$+\mu\eta(X)g(Y,Z)(-\lambda_1\xi - \mu\xi) - 2\mu\eta(X)\eta(Z)(\lambda_1 Y - \mu\eta(Y)\xi + r\mu\eta(X)\eta(Z)Y$$

$$+\mu\eta(Y)\eta(-\lambda_1\xi - \mu\xi) - r\mu\eta(Y)\eta(Z)X + \mu\eta(Z)\eta(-\lambda_1 Y - \mu\eta(Y)\xi)X$$

$$-\mu\eta(Z)\eta(-\lambda_1 X - \mu\eta(X)\xi)Y + 2\lambda_1[(-\lambda_1 g(Y,Z) - \mu\eta(Y)\eta(Z))X - (-\lambda_1 g(X,Z)$$

$$-\mu\eta(X)\eta(Z))Y + g(Y,Z)(-\lambda_1 X - \mu\eta(X)\xi) - g(X,Z)(-\lambda_1 Y - \mu\eta(Y)\xi)$$

$$-\frac{r}{2}(g(Y,Z)X - g(X,Z)Y)] = 0.$$

from which it follows that

$$4\lambda_1\mu\eta(X)\eta(Z)Y + 2\mu^2\eta(X)\eta(Z)Y - 4\lambda_1\mu\eta(Y)\eta(Z)X + r\mu\eta(X)\eta(Z)Y \tag{8.6}$$

$$-r\mu\eta(Y)\eta(Z)X + 2\lambda_1[(-\lambda_1 g(Y,Z) - \mu\eta(X)\eta(Z))X - (-\lambda_1 g(X,Z) - \mu\eta(X)\eta(Z))Y$$

$$+g(Y,Z)(-\lambda_1 X - \mu\eta(X)\xi) - g(X,Z)(-\lambda_1 Y - \mu\eta(Y)\xi) - \frac{r}{2}(g(Y,Z)X - g(X,z)Y)]$$

$$-2\mu^2\eta(Y)\eta(Z)X + \lambda_1\mu g(Y,Z)\eta(X)\xi + \mu^2 g(Y,Z)\eta(X)\xi = 0.$$

Putting $X = Z = \xi$ in (8.6) we get

$$4\lambda_1\mu Y - 4\lambda_1\mu\eta(Y)\xi + r\mu Y - r\mu\eta(Y)\xi \tag{8.7}$$

$$+2\lambda_1[-\lambda_1\eta(Y)\xi - \mu\eta(Y)\xi + \lambda_1 Y + \mu Y - \lambda_1\eta(Y)\xi$$

$$-\mu\eta(Y)\xi + \lambda_1 Y + \mu\eta(Y)\xi - \frac{1}{2}\eta(Y)\xi + \frac{1}{2}Y$$

$$-2\mu^2\eta(Y)\xi + \lambda_1\mu\eta(Y)\xi + \mu^2\eta(Y)\xi = 0.$$

Taking inner product of (8.7) with ξ we get

$$\mu(\lambda_1 + \mu)\eta(Y) = 0. \tag{8.8}$$

By hypothesis $\mu \neq 0$, therefore from the above equation we get

$$\lambda_1 + \mu = 0, ie. \quad \lambda - \frac{scal}{2} + \mu = 0$$

Moreover, for an η-Einstein soliton on a 3-dimensional quasi-Sasakian manifold, we have

$$\lambda + \mu = 2\beta^2 - \frac{r}{2}$$

Thus in this case $\lambda = \beta^2 - \frac{r}{4}$ and $\mu = \beta^2 - \frac{r}{4}$. Therefore we have the following:

Theorem 8.1. *An proper η-Einstein soliton on 3-dimensional quasi-Sasakain manifold Sasakian satisfying the curvature condition Q.R = 0 is of the type* $(g, V, \frac{\beta^2}{2}, \frac{\beta^2}{2})$.

9 η-Einstein soliton on 3-dimensional quasi-Sasakian manifold satisfying the curvature condition $Q.W_2 = 0$.

