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SOME NEW DARBO TYPE FIXED POINT THEOREMS USING GENERALIZED OPERATORS AND EXISTENCE OF A SYSTEM OF FRACTIONAL DIFFERENTIAL EQUATIONS

V. E. Nikam, A. K. Shukla and D. Gopal

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Abstract In this manuscript we implement the generalized Darbo fixed point theorem to ensure the existence of solution for Atangana-Baleanu fractional ordered system of differential equation. For this motive, we define a new class of condensing operators and prove generalized Darbo fixed point theorems using generalized operator and utilized to prove an existence theorem. To validate theoretical results, we provide an illustrative example which can be considered as particular case of Lotka-Volterra model redefined in the sense of Atangana-Baleanu system of fractional differential equation with some local conditions.

1 Introduction

Integral and differential equations plays significant role in various mathematical models. Particularly, the Atangana-Baleanu fractional ordered differential equation plays key role in economics, biology, engineering and applied science [3, 2]. Fixed point theory is widely acceptable tool for the existence of solution to such integral and differential equation. Indeed the Darbo fixed point theorem along with measure of non-compactness is incontestable tool to show the existence of solution for system of integral and differential equation [1, 10, 7]. Following this direction of research work, we proposed a genralized Darbo fixed point theorem using the operator $\mathcal{H}(\Im;\bullet)$. The proposed result generalizes and modifies some well known result available in literature's. Further the generalized Darbo fixed point is utilized to prove the existence solution of Atangana-Baleanu fractional ordered system of differential equation. At the end numerical example which can be considered as particular case of Lotka-Volterra model [5, 12] in the sense of Atangana-Baleanu fractional differentiation is discussed.

1.1 Preliminaries

Through out this paper, we will denote the class of nonempty, bounded, closed and convex sets by $\mathcal{N}.\mathcal{B}.\mathcal{C}.\mathcal{C}$, the term measure of non-compactness by $\mathcal{M}.\mathcal{N}.\mathcal{C}$, $\mathcal{B}_r$ means the open ball with radius r, $conv(.)$ as convex hull, $\mathcal{M}_\mathcal{B}$ and $\mathcal{N}_\mathcal{B}$ means the class of all bounded subsets of the space $\mathcal{B}$ and the symbols $\mathbb{R}$, $\mathbb{R}_+$ & $\mathbb{N}$ are used to denote the set of all real numbers, the set of all positive real numbers and the set of all positive integers respectively. Now, we recall some fundamental definitions in the direction of Darbo fixed point theorem;

Definition 1.1. [6] A mapping $\aleph : \mathcal{M}_E \to \mathbb{R}_+$ is termed as $\mathcal{M}.\mathcal{N}.\mathcal{C}$, if it satisfy the following conditions:

i) (Regularity) $\aleph(\mathcal{S}) = 0 \Leftrightarrow \mathcal{S}$ is pre-compact,

ii) The family $\ker(\aleph) = \{\mathcal{S} \in \mathcal{M}_\mathcal{B} \,|\aleph\,(\mathcal{S})\,=0\}$ is non-empty and $\ker(\aleph) \subset \mathcal{N}_\mathcal{B}$,

iii) (Monotonicity) $\mathcal{S}_1 \subseteq \mathcal{S}_2 \Rightarrow \aleph\,(\mathcal{S}_1) \leq \aleph\,(\mathcal{S}_2)$,

iv) (Invariant under closure) $\aleph\left(\overline{\mathcal{S}}\right) = \aleph\,(\mathcal{S})$,

v) (Invariant under convex hull) $\aleph\,(\mathcal{S}) = \aleph\,(conv(\mathcal{S}))$,

vi) (Convexity) $\aleph\left(\lambda \mathcal{S}_1+(1-\lambda) \mathcal{S}_2\right) \leq \lambda \aleph\left(\mathcal{S}_1\right)+(1-\lambda) \aleph\left(\mathcal{S}_2\right), \forall \lambda \in[0,\ 1]$,

vii) (Generalized Cantor ′s intersection theorem) Let $\langle\mathcal{S}_n\rangle$ be a non-increasing sequence of closed subsets of a class $\mathcal{M}_{\mathcal{B}}$ for the Banach space $\mathcal{B}$. If $\lim_{n \rightarrow \infty} \aleph\left(\mathcal{S}_n\right)=0$, then the countable intersection $\mathcal{S}_{\infty}=\bigcap_{n=1}^{\infty} \mathcal{S}_n$ is non empty.

The family of sets $ker(\aleph)=\{\mathcal{S} \in \mathcal{M}_{\mathcal{B}} \mid \aleph(\mathcal{S})=0\}$ mentioned in assumption $ii)$ is called the kernel of the $\mathcal{M}.\mathcal{N}.\mathcal{C}$. In fact, $\mathcal{S}_{\infty} \subseteq \mathcal{S}_n$ for all $n \in \mathbb{N}$ hence by the virtue of condition $iii)$ we have $\aleph\left(\mathcal{S}_{\infty}\right) \leq \aleph\left(\mathcal{S}_n\right)$. Therefore $\aleph\left(\mathcal{S}_{\infty}\right) \rightarrow 0$ as $n \rightarrow \infty$, which implies $\mathcal{S}_{\infty} \in ker(\aleph)$.
Following are the some fundamental results that plays central role in the direction of development and application of fixed point theorems.

Theorem 1.2. *[11] Let $\mathcal{K}$ be a member of the class $\mathcal{N}.\mathcal{B}.\mathcal{C}.\mathcal{C}$ of a Banach space $\mathcal{B}$, then for any compact and continuous mapping on $\mathcal{K}$ admits at least one fixed point in $\mathcal{K}$.*

Theorem 1.3. *[9] Let $\mathcal{K}$ be a member of the class $\mathcal{N}.\mathcal{B}.\mathcal{C}.\mathcal{C}$ of a Banach space $\mathcal{B}$ and a continuous map $\mathcal{T}: \mathcal{K} \rightarrow \mathcal{K}$ satisfies*

$$\aleph(\mathcal{T} \mathcal{S}) \leq \lambda \aleph(\mathcal{S}),$$

for $\phi \neq \mathcal{S} \subset \mathcal{K}$, where $0 \leq \lambda<1$ and $\aleph$ is $\mathcal{M}.\mathcal{N}.\mathcal{C}$. Then the operator $\mathcal{T}$ admit at least one fixed point in $\mathcal{K}$.

Following are the some definitions which will be utilized to define new condensing mapping;

Definition 1.4. [4] $\mathcal{C}_{\Im}$ denotes the class of all the functions $\Im: \mathbb{R}_{+} \rightarrow \mathbb{R}_{+}$ and $\mathcal{C}_{\mathcal{H}}$ be the class of generalized operators $\mathcal{H}(\bullet ; .): \mathcal{C}_{\Im} \rightarrow \mathcal{C}_{\Im}$, which satisfies the following assumptions:

$\mathcal{H}_i)$ $\mathcal{H}(\Im ; u)>0$ for $u>0$ and $\mathcal{H}(\Im ; 0)=0$.

$\mathcal{H}_{ii})$ $\mathcal{H}(\Im ; u) \leq \mathcal{H}(\Im ; v)$ for $u \leq v$.

$\mathcal{H}_{iii})$ $\lim_{n \rightarrow \infty} \mathcal{H}\left(\Im ; u_n\right)=\mathcal{H}\left(\Im ; \lim_{n \rightarrow \infty} u_n\right)$.

$\mathcal{H}_{iv})$ $\mathcal{H}(\Im ; \max \{u, v\})=\max \{\mathcal{H}(\Im ; u), \mathcal{H}(\Im ; v)\}$ for some $\Im \in \mathcal{C}_{\Im}$.

Definition 1.5. $\mathcal{C}_{\mathcal{Q}}$ be the class of all the continuous functions $\mathcal{Q}:[0, \infty) \rightarrow[0, \infty)$, which validates the following assumptions;

$\mathcal{Q}_i)$ $u_1 \leqslant u_2$ then $\mathcal{Q} u_1 \leqslant \mathcal{Q} u_2$.

$\mathcal{Q}_{ii})$ $\lim_{n \rightarrow \infty} \mathcal{Q}\left(u_n\right)=0 \Rightarrow \lim_{n \rightarrow \infty} u_n=0$.

