Palestine Journal of Mathematics

Vol. 12(4)(2023) , 258–275

Existence and uniqueness results for higher order fractional differential equations and inclusions with multi-strip Hadamard type fractional integral boundary conditions

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Communicated by Ayman Badawi

MSC 2010 Classifications: Primary 34A08, 34A12; Secondary 34A60.

Keywords and phrases: Fractional differential equations, fractional differential inclusions, Hadamard fractional derivative, fractional integral boundary conditions, existence, uniqueness, fixed point.

Abstract In this paper, we obtain some new existence and uniqueness results for higher order fractional differential equations and inclusions with multi-strip Hadamard type fractional integral boundary conditions. In the case of inclusion problem, the existence results are established for convex as well as nonconvex multivalued maps. Our results are based on some fixed point theorems. Finally, an example is given to illustrate our results.

1 Introduction

The topic of fractional differential equations and inclusions has been of great interest for many researchers in view of its theoretical development and widespread applications in various fields of science and engineering, see the books [7, 20, 27, 28]. Many researchers have contributed to the development of the existence, uniqueness and stability theory for fractional equations and inclusions, see [1]-[9], [11]-[12], [17], [18], [20], [23]-[28], [31], [32] and the references cited therein.

Recently, in [2], Ahmad and Nieto studied the existence of solutions of the following fractional differential inclusion of order α with nonlocal boundary conditions

$$\begin{cases} {}^{C}D^{\alpha}x\left(t\right) \in F\left(t,x\left(t\right)\right),\ t\in\left[0,1\right], \\ x\left(0\right)=0,\ x'\left(0\right)=0,\ x''\left(0\right)=0,\ ...,\ x^{(m-2)}\left(0\right)=0,\ x\left(1\right)=\beta x\left(\eta\right), \\ 0<\eta<1,\ \beta\eta^{m-1}\neq 1,\ \beta\in\mathbb{R}, \end{cases}$$

where ${}^{C}D^{\alpha}$ is the Caputo fractional derivative of order $\alpha\in\left(m-1,m\right],\ m\geq 2,\ m\in\mathbb{N}$ and $F:J\times\mathbb{R}\to\mathcal{P}\left(\mathbb{R}\right)$ is a multivalued map.

In [23], the authors discussed the existence of solutions of the following nonlocal boundary value problem of the Hadamard fractional derivatives of the form

$$\begin{cases} {}^{H}\mathfrak{D}^{\alpha}x\left(t\right)=f\left(t,x\right),\ t\in\left[1,T\right], \\ x\left(1\right)=0,\ x'\left(1\right)=0,\ {}^{H}\mathfrak{D}^{\beta}x\left(T\right)=\omega{}^{H}\mathfrak{J}^{\gamma}x\left(\varphi\right),\ 1<\varphi<T, \end{cases}$$

where ${}^{H}\mathfrak{D}^{\alpha}$ and ${}^{H}\mathfrak{D}^{\beta}$ denote the Hadamard fractional derivatives of orders $\alpha\in(2,3]$ and $\beta\in(1,2)$ respectively, ${}^{H}\mathfrak{J}^{\gamma}$ denotes the Hadamard fractional integral of order γ, $\omega>0$ and $f:[1,T]\times\mathbb{R}\to\mathbb{R}$ is a given continuous function.

In [12], the following higher order fractional differential equation with multi-strip Riemann-Liouville fractional integral boundary conditions was studied

$$\begin{cases} {}^{C}D^{\alpha}x\left(t\right)=f\left(t,x\left(t\right)\right),\ 0<t<1,\ n-1<\alpha<n,\ n\geq 2,\ n\in\mathbb{N}, \\ x\left(0\right)=0,\ x'\left(0\right)=0,\ x''\left(0\right)=0,\ ...,\ x^{(m-2)}\left(0\right)=0, \\ x\left(1\right)=\sum\limits_{i=1}^{m}\gamma_i\left[I^{\beta_i}x\left(\eta_i\right)-I^{\beta_i}x\left(\zeta_i\right)\right], \end{cases}$$

where ${}^{C}D^{\alpha}$ is the Caputo fractional derivative of order α, $f : [0,1] \times \mathbb{R} \to \mathbb{R}$ is a jointly continuous function, I^{β_i} is the Riemann-Liouville fractional integral of order $\beta_i > 0$, $i = 1, ..., m$, $0 < \zeta_i < \eta_i < ... < \zeta_m < \eta_m < 1$ and $\gamma_i \in \mathbb{R}$.

Inspired and motivated by the above works, we discuss some existence and uniqueness results for higher order fractional differential equations and inclusions with multi-strip Hadamard type fractional integral boundary conditions. Precisely, we consider the following problems

$$\begin{cases} {}^{H}\mathfrak{D}^{\alpha}x(t) = f(t,x), \ t \in (1,T), \\ x(1) = 0, \ x'(1) = 0, \ x''(1) = 0, \ ..., \ x^{(n-2)}(1) = 0, \\ {}^{H}\mathfrak{D}^{\beta}x(T) = \sum\limits_{i=1}^{k} \theta_i \left[{}^{H}\mathfrak{J}^{\lambda_i}x(\eta_i) - {}^{H}\mathfrak{J}^{\lambda_i}x(\rho_i) \right], \ 1 < \eta_i, \rho_i < T, \ \theta_i \in \mathbb{R}, \end{cases} \tag{1.1}$$

where ${}^{H}\mathfrak{D}^{\alpha}$ and ${}^{H}\mathfrak{D}^{\beta}$ denote the Hadamard fractional derivatives of orders α and β respectively, $n-1 < \alpha \leq n$, $n-2 < \beta < n-1$, $n \geq 2$, $n \in \mathbb{N}$, ${}^{H}\mathfrak{J}^{\lambda_i}$ denotes the Hadamard fractional integral of order $\lambda_i > 0$, $i = 1, ..., k$, $1 < \rho_1 < \eta_1 < ... < \rho_k < \eta_k < T$, $f : [1,T] \times \mathbb{R} \to \mathbb{R}$ is a given continuous function, and