W_2-curvature tensor filed is the curvature tensor introduced by Pokhariyal and Mishra [24].

$$W_2(X, Y)Z = R(X, Y)Z + \frac{1}{dimM - 1}[g(X, Z)QY - g(Y, Z)QX]. \tag{9.1}$$

In this section, we study proper η-Einstein soliton on a 3-dimensional quasi-Sasakian manifold satisfying the curvature condition $Q.W_2 = 0$. Therefore

$$(Q.W_2)(X, Y)Z = 0. \tag{9.2}$$

for all smooth vector filed X, Y, Z on M. The explicit form of the equation (9.2) is

$$Q(W_2(X, Y)Z) - W_2(QX, Y)Z - W_2(X, QY)Z - W_2(X, Y)QZ = 0. \tag{9.3}$$

Using (3.9) in (9.3) we have

$$-\lambda_1 W_2(X, Y)Z - \mu\eta(W_2(X, Y)Z)\xi + \lambda_1 W_2(X, Y)Z + \mu\eta(X)W_2(\xi, Y)Z \tag{9.4}$$

$$+\lambda_1 W_2(X, Y)Z + \mu\eta(Y)W_2(X, \xi)Z + \lambda_1 W_2(X, Y)Z + \mu\eta(Z)W_2(X, Y)\xi = 0,$$

where $\lambda_1 = \left(\lambda - \frac{scal}{2}\right)$. Using (9.1) in (9.4) we have

$$-\mu g(Y, Z)\eta(QX)\xi + \mu g(X, Z)\eta(QY)\xi - \mu\eta(X)\eta(QZ)Y \tag{9.5}$$

$$-\frac{\mu}{2}g(X, Z)\eta(QY)\xi + \frac{\mu}{2}g(Y, Z)\eta(QX)\xi$$

$$+\mu\eta(X)g(Y, Z)Q\xi - 2\mu\eta(X)\eta(Z)QY + r\mu\eta(X)\eta(Z)Y$$

$$\frac{1}{2}\mu\eta(X)\eta(Z)QY - \frac{1}{2}\mu\eta(X)g(Y, Z)Q\xi$$

$$+\mu\eta(Y)\eta(QZ)X + 2\mu\eta(Y)\eta(Z)QX - \mu\eta(Y)g(X,Z)Q\xi$$

$$\frac{\mu}{2}\eta(Y)g(X,Z)Q\xi - \frac{\mu}{2}\eta(Y)\eta(Z)QX$$

$$-r\mu\eta(Y)\eta(Z)X + \mu\eta(Z)\eta(QY)X - \mu\eta(Z)\eta(QX)Y$$

$$+\frac{\mu}{2}\eta(Z)\eta(Y)QY - \frac{\mu}{2}\eta(Z)\eta(Y)QX$$

$$+2\lambda_1[S(Y,Z)X - S(X,Z)Y + g(Y,Z)QX - g(X,Z)QY]$$

$$-\frac{r}{2}[g(Y,Z)X - g(X,Z)Y] + \frac{1}{2}[g(X,Z)QY - g(Y,Z)QX] = 0.$$

Again using (3.9) in (9.5) we obtain

$$\frac{\lambda_1\mu}{2}g(Y,Z)\eta(X)\xi + \frac{\mu^2}{2}g(Y,Z)\eta(X)\xi - \frac{\mu}{2}g(Y,Z)\eta(-\lambda_1X - \mu\eta(X)\xi)\xi \tag{9.6}$$

$$+\frac{\mu}{2}g(X,Z)\eta(-\lambda_1X - \mu\eta(Y)\xi)\xi - \eta(X)\eta(-\lambda_1Z - \mu\eta(Z)\xi)Y$$

$$+\mu\eta(X)g(Y,Z)(-\lambda_1\xi - \mu\xi) - \frac{5}{2}\mu\eta(X)\eta(Z)(\lambda_1Y - \mu\eta(Y)\xi + r\mu\eta(X)\eta(Z)Y$$