2 Generalized Darbo Fixed Point Theorems

Theorem 2.1. *Let $\mathcal{K}$ be an arbitrary member of the class $\mathcal{N}.\mathcal{B}.\mathcal{C}.\mathcal{C}$ of a Banach space $\mathcal{B}$ and $\mathcal{T}: \mathcal{K} \rightarrow \mathcal{K}$ is a continuous function. If $\mathcal{S}$ be any non empty subset of $\mathcal{K}$ such that*

$$\mathcal{H}\left(\Im ; \mathcal{R}_1(\mathcal{Q}(\aleph(\mathcal{T} \mathcal{S})))\right) \leqslant \mathcal{H}\left(\Im ; \mathcal{R}_2(\mathcal{Q}(\aleph(\mathcal{S})))\right)-\mathcal{H}\left(\Im ; \mathcal{R}_3(\mathcal{Q}(\aleph(\mathcal{S})))\right) \tag{2.1}$$

where $\aleph$ is $\mathcal{M}.\mathcal{N}.\mathcal{C}$, $\mathcal{Q} \in \mathcal{C}_{\mathcal{Q}}$, $\mathcal{H}(\Im, \bullet) \in \mathcal{C}_{\mathcal{H}}$ and $\mathcal{R}_1, \mathcal{R}_2, \mathcal{R}_3:[0, \infty) \rightarrow[0, \infty)$ are continuous functions with $\mathcal{H}\left(\Im ; \mathcal{R}_1(u)\right)-\mathcal{H}\left(\Im ; \mathcal{R}_2(u)\right)+\mathcal{H}\left(\Im ; \mathcal{R}_3(u)\right)>0$ for $u>0$. Then $\mathcal{T}$ admits at least one fixed point in $\mathcal{K}$.

Proof. We begin the proof with the construction of sequence of sets $\langle\mathcal{S}_n\rangle$ defined by

$$\mathcal{S}_{n+1}=\overline{conv} \mathcal{T}\left(\mathcal{S}_n\right) \text { by the initial approximation } \mathcal{S}_0=\mathcal{K}.$$

Since, $\mathcal{T}\left(\mathcal{S}_0\right) \subseteq \mathcal{S}_0$ thus $\mathcal{S}_1=\overline{conv} \mathcal{T}\left(\mathcal{S}_0\right) \subseteq \mathcal{S}_0$, continuing in this fashion we get

$$\mathcal{S}_0 \supseteq \mathcal{S}_1 \supseteq \cdots \supseteq \mathcal{S}_n,$$

and

$$\mathcal{S}_{n+1}=\overline{conv} \mathcal{T}\left(\mathcal{S}_n\right) \subseteq \overline{conv} \mathcal{T}\left(\mathcal{S}_{n-1}\right)=\mathcal{S}_n.$$

Hence by the mathematical induction, the sequence $\mathcal{S}_{n+1} = \overline{conv}\mathcal{T}(\mathcal{S}_n)$ of subsets of $\mathcal{K}$ is non-increasing, i.e., $\mathcal{S}_{n+1} \subseteq \mathcal{S}_n$ for all $n \in \mathbb{N}$. If there exists a non-negative integer K such that $\mathcal{Q}(\aleph(\mathcal{S}_m)) > 0$ this produce $\aleph(\mathcal{S}_m) = 0$ then $\mathcal{S}_m$ is pre-compact set and $\mathcal{T}(\mathcal{S}_m) \subseteq \mathcal{S}_m$. Thus Theorem 1.2, implies that $\mathcal{T}$ has at least one fixed point in $\mathcal{K}$.
Now, on the another end we may assume $\mathcal{Q}(\aleph(\mathcal{S}_n)) > 0$ for all $n \in \mathbb{N}$. Using the monotone property of $\aleph$ with assumption $(\mathcal{Q}_i)$ of $\mathcal{Q} \in \mathcal{C}_{\mathcal{Q}}$, we have

$$\mathcal{Q}(\aleph(\mathcal{S}_n)) > \mathcal{Q}(\aleph(\mathcal{S}_{n+1})).$$

This shows that $\langle \mathcal{Q}(\aleph(\mathcal{S}_n))\rangle$, is monotonic decreasing sequence of positive real numbers. Hence there exist a non-negative real number ℓ such that $\lim_{n\to\infty} \mathcal{Q}(\aleph(\mathcal{S}_n)) = \ell$ and $\lim_{n\to\infty} \mathcal{Q}(\aleph(\mathcal{S}_{n+1})) = \ell$. Now, we prove that $\ell = 0$. Assume the contradiction that $\ell > 0$. By the virtue of inequality (2.1), we get

$$\begin{aligned} \mathcal{H}(\Im;\mathcal{R}_1(\mathcal{Q}(\aleph(\mathcal{S}_{n+1})))) &= \mathcal{H}(\Im;\mathcal{R}_1(\mathcal{Q}(\aleph(\overline{conv}\mathcal{T}\mathcal{S}_n)))) \\ &\leqslant \mathcal{H}(\Im;\mathcal{R}_1(\mathcal{Q}(\aleph(\mathcal{T}\mathcal{S}_n)))) \\ &\leqslant \mathcal{H}(\Im;\mathcal{R}_2(\mathcal{Q}(\aleph(\mathcal{S}_n)))) - \mathcal{H}(\Im;\mathcal{R}_3(\mathcal{Q}(\aleph(\mathcal{S}_n)))) \end{aligned} \tag{2.2}$$

Applying the limit as $n \to \infty$, we get

$$\mathcal{H}(\Im;\mathcal{R}_1(\ell)) \leqslant \mathcal{H}(\Im;\mathcal{R}_2(\ell)) - \mathcal{H}(\Im;\mathcal{R}_3(\ell)).$$

This is possible only if $\ell = 0$, i.e., $\lim_{n\to\infty} \mathcal{Q}(\aleph(\mathcal{S}_n)) = 0$. By the assumption $(\mathcal{Q}_{ii})$, we get $\lim_{n\to\infty} \aleph(\mathcal{S}_n) = 0$. Since $\mathcal{S}_{n+1} \subseteq \mathcal{S}_n$ for all $n \in \mathbb{N}$ i.e., $\langle \mathcal{S}_n \rangle$ is decreasing sequence of closed and bounded nested sets. By the axiom vii) of definition the countable intersection $\mathcal{S}_\infty = \bigcap_{n=1}^{\infty} \mathcal{S}_n$, is nonempty, closed, convex and compact. We assures the existence of fixed point in the view of Schauder fixed point theorem for $\mathcal{T} : \mathcal{S}_\infty(\subset \mathcal{K}) \to \mathcal{S}_\infty$. □

Following are some immediate consequences of Theorem 2.1.
If we take $\mathcal{Q} : [0,\infty) \to [0,\infty)$ as $\mathcal{Q}(u) = u + \mathcal{G}(u)$, where $\mathcal{G} : \mathbb{R}_+ \to \mathbb{R}_+$ is non-decreasing continuous function, then we get following theorem

Theorem 2.2. *Let $\mathcal{K}$ be an arbitrary member of the class $\mathcal{N.B.C.C}$ of a Banach space $\mathcal{B}$ and $\mathcal{T} : \mathcal{K} \to \mathcal{K}$ is a continuous function. If $\mathcal{S}$ be any non empty subset of $\mathcal{K}$ such that*

$$\mathcal{H}(\Im;\mathcal{R}_1(\aleph(\mathcal{T}\mathcal{S}) + \mathcal{G}(\aleph(\mathcal{T}\mathcal{S})))) \leqslant \begin{cases} \mathcal{H}(\Im;\mathcal{R}_2(\aleph(\mathcal{S}) + \mathcal{G}(\aleph(\mathcal{S})))) - \\ \mathcal{H}(\Im;\mathcal{R}_3(\aleph(\mathcal{T}\mathcal{S}) + \mathcal{G}(\aleph(\mathcal{S})))) \end{cases}, \tag{2.3}$$

where $\aleph$ is $\mathcal{M.N.C}$, $\mathcal{H}(\Im,\bullet) \in \mathcal{C}_{\mathcal{H}}$, $\mathcal{Q} : \mathbb{R}_+ \to \mathbb{R}_+$ is non-decreasing continuous function and $\mathcal{R}_1, \mathcal{R}_2, \mathcal{R}_3 : [0,\infty) \to [0,\infty)$ are continuous functions with $\mathcal{H}(\Im;\mathcal{R}_1(u)) - \mathcal{H}(\Im;\mathcal{R}_2(u)) + \mathcal{H}(\Im;\mathcal{R}_3(u)) > 0$ for $u > 0$. Then $\mathcal{T}$ admits at least one fixed point in $\mathcal{K}$.

If we take $\mathcal{H}(\Im,\bullet) \in \mathcal{C}_{\mathcal{H}}$ as $\mathcal{H}(\Im, u) = u$, then we get the following theorem.