$$\begin{cases} {}^{H}\mathfrak{D}^{\alpha}x(t) \in F(t,x(t)), \ t \in (1,T), \\ x(1) = 0, \ x'(1) = 0, \ x''(1) = 0, \ ..., \ x^{(n-2)}(1) = 0, \\ {}^{H}\mathfrak{D}^{\beta}x(T) = \sum\limits_{i=1}^{k} \theta_i \left[{}^{H}\mathfrak{J}^{\lambda_i}x(\eta_i) - {}^{H}\mathfrak{J}^{\lambda_i}x(\rho_i) \right], \ 1 < \eta_i, \rho_i < T, \ \theta_i \in \mathbb{R}, \end{cases} \tag{1.2}$$

where $F : J \times \mathbb{R} \to \mathcal{P}(\mathbb{R})$ is a multivalued map and $\mathcal{P}(\mathbb{R})$ is the family of all nonempty subsets of $\mathbb{R}$.

This paper is organized as follows. In section 2, we recall some basic concepts of fractional calculus, multivalued analysis and fixed point theory. In section 3, we discuss the existence and uniqueness of solutions for problem (1.1) by applying Banach and Schauder fixed point theorems. In section 4, we deal with some existence results for the inclusion problem (1.2) involving convex as well as nonconvex multivalued maps. These results are based on the nonlinear alternative of Leray-Schauder type and a fixed point theorem due to Covitz and Nadler. Finally, an example is given in Section 5 to illustrate the usefulness of our main results.

2 Preliminaries

2.1 Fractional calculus

In this subsection, we recall some basic ideas of fractional calculus and present known results needed in our forthcoming analysis.

Let $J = [1,T]$. We denote by $C(J,\mathbb{R})$ the Banach space of all continuous functions from J into $\mathbb{R}$ with the norm

$$\|x\| = \sup\{|x(t)| : t \in J\}.$$

Let $L^1(J,\mathbb{R})$ be the Banach space of measurable functions $x : J \to \mathbb{R}$ that are Lebesgue integrable with norm

$$\|x\|_{L^1} = \int_J |x(t)|\, dt.$$

And $AC(J,\mathbb{R})$ be the space of absolutely continuous valued functions on J, and set

$$AC^n(J) = \left\{ x : J \to E : x, x', x'', ..., x^{n-1} \in C(J,\mathbb{R}) \text{ and } x^{n-1} \in AC(J,\mathbb{R}) \right\}.$$

Definition 2.1 ([20]). The Hadamard fractional integral of order $\alpha > 0$ for a function $x \in L^1(J,\mathbb{R})$ is defined as

$${}^{H}\mathfrak{J}^{\alpha}x(t) = \frac{1}{\Gamma(\alpha)} \int_1^t \left(\log \frac{t}{s} \right)^{\alpha - 1} x(s) \frac{ds}{s}, \ \alpha > 0.$$

Set $\delta = \left(t \frac{d}{dt}\right)$, $\alpha > 0$, $n = [\alpha] + 1$, where $[\alpha]$ denotes the integer part of α. Define the space

$$AC_{\delta}^{n}(J) = \left\{ x : J \to \mathbb{R} : \delta^{n-1}x \in AC(J,\mathbb{R}) \right\},$$

Definition 2.2 ([20]). The Hadamard fractional derivative of order $\alpha > 0$ for a function $x \in AC_\delta^n(J)$ is defined as

$$^H\mathfrak{D}^\alpha x(t) = \delta^n \left({}^H\mathfrak{J}^{n-\alpha}x\right)(t) = \frac{1}{\Gamma(n-\alpha)}\left(t\frac{d}{dt}\right)^n \int_1^t \left(\log\frac{t}{s}\right)^{n-\alpha-1} x(s)\frac{ds}{s}.$$

Lemma 2.3 ([20]). *Let $n-1<\alpha\le n$, $n\in\mathbb{N}$, the general solution of the fractional differential equation $^H\mathfrak{D}^\alpha x(t)=0$ is given by*

$$x(t) = \sum_{k=1}^n c_k (\log t)^{\alpha-k},$$

where $c_k \in \mathbb{R}$, $k=1,2,...,n$ are arbitrary constants.

From the above lemma, it follows that

$$^H\mathfrak{J}^\alpha\, {}^H\mathfrak{D}^\alpha x(t) = x(t) + \sum_{k=1}^n c_k (\log t)^{\alpha-k},$$

for some $c_k \in \mathbb{R}$, $k=1,2,...,n$ are arbitrary constants.

Lemma 2.4 ([20]). *Let $\alpha,\beta,a>0$, then*

$$^H\mathfrak{D}^\alpha \left(\log\frac{t}{a}\right)^{\beta-1}(x) = \frac{\Gamma(\alpha)}{\Gamma(\alpha+\beta)}\left(\log\frac{x}{a}\right)^{\beta+\alpha-1},$$

and

$$^H\mathfrak{J}^\alpha \left(\log\frac{t}{a}\right)^{\beta-1}(x) = \frac{\Gamma(\alpha)}{\Gamma(\alpha-\beta)}\left(\log\frac{x}{a}\right)^{\beta-\alpha-1}.$$

To study the nonlinear problem (1.1), we need the following lemma.