$$+\mu\eta(Y)\eta(-\lambda_1\xi - \mu\xi) - r\mu\eta(Y)\eta(Z)X + \mu\eta(Z)\eta(-\lambda_1Y - \mu\eta(Y)\xi)X$$

$$-\mu\eta(Z)\eta(-\lambda_1X - \mu\eta(X)\xi)Y + 2\lambda_1[(-\lambda_1g(Y,Z) - \mu\eta(Y)\eta(Z))X - (-\lambda_1g(X,Z)$$

$$-\mu\eta(X)\eta(Z))Y + g(Y,Z)(-\lambda_1X - \mu\eta(X)\xi) - g(X,Z)(-\lambda_1Y - \mu\eta(Y)\xi)$$

$$-\frac{r}{2}(g(Y,Z)X - g(X,Z)Y)] + \frac{1}{2}[[g(X,Z)(-\lambda_1Y - \mu\eta(Y)\xi - g(Y,Z)(-\lambda_1X - \mu\eta(X)\xi] = 0.$$

from which it follows that

$$\frac{9}{2}\lambda_1\mu\eta(X)\eta(Z)Y + 2\mu^2\eta(X)\eta(Z)Y - 3\lambda_1\mu\eta(Y)\eta(Z)X + r\mu\eta(X)\eta(Z)Y \tag{9.7}$$

$$-r\mu\eta(Y)\eta(Z)X + 2\lambda_1[(-\lambda_1g(Y,Z) - \mu\eta(X)\eta(Z))X - (-\lambda_1g(X,Z) - \mu\eta(X)\eta(Z))Y$$

$$+g(Y,Z)(-\lambda_1X - \mu\eta(X)\xi) - g(X,Z)(-\lambda_1Y - \mu\eta(Y)\xi) - \frac{r}{2}(g(Y,Z)X - g(X,z)Y)]$$

$$-\lambda_1^2g(X,Z)Y + \lambda_1\mu\eta(Y)g(X,Z)\xi + \lambda_1^2g(Y,Z)X + \lambda_1\mu\eta(X)g(Y,Z)\xi$$

$$-2\mu^2\eta(Y)\eta(Z)X + \frac{3}{2}\mu^2\eta(X)\eta(Y)\eta(Z)\xi = 0.$$

Putting $X = Z = \xi$ in (9.7) we get

$$\frac{9}{2}\lambda_1\mu Y + 2\mu^2Y - 3\lambda - 1\mu\eta(Y)\xi + r\mu Y - r\mu\eta(Y)\xi \tag{9.8}$$

$$+2\lambda_1[-\lambda_1\eta(Y)\xi - \mu\eta(Y)\xi + \lambda_1 Y + \mu Y - \lambda_1\eta(Y)\xi$$

$$-\mu\eta(Y)\xi + \lambda_1 Y + \mu\eta(Y)\xi - \frac{1}{2}\eta(Y)\xi + \frac{1}{2}Y - \lambda_1^2\eta(Y)\xi + \lambda_1^2 Y$$

$$-2\mu^2\eta(Y)\xi + \frac{3}{2}\mu^2\eta(Y)\xi = 0.$$

Taking inner product of (9.8) with ξ we get

$$\mu\left(\frac{5}{2}\lambda_1 + \frac{3}{2}\mu\right)\eta(Y) = 0. \tag{9.9}$$

By hypothesis $\mu \neq 0$, therefore from the above equation we get

$$\frac{5}{2}\lambda_1 + \frac{3}{2}\mu = 0, ie. \quad \frac{1}{2}(5\lambda - scal) + \frac{3}{2}\mu = 0$$

Moreover for an η-Einstein soliton on 3-dimensional quasi-Sasakian manifold we have

$$\lambda + \mu = 2\beta^2 - \frac{r}{2}$$

Thus in this case $\lambda = 2\beta^2$ and $\mu = -2\beta^2$. Therefore we have the following:

Theorem 9.1. *An proper η-Einstein soliton on 3-dimensional quasi-Sasakain manifold Sasakian satisfying the curvature condition $Q.W_2 = 0$ is of the type $(g, V, 2\beta^2, -2\beta^2)$.*

10 Application of η-Einstein soliton on quasi-Sasakain manifold to Algebraical Number Theory

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In 2008, Wylie [34] proved that if (M, g) is a complete Riemannian manifold satisfying (1.2), then M has a finite fundamental group (for more information see Theorem 1.1 [34]).

Remark 10.1. According to Theorem 1 and Theorem 2, proved by Rustanov (see [23]), he has shown that with a non-zero scalar curvature a quasi-Sasakian manifold is complete, it is compact and has a finite fundamental group.

Remark 10.2. [22] A finite fundamental group is the first and simplest homotopy group. The fundamental group is homotopy invariant, topological spaces are homotopy equivalent and have isomorphic fundamental groups. The fundamental group or a homotopy group of a topological space X is denoted by $\pi_1(X)$.

In view of (3.1), non-zero scalar curvature $scal$, implies the existence η-Einstein soliton on 3-dimensional quasi-Sasakain manifold. Therefore, in light of (3.1), Remark 10.1, and Remark 10.2 we gain the following outcomes:

Theorem 10.3. *If a 3-dimensional quasi-Sasakain manifold (M^3, g) admits an η-Einstein soliton with non-zero scalar curvature $scal$, then, it is complete, compact and has a finite fundamental group.*

Theorem 10.4. *If a 3-dimensional quasi-Sasakain manifold (M^3, g) admits an η-Einstein soliton with non-zero scalar curvature $scal$, then, it is complete, compact and has a homotopy group $\pi_1(M^3)$.*

Next, we have the following discussion, which is based on the Number theoretic approach with the homotopy group of a space.

Remark 10.5. For a prime number p, the homotopy p-exponent of a topological space $\mathcal{T}$, denoted by $Exp_p(U)$, is defined to be a largest $e \in \mathbb{N} = \{0, 1, 2, \cdots\}$ such that some homotopy group $\Phi_j(\mathcal{T})$ has an element of order p^e. Cohen et al. [10] proved that the

$$Exp_p(\mathcal{S}^{2n+1}) = n \quad if \quad p \neq 2.$$

For a prime number p and a integer z, the p-adic order of z is given by $Ord_p(z) = sup\{z \in \mathbb{N} : p^z \,|z\}$.

In 2007, Davis and Sun derived an interesting inequality in terms of a homotopy group (Theorem 1.1 Page 2 [12]). According to them, for any prime p and $z = 2, 3, \cdots$ some homotopy group $\pi_i(\mathcal{T})$ contains an element of order $p^{n-1+Ord_p(\lfloor n/p \rfloor !)}$, i.e., then the strong and elegant lower bound for homotopy p-exponent of homotopy group is

$$Exp_p(\pi_i(\mathcal{T})) \geq n - 1 + Ord_p\left(\left\lfloor \frac{n}{p} \right\rfloor !\right), \tag{10.1}$$

where $\pi_i(\mathcal{T})$ is a homotopy group of degree n.
Therefore, using Davis and Sun result (Theorem 1.1 [12]) with Theorem 10.4, we gain an interesting inequality

Theorem 10.6. *For any prime number p and $s = 2, 3, \cdots$, a homotopy group $\pi_1(M^3)$ of a complete 3-dimensional quasi-Sasakain manifold (M^3, g) admits an η-Einstein soliton with non-zero scalar curvature $scal$, contains an element of order $p^{s-1+Ord_p(\lfloor s/p \rfloor !)}$, we obtain the inequality*

$$Exp_p(\pi_1(M^3)) \geq n - 1 + Ord_p\left(\left\lfloor \frac{s}{p} \right\rfloor !\right). \tag{10.2}$$

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