Theorem 2.3. *Let $\mathcal{K}$ be an arbitrary member of the class $\mathcal{N.B.C.C}$ of a Banach space $\mathcal{B}$ and $\mathcal{T} : \mathcal{K} \to \mathcal{K}$ is a continuous function. If $\mathcal{S}$ be any non empty subset of $\mathcal{K}$ such that*

$$\mathcal{R}_1(\mathcal{Q}(\aleph(\mathcal{T}\mathcal{S}))) \leqslant \mathcal{R}_2(\mathcal{Q}(\aleph(\mathcal{S}))) - \mathcal{R}_3(\mathcal{Q}(\aleph(\mathcal{T}\mathcal{S}))), \tag{2.4}$$

where $\aleph$ is $\mathcal{M.N.C}$, $\mathcal{Q} \in \mathcal{C}_{\mathcal{Q}}$ and $\mathcal{R}_1, \mathcal{R}_2, \mathcal{R}_3 : [0,\infty) \to [0,\infty)$ are continuous functions with $\mathcal{R}_1(u) - \mathcal{R}_2(u) + \mathcal{R}_3(u) > 0$ for $u > 0$. Then $\mathcal{T}$ admits at least one fixed point in $\mathcal{K}$.

3 Corollary

Following corollaries are the result available in the literature's [10, 9, 1, 7]

Corollary 3.1. *Let $\mathcal{K}$ be an arbitrary member of the class $\mathcal{N}.\mathcal{B}.\mathcal{C}.\mathcal{C}$ of a Banach space $\mathcal{B}$ and $\mathcal{T}:\mathcal{K}\to\mathcal{K}$ is a continuous function. If $\mathcal{S}$ be any non empty subset of $\mathcal{K}$ such that*

$$\mathcal{R}_1\left(\aleph\left(\mathcal{T}\mathcal{S}\right)+\mathcal{G}\left(\aleph\left(\mathcal{T}\mathcal{S}\right)\right)\right)\leq\mathcal{R}_2\left(\aleph\left(\mathcal{S}\right)+\mathcal{G}\left(\aleph\left(\mathcal{S}\right)\right)\right)-\mathcal{R}_3\left(\aleph\left(\mathcal{S}\right)+\mathcal{G}\left(\aleph\left(\mathcal{S}\right)\right)\right), \tag{3.1}$$

where $\aleph$ is $\mathcal{M}.\mathcal{N}.\mathcal{C}$, $\mathcal{R}_1,\mathcal{R}_2,\mathcal{R}_3:[0,\infty)\to[0,\infty)$ are continuous functions with $\mathcal{R}_1(u)-\mathcal{R}_2(u)+\mathcal{R}_3(u)>0$ for $u>0$ and $\mathcal{G}:\mathbb{R}_+\to\mathbb{R}_+$ is non-decreasing continuous function. Then $\mathcal{T}$ admits at least one fixed point in $\mathcal{K}$.

Proof. In Theorem 2.3, if we take $\mathcal{Q}:[0,\infty)\to[0,\infty)$ as $\mathcal{Q}(u)=u+\mathcal{G}(u)$ then we get above corollary. □

Corollary 3.2. *Let $\mathcal{K}$ be an arbitrary member of the class $\mathcal{N}.\mathcal{B}.\mathcal{C}.\mathcal{C}$ of a Banach space $\mathcal{B}$ and $\mathcal{T}:\mathcal{K}\to\mathcal{K}$ is a continuous map. If for any non empty subset $\mathcal{S}$ of $\mathcal{K}$*

$$\psi\left(\aleph\left(\mathcal{T}\mathcal{S}\right)\right)\leqslant\psi\left(\aleph\left(\mathcal{S}\right)\right)-\varphi\left(\aleph\left(\mathcal{S}\right)\right), \tag{3.2}$$

where $\aleph$ is $\mathcal{M}.\mathcal{N}.\mathcal{C}$, $\psi:[0,\infty)\to[0,\infty)$ is continuous function and $\varphi:[0,\infty)\to[0,\infty)$ is semi-continuous function with $\varphi(u)=0$ if and only if $u=0$. Then $\mathcal{T}$ admits at least one fixed point in $\mathcal{K}$.

Proof. In Theorem 2.3, if we take $\mathcal{Q}:[0,\infty]\to[0,\infty]$ as $\mathcal{Q}(u)=u$, $\mathcal{R}_1,\mathcal{R}_2,\mathcal{R}_3:[0,\infty)\to[0,\infty)$ as

$$\mathcal{R}_1(u)=\psi(u)+\frac{\theta}{2},\mathcal{R}_2(u)=\psi(u)+\theta\,and\,\mathcal{R}_3(u)=\varphi(u)+\frac{\theta}{2},$$

where $\theta>0$ is real number, then we get above corollary. □

Corollary 3.3. *Let $\mathcal{K}$ be an arbitrary member of the class $\mathcal{N}.\mathcal{B}.\mathcal{C}.\mathcal{C}$ of a Banach space $\mathcal{B}$ and $\mathcal{T}:\mathcal{K}\to\mathcal{K}$ is a continuous map. If for any non empty subset $\mathcal{S}$ of $\mathcal{K}$*

$$\aleph\left(\mathcal{T}\mathcal{S}\right)\leqslant\aleph\left(\mathcal{S}\right)-\psi\left(\aleph\left(\mathcal{S}\right)\right), \tag{3.3}$$

where $\aleph$ is $\mathcal{M}.\mathcal{N}.\mathcal{C}$ and $\psi:[0,\infty)\to[0,\infty)$ is semi continuous function with $\psi(u)=0$ if and only if $u=0$. Then $\mathcal{T}$ admits at least one fixed point in $\mathcal{K}$.

Proof. In Theorem 2.3, if we take $\mathcal{Q}:[0,\infty]\to[0,\infty]$ as $\mathcal{Q}(u)=u$, $\mathcal{R}_1,\mathcal{R}_2,\mathcal{R}_3:[0,\infty)\to[0,\infty)$ as

$$\mathcal{R}_1(u)=u+\frac{\theta}{2},\mathcal{R}_2(u)=u+\theta\,and\,\mathcal{R}_3(u)=\psi(u)+\frac{\theta}{2},$$

where $\theta>0$ is real number, then we get above corollary. □

Corollary 3.4. *Let $\mathcal{K}$ be an arbitrary member of the class $\mathcal{N}.\mathcal{B}.\mathcal{C}.\mathcal{C}$ of a Banach space $\mathcal{B}$ and $\mathcal{T}:\mathcal{K}\to\mathcal{K}$ is a continuous map. If for any non empty subset $\mathcal{S}$ of $\mathcal{K}$*

$$\aleph\left(\mathcal{T}\mathcal{S}\right)\leq\psi\left(\aleph\left(\mathcal{S}\right)\right), \tag{3.4}$$

where $\aleph$ is $\mathcal{M}.\mathcal{N}.\mathcal{C}$ and $\psi:(0,\infty)\to(0,\infty)$ as is non decreasing with $\psi(u)<u$ for $u>0$. Then $\mathcal{T}$ admits at least one fixed point in $\mathcal{K}$.

Proof. In Theorem 2.3, if we take $\mathcal{R}_1,\mathcal{R}_2,\mathcal{R}_3:[0,\infty)\to[0,\infty)$ as

$$\mathcal{R}_1(u)=u+\frac{\theta}{2},\mathcal{R}_2(u)=2\psi(u)+\theta\,and\,\mathcal{R}_3(u)=\psi(u)+\frac{\theta}{2}, \tag{3.5}$$

where $\theta>0$ is real number, then we get above corollary. □

Corollary 3.5. *Let $\mathcal{K}$ be an arbitrary member of the class $\mathcal{N}.\mathcal{B}.\mathcal{C}.\mathcal{C}$ of a Banach space $\mathcal{B}$ and $\mathcal{T}:\mathcal{K}\to\mathcal{K}$ is a continuous map. If for any non empty subset $\mathcal{S}$ of $\mathcal{K}$*

$$\aleph\left(\mathcal{T}\mathcal{S}\right)\leqslant\lambda\aleph\left(\mathcal{S}\right), \tag{3.6}$$

where $\aleph$ is $\mathcal{M}.\mathcal{N}.\mathcal{C}$ and $\lambda\in[0,1)$. Then $\mathcal{T}$ admits at least one fixed point in $\mathcal{K}$.

Proof. In equation (3.5), (appered in the proof of corollary (3.4)) if we take $\psi(u)=\lambda u$, then we get above corollary. □

4 Genralized Coupled fixed point Theorem

Definition 4.1. [8] Any pair of element $(\ell_1, \ell_2) \in \mathcal{K} \times \mathcal{K}$ is called coupled fixed point ($\mathcal{CFP}$) for a mapping $\mathcal{T} : \mathcal{K} \times \mathcal{K} \to \mathcal{K}$ if $\mathcal{T}(\ell_1, \ell_2) = \ell_1$ and $\mathcal{T}(\ell_2, \ell_1) = \ell_2$.