Lemma 2.5. *Suppose that*

$$\Lambda = \frac{\Gamma(\alpha)}{\Gamma(\alpha-\beta)}(\log T)^{\alpha-\beta-1} + \sum_{i=1}^k \frac{\theta_i\Gamma(\alpha)}{\Gamma(\alpha+\beta)}\left[(\log\rho_i)^{\alpha+\beta-1} - (\log\eta_i)^{\alpha+\beta-1}\right] \neq 0. \quad (2.1)$$

Then, for any $q\in C(J,\mathbb{R})$, the unique solution of the boundary value problem

$$\begin{cases} ^H\mathfrak{D}^\alpha x(t) = q(t),\ t\in(1,T),\ n-1<\alpha\le n,\ n\ge 2,\ n\in\mathbb{N}, \\ x(1)=0,\ x'(1)=0,\ x''(1)=0,\ ...,\ x^{(n-2)}(1)=0, \\ ^H\mathfrak{D}^\beta x(T) = \sum_{i=1}^k \theta_i\left[{}^H\mathfrak{J}^{\lambda_i}x(\eta_i) - {}^H\mathfrak{J}^{\lambda_i}x(\rho_i)\right], \end{cases} \quad (2.2)$$

is given by

$$\begin{aligned} x(t) &= \left({}^H\mathfrak{J}^\alpha q\right)(t) + \frac{(\log t)^{\alpha-1}}{\Lambda} \\ &\times\left(\sum_{i=1}^k \theta_i\left[\left({}^H\mathfrak{J}^{\alpha+\lambda_i}q\right)(\eta_i) - \left({}^H\mathfrak{J}^{\alpha+\lambda_i}q\right)(\rho_i)\right] - \left({}^H\mathfrak{J}^{\alpha-\beta}q\right)(T)\right) \\ &= \frac{1}{\Gamma(\alpha)}\int_1^t\left(\log\frac{t}{s}\right)^{\alpha-1} q(s)\frac{ds}{s} + \frac{(\log t)^{\alpha-1}}{\Lambda} \\ &\times\left(\sum_{i=1}^k \frac{\theta_i}{\Gamma(\alpha+\lambda_i)}\left[\int_1^{\eta_i}\left(\log\frac{\eta_i}{s}\right)^{\alpha+\lambda_i-1} q(s)\frac{ds}{s} - \int_1^{\rho_i}\left(\log\frac{\rho_i}{s}\right)^{\alpha+\lambda_i-1} q(s)\frac{ds}{s}\right]\right. \\ &\left. - \frac{1}{\Gamma(\alpha-\beta)}\int_1^T\left(\log\frac{T}{s}\right)^{\alpha-\beta-1} q(s)\frac{ds}{s}\right). \end{aligned} \quad (2.3)$$

Proof. Applying the Hadamard fractional integral of order α to both sides of the equation in (2.2), and using Lemma 2.3, we get

$$x(t) = \left({}^H\mathfrak{J}^{\alpha}q\right)(t) - c_1(\log t)^{\alpha-1} - c_2(\log t)^{\alpha-2} - \ldots - c_{n-1}(\log t)^{\alpha-n+1} - c_n(\log t)^{\alpha-n}, \quad (2.4)$$

then

$$x'(t) = \frac{1}{\Gamma(\alpha-1)}\int_1^t \frac{\left(\log\frac{t}{s}\right)^{\alpha-2}}{t} q(s)\frac{ds}{s} - c_1(\alpha-1)\frac{(\log t)^{\alpha-2}}{t} - \ldots - c_n(\alpha-n)\frac{(\log t)^{\alpha-n-1}}{t},$$

and

$$x''(t) = \frac{1}{\Gamma(\alpha-2)}\int_1^t \frac{\left(\log\frac{t}{s}\right)^{\alpha-3} - \left(\log\frac{t}{s}\right)^{\alpha-2}}{t^2} q(s)\frac{ds}{s} - c_1(\alpha-1)(\alpha-2)\frac{(\log t)^{\alpha-3} - (\log t)^{\alpha-2}}{t^2} - \ldots - c_n(\alpha-n)(\alpha-n-1)\frac{(\log t)^{\alpha-n-2} - (\log t)^{\alpha-n-1}}{t^2}, \ldots.$$

Applying the boundary conditions, we have

$$c_2 = \ldots = c_{n-1} = c_n = 0. \quad (2.5)$$

Using (2.5) in (2.4), we get

$$x(t) = \left({}^H\mathfrak{J}^{\alpha}q\right)(t) - c_1(\log t)^{\alpha-1}. \quad (2.6)$$

Applying the Hadamard fractional derivative of order β to (2.6), we get

$${}^H\mathfrak{D}^{\beta}x(t) = \left({}^H\mathfrak{J}^{\alpha-\beta}q\right)(t) - c_1\frac{\Gamma(\alpha)}{\Gamma(\alpha-\beta)}(\log t)^{\alpha-\beta-1}.$$

Again applying the Hadamard fractional integral of order λ_i to (2.6), we obtain

$${}^H\mathfrak{J}^{\lambda_i}x(t) = \left({}^H\mathfrak{J}^{\alpha+\lambda_i}q\right)(t) - c_1\frac{\Gamma(\alpha)}{\Gamma(\alpha+\beta)}(\log t)^{\alpha+\beta-1}.$$

Using the condition

$${}^H\mathfrak{D}^{\beta}x(T) = \sum_{i=1}^{k}\theta_i\left[{}^H\mathfrak{J}^{\lambda_i}x(\eta_i) - {}^H\mathfrak{J}^{\lambda_i}x(\rho_i)\right],$$

we have

$$\left({}^H\mathfrak{J}^{\alpha-\beta}q\right)(T) - c_1\frac{\Gamma(\alpha)}{\Gamma(\alpha-\beta)}(\log T)^{\alpha-\beta-1}$$
$$= \sum_{i=1}^{k}\theta_i\left[\left({}^H\mathfrak{J}^{\alpha+\lambda_i}q\right)(\eta_i) - \left({}^H\mathfrak{J}^{\alpha+\lambda_i}q\right)(\rho_i) - c_1\frac{\Gamma(\alpha)}{\Gamma(\alpha+\beta)}(\log\eta_i)^{\alpha+\beta-1} + c_1\frac{\Gamma(\alpha)}{\Gamma(\alpha+\beta)}(\log\rho_i)^{\alpha+\beta-1}\right]. \quad (2.7)$$

By solving (2.7), we find that

$$c_1 = \frac{1}{\Lambda}\left(\left({}^H\mathfrak{J}^{\alpha-\beta}q\right)(T) - \sum_{i=1}^{k}\theta_i\left[\left({}^H\mathfrak{J}^{\alpha+\lambda_i}q\right)(\eta_i) - \left({}^H\mathfrak{J}^{\alpha+\lambda_i}q\right)(\rho_i)\right]\right),$$

where Λ is given by (2.1). Replacing the value of c_1 into (2.6), we get the integral equation (2.3). The converse follows by direct computation which completes the proof. □

2.2 Multivalued analysis

Here, we outline some basic definitions and results for multivalued maps.