Theorem 4.2. *[6] Assume $\aleph_1, \aleph_2, ..., \aleph_n$, are $\mathcal{M.N.C}$ in Banach spaces $\mathcal{B}_1, \mathcal{B}_2, \cdots, \mathcal{B}_n$, respectively, and the mapping $\mu : [0, \infty) \times [0, \infty) \times \cdots \times [0, \infty) \to [0, \infty)$ be a convex function and $\mu(\iota_1, \iota_2, \cdots, \iota_n) = 0$ if and only if $\iota_k = 0$ for $k = 1, 2, \cdots, n$. Then*

$$\overline{\aleph}(\mathcal{S}) = \mu(\aleph_1(\mathcal{S}_1), \aleph_2(\mathcal{S}_2), ..., \aleph_n(\mathcal{S}_n)),$$

is a measure of noncompactness in $\mathcal{B}_1 \times \mathcal{B}_2 \times ... \times \mathcal{B}_n$, where $\mathcal{S}_k$ denotes the natural projection of $\mathcal{S}$ into $\mathcal{S}_k$, for $k = 1, 2, \cdots, n$.

Example 4.3. μ is $\mathcal{MNC}$ if we take $\mu : [0, \infty) \times [0, \infty) \to [0, \infty)$ as $\mu(\iota_1, \iota_2) = \iota_1 + \iota_2$.

Theorem 4.4. *Let $\mathcal{K}$ be an arbitrary member of the class $\mathcal{N.B.C.C}$ of a Banach space $\mathcal{B}$ and $\mathcal{T} : \mathcal{K} \times \mathcal{K} \to \mathcal{K}$ is a continuous function. If $\mathcal{S}_1, \mathcal{S}_2$ are non empty subset of $\mathcal{K}$ such that*

$$\mathcal{H}(\Im; \mathcal{R}_1(\aleph(\mathcal{T}(\mathcal{S}_1 \times \mathcal{S}_2)))) \leqslant \frac{1}{2}(\mathcal{H}(\Im; \mathcal{R}_2(\aleph(\mathcal{S}_1 \times \mathcal{S}_2))) - \mathcal{H}(\Im; \mathcal{R}_3(\aleph(\mathcal{S}_2 \times \mathcal{S}_1)))), \quad (4.1)$$

where $\aleph$ is $\mathcal{M.N.C}$ $\mathcal{H}(\Im, \bullet) \in \mathcal{C}_{\mathcal{H}}$ such that $\mathcal{H}(\Im, f + g) \leqslant \mathcal{H}(\Im, f) + \mathcal{H}(\Im, g)$ and $\mathcal{R}_1, \mathcal{R}_2, \mathcal{R}_3 : [0, \infty) \to [0, \infty)$ are continuous functions with $\mathcal{H}(\Im; \mathcal{R}_1(u)) - \mathcal{H}(\Im; \mathcal{R}_2(u)) + \mathcal{H}(\Im; \mathcal{R}_3(u)) > 0$ for $u > 0$ and $\mathcal{R}_1(u + v) \leqslant \mathcal{R}_1(u) + \mathcal{R}_1(v)$. Then $\mathcal{T}$ admits at least one fixed point in $\mathcal{K}$.

Proof. We define the mapping $\widehat{\mathcal{T}} : \mathcal{K} \times \mathcal{K} \to \mathcal{K} \times \mathcal{K}$ by

$$\widehat{\mathcal{T}}(\ell_1, \ell_2) = (\mathcal{T}(\ell_1, \ell_2), \mathcal{T}(\ell_2, \ell_1)).$$

clearly $\widehat{\mathcal{T}}$ is continuous, because $\mathcal{T}$ is continuous. To show the existence of fixed point, we validate the conditions of Theorem (2.3) for the mapping $\widehat{\mathcal{T}}$. Let $\mathcal{S}$ be any subset of $\mathcal{K} \times \mathcal{K}$. Now, by Theorem 4.2 we have $\mu(\mathcal{S}) = \aleph_1(\mathcal{S}_1) + \aleph_2(\mathcal{S}_2)$ is $\mathcal{M.N.C}$, where $\mathcal{S}_1, \mathcal{S}_2$ are natural projections of $\mathcal{S}$ on $\mathcal{K}$. Now, by using inequality (4.1) we have

$$\begin{aligned}
\mathcal{H}\left(\Im; \mathcal{R}_1\left(\mu\left(\widehat{\mathcal{T}}\mathcal{S}\right)\right)\right) &\leqslant \mathcal{H}(\Im; \mathcal{R}_1(\mu(\mathcal{T}(\mathcal{S}_1 \times \mathcal{S}_2) \times \mathcal{T}(\mathcal{S}_2 \times \mathcal{S}_1)))) \\
&\leqslant \mathcal{H}(\Im; \mathcal{R}_1(\aleph(\mathcal{T}(\mathcal{S}_1 \times \mathcal{S}_2)) + \aleph(\mathcal{T}(\mathcal{S}_2 \times \mathcal{S}_1)))) \\
&\leqslant \mathcal{H}(\Im; \mathcal{R}_1(\aleph(\mathcal{T}(\mathcal{S}_1 \times \mathcal{S}_2)))) + \mathcal{H}(\Im; \mathcal{R}_1(\aleph(\mathcal{T}(\mathcal{S}_2 \times \mathcal{S}_1)))) \\
&\leqslant \frac{1}{2}(\mathcal{H}(\Im; \mathcal{R}_2(\aleph(\mathcal{S}_1 \times \mathcal{S}_2))) - \mathcal{H}(\Im; \mathcal{R}_3(\aleph(\mathcal{S}_2 \times \mathcal{S}_1)))) \\
&+ \frac{1}{2}(\mathcal{H}(\Im; \mathcal{R}_2(\aleph(\mathcal{S}_2 \times \mathcal{S}))) - \mathcal{H}(\Im; \mathcal{R}_3(\aleph(\mathcal{S}_1 \times \mathcal{S}_2)))) \\
&\leqslant \frac{1}{2}(\mathcal{H}(\Im; \mathcal{R}_2(\aleph(\mathcal{S}_1) + \aleph(\mathcal{S}_2))) - \mathcal{H}(\Im; \mathcal{R}_3(\aleph(\mathcal{S}_2) + \aleph(\mathcal{S}_2)))) \\
&+ \frac{1}{2}(\mathcal{H}(\Im; \mathcal{R}_2(\aleph(\mathcal{S}_2) + \aleph(\mathcal{S}_2))) - \mathcal{H}(\Im; \mathcal{R}_3(\aleph(\mathcal{S}_1) + \aleph(\mathcal{S}_2)))) \\
&\leqslant \mathcal{H}(\Im; \mathcal{R}_2(\mu(\mathcal{S}))) - \mathcal{H}(\Im; \mathcal{R}_3(\mu(\mathcal{S}))).
\end{aligned} \quad (4.2)$$

Hence, in the view of Theorem 2.3 we ensure that $\widehat{\mathcal{T}}$ has atlest one fixed point in $\mathcal{K} \times \mathcal{K}$. □

5 Application to System of Fractional Differential Equations

In this section we exhibit the utility of our result for the existence of solution to system of fractional ordered Atangana-Baleanu differential equations.

$$\begin{aligned}
{}_0^{\mathcal{ABC}}\mathcal{D}_\ell^\alpha[\gamma(\ell)] &= \mathcal{U}(\ell, \gamma(\ell), \xi(\ell)) \\
{}_0^{\mathcal{ABC}}\mathcal{D}_\ell^\alpha[\xi(\ell)] &= \mathcal{U}(\ell, \xi(\ell), \gamma(\ell)),
\end{aligned} \quad (5.1)$$

where $\ell \in [0, \mathcal{Z}]$, with the initial conditions

$$\gamma(0) = 0 \text{ and } \xi(0) = 0. \tag{5.2}$$

To demonstrate the system of fractional ordered Atangana-Baleanu differential equations we recall fractional ordered Atangana-Baleanu differentiation and integration from [3, 2].

Definition 5.1. The Atangana-Baleanu fractional differentiation of order $\alpha \in [0, 1)$ for the function $f(t) \in C^1(a, b)$, where $a < b$ is defined as

$$ {}_{0}^{\mathcal{ABC}}\mathcal{D}_{\ell}^{\alpha}\left[\varsigma(\ell)\right] = \frac{\mathcal{C}(\alpha)}{1-\alpha}\int_{0}^{\ell}\varsigma'(\tau)\,E_{\alpha}\left[-\alpha\frac{(\ell-\tau)^{\alpha}}{1-\tau}\right]d\tau, \tag{5.3}$$

where, $\mathcal{C}(\alpha)$ is used for normalization factor with $\mathcal{C}(0) = \mathcal{C}(1) = 1$.