Let $(X, \|.\|)$ be a Banach space. A mutlivalued map $G : X \to \mathcal{P}(X)$

(i) is convex (closed) valued if $G(x)$ is convex (closed) for all $x \in X$,

(ii) is bounded on bounded sets if $G(B) = \cup_{x\in B} G(x)$ is bounded in X for all bounded set B of X, i.e., $\sup_{x\in B} \{\sup\{|y| : y \in G(x)\}\} < \infty$,

(iii) is called upper semi-continuous (u.s.c for short) on X if for each $x_0 \in X$ the set $G(x_0)$ is nonempty, closed subset of X, and for each open set $\mathcal{U}$ of X containing $F(x_0)$, there exists an open neighborhood $\mathcal{V}$ of x_0 such that $G(\mathcal{V}) \subseteq \mathcal{U}$,

(iv) is said to be completely continuous if $G(B)$ is relatively compact for every bounded subset B of X,

(v) is said to be measurable if for every $y \in \mathbb{R}$, the function

$$t \to d(y, G(t)) = \inf\{|y - z| : z \in G(t)\},$$

is measurable,

(vi) has a fixed point if there exists $x \in X$ such that $x \in G(x)$. The fixed point set of the multivalued operator F will be denoted by $FixG$.

For each $y \in C(J, \mathbb{R})$, the set of selections for the multivalued F is defined by

$$S_{F,y} = \left\{v \in L^1(J, \mathbb{R}) : v(t) \in F(t, y(t)) \text{ for a.e. } t \in J\right\}.$$

In the following, we denote by $\mathcal{P}_p$ the set of all nonempty subsets of X which have the property "p" where "p" will be bounded (b), closed (cl), convex (c), compact (cp) etc. Thus $\mathcal{P}_b(X) = \{Y \in \mathcal{P}(X) :$
Y is bounded$\}$, $\mathcal{P}_{cl}(X) = \{Y \in \mathcal{P}(X) : Y$ is closed$\}$, $\mathcal{P}_{cp}(X) = \{Y \in \mathcal{P}(X) : Y$ is compact$\}$, and $\mathcal{P}_{cp,c}(X) = \{Y \in \mathcal{P}(X) : Y$ is compact and convex$\}$.

Definition 2.6. A multivalued map $F : J \times \mathbb{R} \to \mathcal{P}(\mathbb{R})$is said to be Carathéodory if

(i) $t \to F(t, x)$ is measurable for each $x \in \mathbb{R}$,

(ii) $x \to F(t, x)$ is upper semi-continuous for almost all $t \in J$.

Further a Carathéodory function F is called L^1-Carathéodory if

(iii) for each $\rho > 0$, there exists $\varphi_\rho \in L^1(J, \mathbb{R}^+)$ such that

$$\|F(t, x)\| = \sup\{|v| : v \in F(t, x)\} \leq \varphi_\rho(t),$$

for all $|x| \leq \rho$ and for a.e. $t \in J$.

We define the graph of G to be the set $Gr(G) = \{(x, y) \in X \times Y,\ y \in G(x)\}$ and recall two results for closed graphs and upper-semicontinuity.

Lemma 2.7 ([10] Proposition 1.2). *If* $\mathrm{G} : X \to \mathcal{P}_{cl}(X)$ *is u.s.c., then* $\mathrm{Gr}(\mathrm{G})$ *is a closed subset of* $X \times Y$*,i.e., for every sequence* $\{x_n\}_{n\in\mathbb{N}} \subset X$ *and* $\{y_n\}_{n\in\mathbb{N}} \subset Y$*, if when* $n \to \infty$*,* $x_n \to x_*$*,* $y_n \to y_*$ *and* $y_n \in \mathrm{G}(x_n)$*, then* $y_* \in \mathrm{G}(x_*)$*. Conversely, if* G *is completely continuous and has a closed graph, then it is upper semi-continuous.*

Lemma 2.8 ([22]). *Let* X *be a separable Banach space. Let* $\mathrm{F} : J \times X \to \mathcal{P}_{cp,c}(X)$ *be an* L^1*-Carathéodory multivalued map and let* Θ *be a linear continuous mapping from* $L^1(J, X)$ *to* $C(J, X)$*. Then the operator*

$$\Theta \circ S_{\mathrm{F}} : C(J, X) \to \mathcal{P}_{cp,c}(C(J, X)),\ x \to (\Theta \circ S_{\mathrm{F}})(x) = \Theta(S_{\mathrm{F},x}),$$

is a closed graph operator in $C(J, X) \times C(J, X)$*.*

For more details on multivalued maps and the proof of the known results cited in this section, we refer the interested reader to the books by Deimling [10], Gorniewicz [16] and Hu and Papageorgiou [19].

2.3 Fixed point theorems

In this part, we collect the fixed point theorems which are used in the proofs of our main results.