Definition 5.2. The fractional integration of order $\alpha \in [0, 1]$ with respect to Atangana-Baleanu is defined as

$$ {}^{\mathcal{ABC}}I_{\ell}^{\alpha}\left[\varsigma(\ell)\right] = \frac{1-\alpha}{C(\alpha)}\varsigma(\theta) + \frac{\alpha}{C(\alpha)\Gamma(\alpha)}\int_{0}^{\ell}\varsigma(\tau)(\ell-\tau)^{\alpha-1}d\tau. \tag{5.4}$$

Assume that $\mathcal{C}[0, \mathcal{Z}]$ be the space of all continuous functions on $\mathcal{C}[0, \mathcal{Z}]$ equipped with the norm

$$||\varsigma|| = \max\left\{|\varsigma(\ell)| : \ell \in [0, \mathcal{Z}]\right\}; \varsigma \in \mathcal{C}([0, \mathcal{Z}]).$$

For $\epsilon > 0$ be any real number, then the definition of modulus of continuity for $\varsigma \in \mathcal{C}[0, \mathcal{Z}]$ is given by

$$\omega(\varsigma, \varepsilon) = \max\left\{|\varsigma(\ell_1) - \varsigma(\ell_2)|; \ell_1, \ell_2 \in [0, \mathcal{Z}], |\ell_1 - \ell_2| \le \varepsilon\right\}.$$

The continuity of ς on $[0, \mathcal{Z}]$ implies $\omega(\varsigma, \varepsilon) \to 0$ as $\epsilon \to 0$. The $\mathcal{M}.\mathcal{N}.\mathcal{C}$ for a bounded subset $\mathcal{S}$ of $[0, \mathcal{Z}]$ is defined as

$$\aleph(\mathcal{S}) = \lim_{\varepsilon\to 0}\left\{\sup_{\varsigma\in\mathcal{S}}\omega(\varsigma, \varepsilon)\right\}. \tag{5.5}$$

Equation (5.5) is measure of non compactness [7]. Now, we assume the following hypothesis for the existence of solution of system of differential equation (5.1) with (5.2).

$\mathcal{ABC}$-1) For the continuous function $\mathcal{U} : [0, \mathcal{Z}] \times \mathcal{C}[0, \mathcal{Z}] \times \mathcal{C}[0, \mathcal{Z}] \to \mathbb{R}$ there exist a real valued function $\Phi : \mathbb{R}_+ \to \mathbb{R}_+$, such that

$$|\mathcal{U}(\ell, \vartheta_1(\ell), \vartheta_2(\ell)) - \mathcal{U}(\ell, \omega_1(\ell), \omega_2(\ell))|)) \le \frac{1}{2}\Phi\left(\max_{1i2}\left\{|\vartheta_i(\ell) - \omega_i(\ell)| : \ell \in [0, \mathcal{Z}]\right\}\right).$$

$\mathcal{ABC}$-2) There exist finite real numbers $\mathcal{M}_{\mathcal{U}}$. such that

$$\mathcal{M}_{\mathcal{U}} = \sup\left\{|\mathcal{U}(\ell, 0, 0)| : \ell \in [0, \mathcal{Z}]\right\}.$$

$\mathcal{ABC}$-3) There exist a a positive real number r_0, with inequality

$$\mathcal{H}\left(\Im; \frac{1-\alpha}{C(\alpha)}\left(\frac{1}{2}\Phi(r_0) + \mathcal{M}_{\mathcal{U}}\right)\right) + \mathcal{H}\left(\Im; \frac{\alpha\left(\frac{1}{2}\Phi(r_0) + \mathcal{M}_{\mathcal{U}}\right)\mathcal{Z}^{\alpha}}{C(\alpha)\Gamma(\alpha)}\right) \leqslant r_0. \tag{5.6}$$

Theorem 5.3. *The system of fractional ordered system (5.1) with (5.2) has atleast one solution, if we consider the assumption $\mathcal{ABC} - 1$) to $\mathcal{ABC} - 3$).*

Proof. Consider the operator $\mathcal{T} : \mathcal{C}[0, \mathcal{Z}] \times \mathcal{C}[0, \mathcal{Z}] \to \mathcal{C}[0, \mathcal{Z}]$ is defined by following manner;

$$\mathcal{T}(\gamma, \xi)(\ell) = \gamma(0) + \frac{1-\alpha}{C(\alpha)}\mathcal{U}(\ell, \gamma(\ell), \xi(\ell)) + \frac{\alpha}{C(\alpha)\Gamma(\alpha)}\int_{0}^{\ell}\frac{\mathcal{U}(\tau, \gamma(\tau), \xi(\tau))}{(\ell-\tau)^{1-\alpha}}d\tau. \tag{5.7}$$

We observe that, being a operator of primitive of continuous function indeed the operator $\mathcal{T}$ continuous. Now, for $\ell \in [0, \mathcal{Z}]$, by the virtue of assumption we have

$$\begin{aligned}
&\mathcal{H}\left(\Im; \left|\mathcal{T}(\gamma,\xi)(\ell)\right|\right) \\
&\leqslant \mathcal{H}\left(\Im; \frac{1-\alpha}{C(\alpha)}\left|\mathcal{U}(\ell,\gamma(\ell),\xi(\ell)) - \mathcal{U}(\ell,0,0) + \mathcal{U}(\ell,0,0)\right|\right) \\
&+ \mathcal{H}\left(\Im; \frac{\alpha}{C(\alpha)\Gamma(\alpha)}\int_0^\ell \frac{\left|\mathcal{U}(\tau,\gamma(\tau),\xi(\tau)) - \mathcal{U}(\ell,0,0) + \mathcal{U}(\ell,0,0)\right|}{(\ell-\tau)^{1-\alpha}}d\tau\right) \\
&\leqslant \mathcal{H}\left(\Im; \frac{1-\alpha}{C(\alpha)}\left(\frac{1}{2}\Phi\left(\max\left\{\left|\gamma(\ell)\right|, \left|\xi(\ell)\right| : \ell \in [0,\mathcal{Z}]\right\}\right) + \left|\mathcal{U}(\ell,0,0)\right|\right)\right) \\
&+ \mathcal{H}\left(\Im; \frac{\alpha}{C(\alpha)\Gamma(\alpha)}\int_0^\ell \frac{\frac{1}{2}\Phi\left(\max\left\{\left|\gamma(\tau)\right|, \left|\xi(\tau)\right| : \tau \in [0,\mathcal{Z}]\right\}\right) + \left|\mathcal{U}(\tau,0,0)\right|}{(\ell-\tau)^{1-\alpha}}d\tau\right) \qquad (5.8) \\
&\leqslant \mathcal{H}\left(\Im; \frac{1-\alpha}{C(\alpha)}\frac{1}{2}\Phi\left(\max\left\{\|\gamma\|, \|\xi\|\right\}\right) + \mathcal{M}_{\mathcal{U}}\right) \\
&+ \mathcal{H}\left(\Im; \frac{\alpha}{C(\alpha)\Gamma(\alpha)}\int_0^\ell \frac{\frac{1}{2}\Phi\left(\max\left\{\|\gamma\|, \|\xi\|\right\}\right) + \mathcal{M}_{\mathcal{U}}}{(\ell-\tau)^{1-\alpha}}d\tau\right) \\
&\leqslant \mathcal{H}\left(\Im; \frac{1-\alpha}{C(\alpha)}\left(\frac{1}{2}\Phi\left(\max\left\{\|\gamma\|, \|\xi\|\right\}\right) + \mathcal{M}_{\mathcal{U}}\right)\right) \\
&+ \mathcal{H}\left(\Im; \frac{\alpha\left(\frac{1}{2}\Phi\left(\max\left\{\|\gamma\|, \|\xi\|\right\}\right) + \mathcal{M}_{\mathcal{U}}\right)\mathcal{Z}^\alpha}{C(\alpha)\Gamma(\alpha)}\right).
\end{aligned}$$