Theorem 2.9 (Banach fixed point theorem [29]). *Let Ω be a non-empty closed subset of a Banach space $(S, \|.\|)$, then any contraction mapping Φ of Ω into itself has a unique fixed point.*

Theorem 2.10 (Schauder fixed point theorem [29]). *Let Ω be a nonempty bounded closed convex subset of a Banach space S and $\Phi : \Omega \to \Omega$ be a continuous compact operator. Then has a fixed point in Ω.*

Theorem 2.11 (Nonlinear alternative of Kakutani maps [15]). *Let C be a closed convex subset of a Banach space E and $\mathcal{U}$ be an open subset of C with $0 \in \mathcal{U}$. Suppose that $N : \overline{\mathcal{U}} \to \mathcal{P}_{cp,c}(C)$ is an upper semi-continuous compact map. Then either*

(i) N has a fixed point in $\overline{\mathcal{U}}$, or
(ii) there is a $x \in \partial\mathcal{U}$ and $\mu \in (0,1)$ with $x \in \mu N(x)$.

Theorem 2.12 (Covitz and Nadler fixed point theorem [14]). *Let (X, d) be complete metric space. If $N : X \to \mathcal{P}_{cl}(X)$ is a contraction, then $FixN \neq \emptyset$.*

3 Existence results for single-valued problems

In what follows, we apply the fixed point theorems of Banach and Schauder to prove the existence and uniqueness results for problem (1.1).

By Lemma 2.5, we define an operator $\Phi : C(J, \mathbb{R}) \to C(J, \mathbb{R})$ by

$$\begin{aligned}(\Phi x)(t) = {} & \frac{1}{\Gamma(\alpha)} \int_1^t \left(\log \frac{t}{s}\right)^{\alpha-1} f(s, x(s)) \frac{ds}{s} \\ & + \frac{(\log(t))^{\alpha-1}}{\Lambda} \left(\sum_{i=1}^{k} \frac{\theta_i}{\Gamma(\alpha+\lambda_i)} \left[\int_1^{\eta_i} \left(\log \frac{\eta_i}{s}\right)^{\alpha+\lambda_i-1} f(s, x(s)) \frac{ds}{s} \right.\right. \\ & \left. - \int_1^{\rho_i} \left(\log \frac{\rho_i}{s}\right)^{\alpha+\lambda_i-1} f(s, x(s)) \frac{ds}{s} \right] \\ & \left. - \frac{1}{\Gamma(\alpha-\beta)} \int_1^T \left(\log \frac{T}{s}\right)^{\alpha-\beta-1} f(s, x(s)) \frac{ds}{s} \right). \end{aligned} \tag{3.1}$$

Clearly, the fractional integral equation (2.3) can be written as the following operator equation

$$x = \Phi x. \tag{3.2}$$

Thus, the existence of a solution for (1.1) is equivalent to the existence of a fixed point for the operator Φ which satisfies the operator equation (3.2).

Our first result, dealing with the existence of a unique solution, is based on the Banach contraction mapping principle.

Theorem 3.1. *Let $f : J \times \mathbb{R} \to \mathbb{R}$ be a continuous function and (2.1) holds. Assume that*

(H1) There exists a constant $L_f \in \mathbb{R}^+$ such that

$$|f(t, x) - f(t, y)| \leq L_f |x - y|, \; t \in J, \; x, y \in \mathbb{R}.$$

(H2)

$$\kappa = \frac{L_f (\log T)^{\alpha}}{\Gamma(\alpha+1)} + \frac{L_f (\log T)^{\alpha-1}}{|\Lambda|} \left(2 \sum_{i=1}^{k} \frac{|\theta_i| (\log T)^{\alpha+\lambda_i}}{\Gamma(\alpha+\lambda_i+1)} + \frac{(\log T)^{\alpha-\beta}}{\Gamma(\alpha-\beta+1)} \right) < 1.$$

Then the boundary value problem (1.1) has a unique solution on J.

Proof. For $x, y \in C(J, \mathbb{R})$ and for each $t \in J$, from the definition of Φ and assumption (H1), we obtain

$$\begin{aligned}
&|(\Phi x)(t) - (\Phi y)(t)| \\
&\leq \frac{1}{\Gamma(\alpha)} \int_1^t \left(\log \frac{t}{s}\right)^{\alpha-1} |f(s, x(s)) - f(s, y(s))| \frac{ds}{s} \\
&+ \frac{(\log t)^{\alpha-1}}{|\Lambda|} \left(\sum_{i=1}^{k} \frac{|\theta_i|}{\Gamma(\alpha+\lambda_i)} \left[\int_1^{\eta_i} \left(\log \frac{\eta_i}{s}\right)^{\alpha+\lambda_i-1} |f(s, x(s)) - f(s, y(s))| \frac{ds}{s} \right.\right. \\
&\left. + \int_1^{\rho_i} \left(\log \frac{\rho_i}{s}\right)^{\alpha+\lambda_i-1} |f(s, x(s)) - f(s, y(s))| \frac{ds}{s} \right] \\
&\left. + \frac{1}{\Gamma(\alpha-\beta)} \int_1^T \left(\log \frac{T}{s}\right)^{\alpha-\beta-1} |f(s, x(s)) - f(s, y(s))| \frac{ds}{s} \right) \\
&\leq \left(\frac{L_f (\log T)^{\alpha}}{\Gamma(\alpha+1)} + \frac{L_f (\log T)^{\alpha-1}}{|\Lambda|} \left(2 \sum_{i=1}^{k} \frac{|\theta_i| (\log T)^{\alpha+\lambda_i}}{\Gamma(\alpha+\lambda_i+1)} + \frac{(\log T)^{\alpha-\beta}}{\Gamma(\alpha-\beta+1)} \right) \right) \|x - y\|.
\end{aligned}$$

Thus

$$\|\Phi x - \Phi y\| \leq \kappa \|x - y\|.$$

From (H2), Φ is a contraction. As a consequence of the Banach fixed point theorem, we get that Φ has a unique fixed point which is a unique solution of the problem (1.1) on J. □

Next, we prove an existence result for the problem (1.1) by using the Schauder fixed point theorem.

Theorem 3.2. *Let $f : J \times \mathbb{R} \to \mathbb{R}$ be a continuous function and (2.1) holds. Assume that*
(H3) $|f(t, x(t))| \leq \Psi(t)$, $\forall (t, x) \in J \times C(J, \mathbb{R})$, $\Psi \in L^1(J, \mathbb{R}^+)$.
Then the boundary value problem (1.1) has at least one solution on J.

Proof. We consider the non-empty closed bounded convex subset $\Omega = \{x \in C(J, \mathbb{R}) : \|x\| \leq M\}$ of $C(J, \mathbb{R})$, where M is chosen such

$$M \geq \Lambda_1,$$

where

$$\Lambda_1 = \frac{\Psi^* (\log T)^{\alpha}}{\Gamma(\alpha+1)} + \frac{\Psi^* (\log T)^{\alpha-1}}{|\Lambda|} \left(2 \sum_{i=1}^{k} \frac{|\theta_i| (\log T)^{\alpha+\lambda_i}}{\Gamma(\alpha+\lambda_i+1)} + \frac{(\log T)^{\alpha-\beta}}{\Gamma(\alpha-\beta+1)} \right),$$

with $\Psi^* = \sup\{\Psi(t) : t \in J\}$. The continuity of f implies the continuity of the operator Φ. Now, we need to show that the operator Φ is compact by applying the well known Arzela-Ascoli theorem. So we will show that $\Phi(\Omega) \subset \Omega$ and $\Phi(\Omega)$ is uniformly bounded and equicontinuous set. For $x \in \Omega$, it follows that

$$\begin{aligned}
|(\Phi x)(t)| &\leq \frac{1}{\Gamma(\alpha)} \int_1^t \left(\log \frac{t}{s}\right)^{\alpha-1} |f(s, x(s))| \frac{ds}{s} \\
&+ \frac{(\log (t))^{\alpha-1}}{|\Lambda|} \left(\sum_{i=1}^{k} \frac{|\theta_i|}{\Gamma(\alpha+\lambda_i)} \left[\int_1^{\eta_i} \left(\log \frac{\eta_i}{s}\right)^{\alpha+\lambda_i-1} |f(s, x(s))| \frac{ds}{s} \right.\right. \\
&\left. + \int_1^{\rho_i} \left(\log \frac{\rho_i}{s}\right)^{\alpha+\lambda_i-1} |f(s, x(s))| \frac{ds}{s} \right] \\
&\left. + \frac{1}{\Gamma(\alpha-\beta)} \int_1^T \left(\log \frac{T}{s}\right)^{\alpha-\beta-1} |f(s, x(s))| \frac{ds}{s} \right) \\
&\leq \frac{\Psi^* (\log T)^{\alpha}}{\Gamma(\alpha+1)} + \frac{\Psi^* (\log T)^{\alpha-1}}{|\Lambda|} \left(2 \sum_{i=1}^{k} \frac{|\theta_i| (\log T)^{\alpha+\lambda_i}}{\Gamma(\alpha+\lambda_i+1)} + \frac{(\log T)^{\alpha-\beta}}{\Gamma(\alpha-\beta+1)} \right),
\end{aligned}$$

and consequently

$$\|\Phi x\| \leq \Lambda_1 \leq M,$$

which implies that $\Phi(\Omega) \subset \Omega$ and the set $\Phi(\Omega)$ is uniformly bounded. Next, we are going to prove that $\Phi(\Omega)$ is equicontinuous set. For $t_1, t_2 \in J$ such that $t_1 < t_2$ and for $u \in \Omega$, we obtain

$$\begin{aligned}
&|(\Phi x)(t_2) - (\Phi y)(t_1)| \\
&\leq \frac{1}{\Gamma(\alpha)} \int_1^{t_1} \left(\left(\log \frac{t_2}{s}\right)^{\alpha-1} - \left(\log \frac{t_1}{s}\right)^{\alpha-1} \right) |f(s, x(s))| \frac{ds}{s} \\
&+ \frac{1}{\Gamma(\alpha)} \int_{t_1}^{t_2} \left(\log \frac{t_2}{s}\right)^{\alpha-1} |f(s, x(s))| \frac{ds}{s} \\
&+ \frac{\left((\log t_2)^{\alpha-1} - (\log t_1)^{\alpha-1}\right)}{|\Lambda|} \left(\sum_{i=1}^{k} \frac{|\theta_i|}{\Gamma(\alpha+\lambda_i)} \left[\int_1^{\eta_i} \left(\log \frac{\eta_i}{s}\right)^{\alpha+\lambda_i-1} |f(s, x(s))| \frac{ds}{s} \right.\right. \\
&\left.\left. + \int_1^{\rho_i} \left(\log \frac{\rho_i}{s}\right)^{\alpha+\lambda_i-1} |f(s, x(s))| \frac{ds}{s} \right] + \frac{1}{\Gamma(\alpha-\beta)} \int_1^{T} \left(\log \frac{T}{s}\right)^{\alpha-\beta-1} |f(s, x(s))| \frac{ds}{s} \right) \\
&\leq \frac{\Psi^*}{\Gamma(\alpha)} \int_1^{t_1} \left(\left(\log \frac{t_2}{s}\right)^{\alpha-1} - \left(\log \frac{t_1}{s}\right)^{\alpha-1} \right) \frac{ds}{s} + \frac{\Psi^*}{\Gamma(\alpha)} \int_{t_1}^{t_2} \left(\log \frac{t_2}{s}\right)^{\alpha-1} \frac{ds}{s} \\
&+ \frac{\Psi^* \left((\log t_2)^{\alpha-1} - (\log t_1)^{\alpha-1}\right)}{|\Lambda|} \left(2 \sum_{i=1}^{k} \frac{|\theta_i|}{\Gamma(\alpha+\lambda_i)} \int_1^{T} \left(\log \frac{T}{s}\right)^{\alpha+\lambda_i-1} \frac{ds}{s} \right. \\
&\left. + \frac{1}{\Gamma(\alpha-\beta)} \int_1^{T} \left(\log \frac{T}{s}\right)^{\alpha-\beta-1} \frac{ds}{s} \right) \\
&\leq \frac{\Psi^*}{\Gamma(\alpha+1)} \left((\log t_2)^{\alpha} - (\log t_1)^{\alpha}\right) \\
&+ \frac{\Psi^* \left((\log t_2)^{\alpha-1} - (\log t_1)^{\alpha-1}\right)}{|\Lambda|} \left(2 \sum_{i=1}^{k} \frac{|\theta_i| (\log T)^{\alpha+\lambda_i}}{\Gamma(\alpha+\lambda_i+1)} + \frac{(\log T)^{\alpha-\beta}}{\Gamma(\alpha-\beta+1)} \right).
\end{aligned}$$