Thus, by the virtue of inequality (5.6) from assumption $\mathcal{ABC}-3$), the mappings $\mathcal{T}$ is bounded and map the ball $\mathcal{B}_{r_0}$ into itself. Now, we show that the mapping $\mathcal{T}$ is continuous on $[0, \mathcal{Z}]$. To prove the continuity, we take any $\epsilon > 0$ and $\ell_1, \ell_2 \in [0, \mathcal{Z}]$ with $|\ell_1 - \ell_2| < \epsilon$. Then for γ and ξ

from $\mathcal{C}[0,\mathcal{Z}]$ we derive the following expression

$$
\begin{aligned}
&\mathcal{H}\ (\Im; |\mathcal{T}_1(\gamma,\xi)\ (\ell_1) - \mathcal{T}_1(\gamma,\xi)(\ell_2)|) \\
&\quad\leqslant \mathcal{H}\left(\Im; \frac{1-\alpha}{C(\alpha)} |\mathcal{U}(\ell_1,\gamma(\ell_1),\xi(\ell_1)) - \mathcal{U}(\ell_2,\gamma(\ell_2),\xi(\ell_2))|\right) \\
&\quad + \mathcal{H}\left(\Im; \frac{\alpha}{C(\alpha)\Gamma(\alpha)} \left| \int_0^{\ell_1} \frac{\mathcal{U}(\tau,\gamma(\tau),\xi(\tau))}{(\ell_1-\tau)^{1-\alpha}} d\tau - \int_0^{\ell_2} \frac{\mathcal{U}(\tau,\gamma(\tau),\xi(\tau))}{(\ell_2-\tau)^{1-\alpha}} d\tau \right|\right) \\
&\quad\leqslant \mathcal{H}\left(\Im; \frac{1-\alpha}{C(\alpha)} \left(\frac{1}{2}\Phi(\max\{|\gamma(\ell_1)-\gamma(\ell_2)|, |\xi(\ell_1)-\xi(\ell_2)|\})\right)\right) \\
&\quad + \left(\Im; \frac{1-\alpha}{C(\alpha)} (\Delta(\mathcal{U},\varepsilon))\right) \\
&\quad + \mathcal{H}\left(\Im; \frac{\alpha\left(\frac{1}{2}\Phi(\max\{||\gamma||, ||\xi||\}) + \mathcal{M}_{\mathcal{U}}\right)}{C(\alpha)\Gamma(\alpha)} \int_0^{\ell_1} \frac{1}{(\ell_1-\tau)^{1-\alpha}} - \frac{1}{(\ell_2-\tau)^{1-\alpha}} d\tau\right) \\
&\quad + \mathcal{H}\left(\Im; \frac{\alpha\left(\frac{1}{2}\Phi(\max\{||\gamma||, ||\xi||\}) + \mathcal{M}_{\mathcal{U}} + \mathcal{M}_{\mathcal{U}}\right)}{C(\alpha)\Gamma(\alpha)} \int_{\ell_2}^{\ell_1} \frac{1}{(\ell_2-\tau)^{1-\alpha}} d\tau\right) \\
&\quad\leqslant \mathcal{H}\left(\Im; \frac{1-\alpha}{C(\alpha)} \left(\frac{1}{2}\Phi(\max\{\Delta(\gamma,\varepsilon), \Delta(\xi,\varepsilon)\})\right)\right) \\
&\quad + \left(\Im; \frac{1-\alpha}{C(\alpha)} (\Delta_{\mathcal{U}}([0,\mathcal{Z}],\varepsilon))\right) + \mathcal{H}\left(\Im; \frac{\alpha\left(\frac{1}{2}\Phi(\max\{||\gamma||, ||\xi||\}) + \mathcal{M}_{\mathcal{U}}\right)}{C(\alpha)\Gamma(\alpha)} ({\ell_2}^{\alpha} - {\ell_1}^{\alpha})\right) \\
&\quad + \mathcal{H}\left(\Im; \frac{\alpha(\zeta_{\mathcal{U}}||\gamma|| - \eta_{\mathcal{U}}\mathcal{T}_1(||\xi||) + \mathcal{M}_{\mathcal{U}})}{C(\alpha)\Gamma(\alpha)} (\ell_2-\ell_1)^{\alpha}\right),
\end{aligned}
\tag{5.9}
$$

where

$$
\begin{cases}
\Delta_{\mathcal{U}}([0,\mathcal{Z}],\varepsilon) = \sup\left\{ |\mathcal{U}(\ell_1,\gamma(\ell_2),\xi(\ell_2)) - \mathcal{U}(\ell_2,\gamma(\ell_2),\xi(\ell_2))| : \begin{array}{l} \ell_1,\ell_2 \in [0,\mathcal{Z}], \\ |\ell_1-\ell_2| < \varepsilon, \\ \gamma,\xi \in \mathcal{C}[0,\mathcal{Z}] \end{array} \right\}, \\
\Delta(\gamma,\varepsilon) = \sup\{|\gamma(\ell_1)-\gamma(\ell_2)| : \ell_1,\ell_2 \in [0,\mathcal{Z}], |\ell_1-\ell_2| < \varepsilon, \gamma \in \mathcal{C}[0,\mathcal{Z}]\}, \\
\Delta(\xi,\varepsilon) = \sup\{|\xi(\ell_1)-\xi(\ell_2)| : \ell_1,\ell_2 \in [0,\mathcal{Z}], |\ell_1-\ell_2| < \varepsilon, \xi \in \mathcal{C}[0,\mathcal{Z}]\}.
\end{cases}
\tag{5.10}
$$

Thus, equations (5.9) with (5.10) shows that both the mapping $\mathcal{T}$ is continuous on the $[0,\mathcal{Z}]$. By the virtue of boundedness and continuity of $\mathcal{T}$along with assumption $\mathcal{ABC}-3)$ we conformed, $\mathcal{T}$ maps the closed ball $\mathcal{B}_{r_0}$ to itself.

Further, we prove that $\mathcal{T}$ is continuous on $\mathcal{B}_{r_0}$. For this take $\epsilon_1,\epsilon_2 > 0$ for γ_1,γ_2 & ξ_1,ξ_2 from $\mathcal{B}_{r_0}$ such that $||\gamma_1-\gamma_2|| < \varepsilon_1$ & $||\xi_1-\xi_2|| < \varepsilon_2$. Then for $\varepsilon = \max\{\varepsilon_1,\varepsilon_2\}$ and $\ell \in [0,\mathcal{Z}]$, we derive the following expression

$$\begin{aligned}
&\mathcal{H}\left(\Im;\left|\mathcal{T}_1(\gamma,\xi)(\ell_1)-\mathcal{T}_1(\gamma,\xi)(\ell_2)\right|\right)\\
&\leqslant \mathcal{H}\left(\Im;\frac{1-\alpha}{C(\alpha)}\left|\mathcal{U}(\ell,\gamma_1(\ell),\xi_1(\ell))-\mathcal{U}(\ell,\gamma_2(\ell),\xi_2(\ell))\right|\right)\\
&+\mathcal{H}\left(\Im;\frac{\alpha}{C(\alpha)\Gamma(\alpha)}\left|\int_0^\ell\frac{\mathcal{U}(\tau,\gamma_1(\tau),\xi_1(\tau))}{(\ell-\tau)^{1-\alpha}}d\tau-\int_0^\ell\frac{\mathcal{U}(\tau,\gamma_2(\tau),\xi_2(\tau))}{(\ell-\tau)^{1-\alpha}}d\tau\right|\right)\\
&\leqslant\left(\Im;\frac{1-\alpha}{\mathcal{C}(\alpha)}\left(\frac{1}{2}\Phi\left(\max\left\{\left|\gamma_1(\ell)-\gamma_2(\ell)\right|,\left|\xi_1(\ell)-\xi_2(\ell)\right|\right\}\right)\right)\right)\\
&+\mathcal{H}\left(\Im;\frac{\alpha}{\mathcal{C}(\alpha)\Gamma(\alpha)}\int_0^\ell\frac{\frac{1}{2}\Phi\left(\max\left\{\left|\gamma_1(\tau)-\gamma_2(\tau)\right|,\left|\xi_1(\tau)-\xi_2(\tau)\right|\right\}\right)}{(\ell-\tau)^{1-\alpha}}d\tau\right)\\
&+\mathcal{H}\left(\Im;\frac{1-\alpha}{\mathcal{C}(\alpha)}\left(\frac{1}{2}\Phi\left(\max\left\{||\gamma_1-\gamma_2||,||\xi_1-\xi_2||\right\}\right)\right)\right)\\
&+\mathcal{H}\left(\Im;\frac{\alpha}{\mathcal{C}(\alpha)\Gamma(\alpha)}\int_0^\ell\frac{\frac{1}{2}\Phi\left(\max\left\{||\gamma_1-\gamma_2||,||\xi_1-\xi_2||\right\}\right)}{(\ell-\tau)^{1-\alpha}}d\tau\right)\\
&\leqslant\mathcal{H}\left(\Im;\frac{1-\alpha}{\mathcal{C}(\alpha)}\left(\frac{1}{2}\Phi\left(\max\left\{||\gamma_1-\gamma_2||,||\xi_1-\xi_2||\right\}\right)\right)\right)\\
&+\mathcal{H}\left(\Im;\frac{\alpha\left(\frac{1}{2}\Phi\left(\max\left\{||\gamma_1-\gamma_2||,||\xi_1-\xi_2||\right\}\right)\right)}{\mathcal{C}(\alpha)\Gamma(\alpha)}\mathcal{Z}^\alpha\right).
\end{aligned}\tag{5.11}$$

Equation (5.11) implies that, $\mathcal{H}\left(\Im;\left|\mathcal{T}(\gamma_1,\xi_1)(\ell)-\mathcal{T}(\gamma_2,\xi_2)(\ell)\right|\right)\to 0$ as $\epsilon\to 0$. Hence by the definition of $\mathcal{H}(\Im,\bullet)$ we assure that $\left|\mathcal{T}(\gamma_1,\xi_1)(\ell)-\mathcal{T}(\gamma_2,\xi_2)(\ell)\right|\to 0$ as $\epsilon\to 0$. This shows that $\mathcal{T}$ is continuous on $\mathcal{B}_{r_0}$. Now, using the expression from (5.9), we obtain following expression

$$\begin{aligned}
\mathcal{H}\left(\Im;\left|\mathcal{T}_1(\gamma,\xi)(\ell_1)-\mathcal{T}_1(\gamma,\xi)(\ell_2)\right|\right)&\leqslant\mathcal{H}\left(\Im;\frac{1-\alpha}{\mathcal{C}(\alpha)}\frac{1}{2}\Phi\left(\max\left\{\Delta(\gamma,\varepsilon),\Delta(\xi,\varepsilon)\right\}\right)\right)\\
&+\mathcal{H}\left(\Im;\frac{1-\alpha}{\mathcal{C}(\alpha)}\left(\Delta(\mathcal{U},\varepsilon)\right)\right)\\
&+\mathcal{H}\left(\Im;\frac{\alpha\left(\frac{1}{2}\Phi\left(\max\left\{||\gamma||,||\xi||\right\}\right)+\mathcal{M}_{\mathcal{U}}\right)}{\mathcal{C}(\alpha)\Gamma(\alpha)}\left({\ell_2}^\alpha-{\ell_1}^\alpha\right)\right)\\
&+\mathcal{H}\left(\Im;\frac{\alpha\left(\frac{1}{2}\Phi\left(\max\left\{||\gamma||,||\xi||\right\}\right)+\mathcal{M}_{\mathcal{U}}\right)}{\mathcal{C}(\alpha)\Gamma(\alpha)}(\ell_2-\ell_1)^\alpha\right).
\end{aligned}\tag{5.12}$$

Applying $\epsilon\to 0$, $\ell_1\to\ell_2$, so by the virtue of definition (1.4), equation (5.10) and uniform continuity of $\mathcal{U}$ on $[0,\mathcal{Z}]$, we obtain

$$\begin{cases}
\mathcal{H}\left(\Im;\frac{1-\alpha}{C(\alpha)}\left(\Delta(\mathcal{U},\varepsilon)\right)\right)\to 0\\
\mathcal{H}\left(\Im;\frac{\alpha\left(\frac{1}{2}\Phi(\max\{||\gamma||,||\xi||\})+\mathcal{M}_{\mathcal{U}}\right)}{\mathcal{C}(\alpha)\Gamma(\alpha)}\left({\ell_2}^\alpha-{\ell_1}^\alpha\right)\right)\to 0\\
\mathcal{H}\left(\Im;\frac{\alpha\left(\frac{1}{2}\Phi(\max\{||\gamma||,||\xi||\})+\mathcal{M}_{\mathcal{U}}\right)}{\mathcal{C}(\alpha)\Gamma(\alpha)}(\ell_2-\ell_1)^\alpha\right)\to 0.
\end{cases}$$

With the above estimation, if we take $\sup\limits_{\gamma\in\mathcal{S}}$, then by the virtue of equation (5.5), we get

$$\mathcal{H}\left(\Im;\aleph\left(\mathcal{T}(\mathcal{S}_1\times\mathcal{S}_2)\right)\right)\leqslant\mathcal{H}\left(\Im;\frac{1-\alpha}{C(\alpha)}\frac{1}{2}\Phi\left(\max\left\{\aleph(\mathcal{S}_1),\aleph(\mathcal{S}_2)\right\}\right)\right).\tag{5.13}$$

By taking $\mathcal{H} \in \mathcal{C}_{\mathcal{H}}$ as $\mathcal{H}(\Im, h) = h$, $\mathcal{Q} : [0, \infty) \to [0, \infty)$ as $\mathcal{Q}(u) = u$ and

$$\mathcal{R}_1(u) = \frac{u}{2} + \frac{\lambda}{2}, \mathcal{R}_2(u) = \frac{1}{2}\frac{1-\alpha}{\mathcal{C}(\alpha)}\Phi(u) + \lambda \text{ and } \mathcal{R}_3(u) = \frac{1}{4}\frac{1-\alpha}{\mathcal{C}(\alpha)}\Phi(u) + \frac{\lambda}{2},$$

in equation (5.13), we get

$$\mathcal{H}(\Im; \mathcal{R}_1(\aleph(\mathcal{T}(\mathcal{S}_1 \times \mathcal{S}_2)))) \leqslant \frac{1}{2}\left(\begin{array}{c} \mathcal{H}(\Im; \mathcal{R}_2(\Phi(\max\{\aleph(\mathcal{S}_1), \aleph(\mathcal{S}_2)\}))) \\ -\mathcal{H}(\Im; \mathcal{R}_3(\Phi(\max\{\aleph(\mathcal{S}_1), \aleph(\mathcal{S}_2)\}))). \end{array} \right. \tag{5.14}$$

It is clear that,

$$\begin{aligned} \mathcal{R}_1(u) - \mathcal{R}_2(u) + \mathcal{R}_3(u) &= \frac{u}{2} + \frac{\lambda}{2} - \frac{1}{2}\frac{1-\alpha}{\mathcal{C}(\alpha)}\Phi(u) + \lambda + \frac{1}{4}\frac{1-\alpha}{\mathcal{C}(\alpha)}\Phi(u) + \frac{\lambda}{2} \\ &= \frac{u}{2} - \frac{1}{4}\frac{1-\alpha}{\mathcal{C}(\alpha)}\Phi(u). \end{aligned} \tag{5.15}$$

Note that, if we consider $\Phi : (0, \infty) \to (0, \infty)$ as $\Phi(u) = \vartheta u$, then for sufficient value of ϑ we have $\frac{u}{2} - \frac{1}{4}\frac{1-\alpha}{\mathcal{C}(\alpha)}\Phi(u) > 0$. Also,

$$\begin{aligned} \mathcal{R}_1(\aleph(\mathcal{T}(\mathcal{S}_1 \times \mathcal{S}_2))) &\leqslant \mathcal{R}_1\left(\frac{1-\alpha}{C(\alpha)}\frac{1}{2}\Phi(\max\{\aleph(\mathcal{S}_1), \aleph(\mathcal{S}_2)\})\right) \\ &= \frac{1}{4}\frac{1-\alpha}{\mathcal{C}(\alpha)}\Phi(\max\{\aleph(\mathcal{S}_1), \aleph(\mathcal{S}_2)\}) + \frac{\lambda}{2}, \end{aligned} \tag{5.16}$$

and

$$\begin{aligned} \mathcal{R}_2(\aleph(\mathcal{S})) - \mathcal{R}_3(\aleph(\mathcal{S})) &= \frac{1}{2}\frac{1-\alpha}{\mathcal{C}(\alpha)}\Phi(\max\{\aleph(\mathcal{S}_1), \aleph(\mathcal{S}_2)\}) \\ &\quad + \lambda - \frac{1}{4}\frac{1-\alpha}{\mathcal{C}(\alpha)}\Phi(\max\{\aleph(\mathcal{S}_1), \aleph(\mathcal{S}_2)\}) - \frac{\lambda}{2} \\ &= \frac{1}{4}\frac{1-\alpha}{\mathcal{C}(\alpha)}\Phi(\max\{\aleph(\mathcal{S}_1), \aleph(\mathcal{S}_2)\}) + \frac{\lambda}{2}. \end{aligned} \tag{5.17}$$

Equations (5.14)-(5.17) justifies that the operator $\mathcal{T}$ satisfies all the conditions of Theorem 4.4, hence the operator $\mathcal{T}$ has at least one coupled fixed point, which is an solution for the system (5.1) with (5.2).

Example 5.4. Consider

$$\begin{aligned} {}_0^{\mathcal{ABC}}\mathcal{D}_\ell^\alpha[\gamma(\ell)] &= a\gamma + b\gamma\xi \\ {}_0^{\mathcal{ABC}}\mathcal{D}_\ell^\alpha[\xi(\ell)] &= a\xi + b\gamma\xi, \end{aligned} \tag{5.18}$$

with

$$\gamma(0) = 0 \text{ and } \xi(0) = 0, \tag{5.19}$$

where $a,\ b \in \mathbb{R}$, is the system of differential equation in the space $\mathcal{C}[0, 1]$.