As $t_1 \to t_2$, the right-hand side of the above inequality tends to zero and the convergence is independent of x in Ω, which means $\Phi(\Omega)$ is equicontinuous. The Arzela-Ascoli Theorem implies that Φ is compact.

Hence, by the Schauder fixed point theorem, the operator Φ has at least one fixed point $x \in \Omega$. Therefore, the problem (1.1) has at least one solution on J. □

4 Existence results for multivalued problems

In this part, we use the fixed point theorems for multivalued maps to demonstrate the existence results for problem (1.2).

Definition 4.1. A function $x \in AC^n(J, \mathbb{R})$ is said to be a solution of the problem (1.2) if there exists a function $v \in L^1(J, \mathbb{R})$ with $v(t) \in F(t, x(t))$ a.e. on J such that

$$\begin{aligned}
x(t) &= \frac{1}{\Gamma(\alpha)} \int_1^{t} \left(\log \frac{t}{s}\right)^{\alpha-1} v(s) \frac{ds}{s} \\
&+ \frac{(\log t)^{\alpha-1}}{\Lambda} \left(\sum_{i=1}^{k} \frac{\theta_i}{\Gamma(\alpha+\lambda_i)} \left[\int_1^{\eta_i} \left(\log \frac{\eta_i}{s}\right)^{\alpha+\lambda_i-1} v(s) \frac{ds}{s} \right.\right. \\
&\left.\left. - \int_1^{\rho_i} \left(\log \frac{\rho_i}{s}\right)^{\alpha+\lambda_i-1} v(s) \frac{ds}{s} \right] - \frac{1}{\Gamma(\alpha-\beta)} \int_1^{T} \left(\log \frac{T}{s}\right)^{\alpha-\beta-1} v(s) \frac{ds}{s} \right).
\end{aligned}$$

4.1 The upper semi-continuous case

Our first result, dealing with the convex valued F, is based on Leray-Schauder nonlinear alternative for multivalued maps.

Theorem 4.2. *Let (2.1) holds. Set*

$$\Lambda_2 = \frac{(\log T)^{\alpha}}{\Gamma(\alpha+1)} + \frac{(\log T)^{\alpha-1}}{|\Lambda|}\left(2\sum_{i=1}^{k}\frac{|\theta_i|(\log T)^{\alpha+\lambda_i}}{\Gamma(\alpha+\lambda_i+1)} + \frac{(\log T)^{\alpha-\beta}}{\Gamma(\alpha-\beta+1)}\right), \tag{4.1}$$

and assume that

(A1) $\mathrm{F} : J \times \mathbb{R} \to \mathcal{P}_{cp,c}(\mathbb{R})$ *is a* L^1*-Carathéodory multivalued map,*

(A2) there exist a continuous nondecreasing function $Q : [0,\infty) \to (0,\infty)$ *and a function* $P \in C(J,\mathbb{R}^+)$ *such that*

$$\|\mathrm{F}(t,x)\|_{\mathcal{P}} = \sup\{|y| : y \in \mathrm{F}(t,x)\} \le P(t)\,Q(|x|),$$

for each $(t,x) \in J \times \mathbb{R}$*,*

(A3) there exists a constant $M > 0$ *such that*

$$\frac{M}{\Lambda_2\|P\|\,Q(M)} > 1. \tag{4.2}$$

Then the boundary value problem (1.2) has at least one solution on J*.*

Proof. Firstly, we transform the problem (1.2) into a fixed point problem. Consider the multivalued map $N : C(J,\mathbb{R}) \to \mathcal{P}(C(J,\mathbb{R}))$ defined by

$$N(x) = \left\{\begin{array}{l} h \in C(J,\mathbb{R}), \\ h(t) = \left\{\begin{array}{l} \frac{1}{\Gamma(\alpha)}\int_1^t \left(\log\frac{t}{s}\right)^{\alpha-1} v(s)\frac{ds}{s} \\ +\frac{(\log t)^{\alpha-1}}{\Lambda}\left(\sum_{i=1}^{k}\frac{\theta_i}{\Gamma(\alpha+\lambda_i)}\left[\int_1^{\eta_i}\left(\log\frac{\eta_i}{s}\right)^{\alpha+\lambda_i-1} v(s)\frac{ds}{s}\right.\right. \\ \left.-\int_1^{\rho_i}\left(\log\frac{\rho_i}{s}\right)^{\alpha+\lambda_i-1} v(s)\frac{ds}{s}\right] \\ \left.-\frac{1}{\Gamma(\alpha-\beta)}\int_1^T\left(\log\frac{T}{s}\right)^{\alpha-\beta-1} v(s)\frac{ds}{s}\right), \end{array}\right. \end{array}\right\}, \tag{4.3}$$

for $v \in S_{F,x}$. Clearly the fixed points of N are solutions of the problem (1.2). Now we proceed to show that the operator N satisfies all condition of Theorem 2.11. This is done in several steps.

Step 1. $N(x)$ is convex for each $x \in C(J,\mathbb{R})$.