The equation (5.18) is special case of (5.1) with (5.2) with

$$\mathcal{U}(\ell, \gamma(\ell), \xi(\ell)) = a\gamma + b\gamma\xi \ \&\ \mathcal{U}(\ell, \xi(\ell), \gamma(\ell)) = a\xi + b\gamma\xi.$$

To prove the existence for the solution of (5.18) we should validate the conditions of Theorem 5.3.

$$\begin{aligned}
|\mathcal{U}(\ell,\gamma_1(\ell),\gamma_2(\ell)) - \mathcal{U}(\ell,\xi_1(\ell),\xi_2(\ell))|)) &\leqslant |a\gamma_1(\ell) - a\xi_1(\ell)| \\
&+ |b\gamma_1(\ell)\gamma_2(\ell) - \xi_1(\ell)\xi_2(\ell)| \\
&\leqslant |a|\,|\gamma_1(\ell) - \xi_1(\ell)| \\
&+ |b\gamma_1(\ell)\gamma_2(\ell) - b\gamma_1(\ell)\xi_2(\ell)| \\
&+ |b\gamma_1(\ell)\xi_2(\ell) - \xi_1(\ell)\xi_2(\ell)| \\
&\leqslant |a|\,|\gamma_1(\ell) - \xi_1(\ell)| + |b|\,\|\gamma_1\|\,|\gamma_2(\ell) - \xi_2(\ell)| \\
&+ |b|\,\|\xi_2\|\,|\gamma_1(\ell) - \xi_1(\ell)| \\
&\leqslant (|a| + |b|\,\|\xi_2\|)\,|\gamma_1(\ell) - \xi_1(\ell)| \\
&+ |b|\,\|\gamma_1\|\,|\gamma_2(\ell) - \xi_2(\ell)| \\
&\leqslant \frac{1}{2}\Phi\left(\max_{1\leqslant i\leqslant 2}\{|\gamma_i(\ell) - \xi_i(\ell)| : \ell \in [0,1]\}\right),
\end{aligned} \tag{5.20}$$

where $\Phi : (0,\infty) \to (0,\infty)$ is considered as $\Phi(u) = (|a| + |b|\,\|\xi_2\| + |b|\,\|\gamma_1\|)\,u$. Hence assumption $\mathcal{ABC}-1)$ is satisfied. Now, $\mathcal{M}_{\mathcal{U}} = \sup\{|\mathcal{U}(\ell,0,0)| : \ell \in [0,\mathcal{Z}]\} = 0$. Further, in the inequality

$$\mathcal{H}\left(\Im; \frac{1-\alpha}{C(\alpha)}\left(\frac{1}{2}\Phi(r_0) + \mathcal{M}_{\mathcal{U}}\right)\right) + \mathcal{H}\left(\Im; \frac{\alpha\left(\frac{1}{2}\Phi(r_0) + \mathcal{M}_{\mathcal{U}}\right)}{C(\alpha)\,\Gamma(\alpha)}\right) \leqslant r_0,$$

from assumption $\mathcal{ABC}-3)$ takes the following form, if we consider $\mathcal{H} \in \mathcal{C}_{\mathcal{H}}$ as $\mathcal{H}(\Im; h) = h$,

$$\frac{1-\alpha}{\mathcal{C}(\alpha)}\frac{1}{2}\left(r_0 + 2{r_0}^2\right) + \frac{\alpha}{\mathcal{C}(\alpha)\,\Gamma(\alpha)}\frac{1}{2}\left(r_0 + 2{r_0}^2\right) \leqslant r_0. \tag{5.21}$$

From the above estimate we take $\mathcal{M}_{\mathcal{U}} = 0$, $\Phi(u) = (|a| + |b|\,\|\xi_2\| + |b|\,\|\gamma_1\|)\,u$, in above inequality (5.21) and we define the mapping for r_0 and α in the following sense

$$\mathcal{F}(r_0,\alpha) = 2\mathcal{C}(\alpha)\,r_0 - \left(|a|\,r_0 + 2\,|b|\,{r_0}^2\right)\left(1 - \alpha + \frac{\alpha}{\Gamma(\alpha)}\right).$$

For $a = 1$ & $b = 1$ the above function becomes

$$\mathcal{F}(r_0,\alpha) = 2\mathcal{C}(\alpha)\,r_0 - \left(r_0 + 2{r_0}^2\right)\left(1 - \alpha + \frac{\alpha}{\Gamma(\alpha)}\right). \tag{5.22}$$

An inequality (5.21) has positive solution if the function (5.22) is positive for some values of r_0 and α. For this we plot the graph of $\mathcal{F}(\alpha, r_0)$ for $r_0 \in (0.5, 2.5)$ and $\alpha \in (0,1)$.

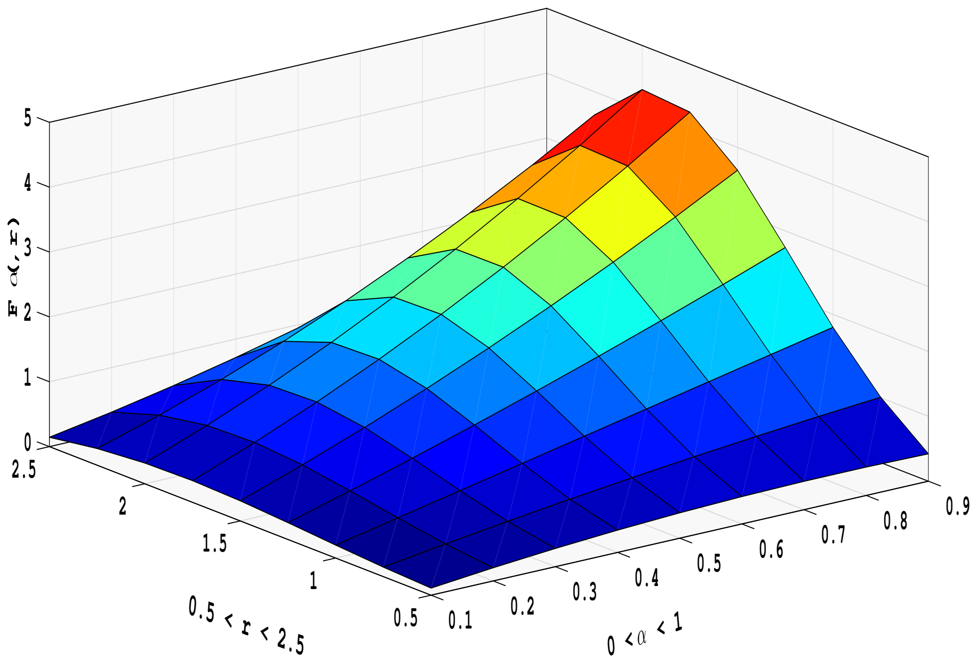

Figure 1.

The Figure 1 conformed that the inequality (5.21) have positive solution for $r_0 \in (0.5, 2.5)$ and $\alpha \in (0, 1)$. All the assumptions of Theorem 5.3 satisfied, hence the system of differential equation (5.18) with (5.19), which is of Lotka-Volterra model in the sense of Atangana-Baleanu fractional differential equation, has atleast one solution in the space $\mathcal{C}[0, 1]$. □

6 Conclusion

In this work we apply the generalized Darbo fixed point theorem to ensure the existence of solution for system of differential equation with Atangana-Baleanu fractional order. In section 3 we prove that Theorem 2.1 is generalization of results in [10]. Moreover, if we take $\mathcal{R}_1, \mathcal{R}_2, \mathcal{R}_3 : [0, \infty) \to [0, \infty)$ in the Theorem 2.3 of section 2, as

$$\mathcal{R}_1(u) = \psi(u) + \frac{\theta}{2},\ \mathcal{R}_2(u) = \psi(u) + \theta \text{ and } \mathcal{R}_3(u) = \varphi(u) + \frac{\theta}{2},$$

where $\theta > 0$ is real number and $\varphi, \psi : [0, \infty) \to [0, \infty)$ are real valued functions with some conditions, then we get the results from literature's [1, 9, 7]. Further, we proved a generalized Darbo type coupled fixed point theorem and used to prove the existence of solution. In future, we implement some new Darbo type fixed point theorem to elaborate the stability behavior of the model via delay differential equation solutions.

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Author information

V. E. Nikam, A. K. Shukla and D. Gopal, Department of Mathematics and Humanities, S. V. National Institute of Technology, Surat, Gujrat 395007, India.
E-mail: nikamvishal832@gmail.com

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