Indeed, if h_1 and h_2 belong to $N(x)$, then there exist $v_1, v_2 \in S_{F,x}$ such that for each $t \in J$, we have

$$\begin{aligned} h_j(t) = {} & \frac{1}{\Gamma(\alpha)}\int_1^t\left(\log\frac{t}{s}\right)^{\alpha-1} v_j(s)\frac{ds}{s} \\ & + \frac{(\log t)^{\alpha-1}}{\Lambda}\left(\sum_{i=1}^{k}\frac{\theta_i}{\Gamma(\alpha+\lambda_i)}\left[\int_1^{\eta_i}\left(\log\frac{\eta_i}{s}\right)^{\alpha+\lambda_i-1} v_j(s)\frac{ds}{s}\right.\right. \\ & \left. - \int_1^{\rho_i}\left(\log\frac{\rho_i}{s}\right)^{\alpha+\lambda_i-1} v_j(s)\frac{ds}{s}\right] \\ & \left. - \frac{1}{\Gamma(\alpha-\beta)}\int_1^T\left(\log\frac{T}{s}\right)^{\alpha-\beta-1} v_j(s)\frac{ds}{s}\right),\ j = 1,2. \end{aligned}$$

Let $0 \le \sigma \le 1$. Then, for each $t \in J$, we have

$$\begin{aligned}
&[\sigma h_1 + (1-\sigma) h_2](t) \\
&= \frac{1}{\Gamma(\alpha)} \int_1^t \left(\log \frac{t}{s}\right)^{\alpha-1} (\sigma v_1(s) + (1-\sigma) v_2(s)) \frac{ds}{s} \\
&+ \frac{(\log(t))^{\alpha-1}}{\Lambda} \left(\sum_{i=1}^{k} \frac{\theta_i}{\Gamma(\alpha+\lambda_i)} \left[\int_1^{\eta_i} \left(\log \frac{\eta_i}{s}\right)^{\alpha+\lambda_i-1} (\sigma v_1(s) + (1-\sigma) v_2(s)) \frac{ds}{s} \right.\right. \\
&\left. - \int_1^{\rho_i} \left(\log \frac{\rho_i}{s}\right)^{\alpha+\lambda_i-1} (\sigma v_1(s) + (1-\sigma) v_2(s)) \frac{ds}{s} \right] \\
&\left. - \frac{1}{\Gamma(\alpha-\beta)} \int_1^T \left(\log \frac{T}{s}\right)^{\alpha-\beta-1} (\sigma v_1(s) + (1-\sigma) v_2(s)) \frac{ds}{s} \right).
\end{aligned}$$

Since F has convex values, so $S_{F,x}$ is convex and $\sigma v_1(s) + (1-\sigma) v_2(s) \in S_{F,x}$. Thus, $\sigma h_1 + (1-\sigma) h_2 \in N(x)$.

Step 2. $N(x)$ maps bounded sets into bounded sets in $C(J, \mathbb{R})$.

For a positive constant r, let $B_r = \{x \in C(J, \mathbb{R}) : \|x\| \le r\}$ be a bounded set in $C(J, \mathbb{R})$. Then for each $h \in N(x)$, $x \in B_r$, there exists $v \in S_{F,x}$ such that

$$\begin{aligned}
h(t) &= \frac{1}{\Gamma(\alpha)} \int_1^t \left(\log \frac{t}{s}\right)^{\alpha-1} v(s) \frac{ds}{s} \\
&+ \frac{(\log t)^{\alpha-1}}{\Lambda} \left(\sum_{i=1}^{k} \frac{\theta_i}{\Gamma(\alpha+\lambda_i)} \left[\int_1^{\eta_i} \left(\log \frac{\eta_i}{s}\right)^{\alpha+\lambda_i-1} v(s) \frac{ds}{s} \right.\right. \\
&\left.\left. - \int_1^{\rho_i} \left(\log \frac{\rho_i}{s}\right)^{\alpha+\lambda_i-1} v(s) \frac{ds}{s} \right] - \frac{1}{\Gamma(\alpha-\beta)} \int_1^T \left(\log \frac{T}{s}\right)^{\alpha-\beta-1} v(s) \frac{ds}{s} \right).
\end{aligned}$$

In view of (A2), for each $t \in J$, we have

$$\begin{aligned}
&|h(t)| \\
&\le \frac{1}{\Gamma(\alpha)} \int_1^t \left(\log \frac{t}{s}\right)^{\alpha-1} |v(s)| \frac{ds}{s} \\
&+ \frac{(\log t)^{\alpha-1}}{|\Lambda|} \left(\sum_{i=1}^{k} \frac{|\theta_i|}{\Gamma(\alpha+\lambda_i)} \left[\int_1^{\eta_i} \left(\log \frac{\eta_i}{s}\right)^{\alpha+\lambda_i-1} |v(s)| \frac{ds}{s} \right.\right. \\
&\left.\left. + \int_1^{\rho_i} \left(\log \frac{\rho_i}{s}\right)^{\alpha+\lambda_i-1} |v(s)| \frac{ds}{s} \right] + \frac{1}{\Gamma(\alpha-\beta)} \int_1^T \left(\log \frac{T}{s}\right)^{\alpha-\beta-1} |v(s)| \frac{ds}{s} \right) \\
&\le \frac{\|P\| Q(r) (\log T)^{\alpha}}{\Gamma(\alpha+1)} + \frac{\|P\| Q(r) (\log T)^{\alpha-1}}{|\Lambda|} \left(2 \sum_{i=1}^{k} \frac{|\theta_i| (\log T)^{\alpha+\lambda_i}}{\Gamma(\alpha+\lambda_i+1)} + \frac{(\log T)^{\alpha-\beta}}{\Gamma(\alpha-\beta+1)} \right),
\end{aligned}$$

thus

$$\|h\| \le \Lambda_2 \|P\| Q(r).$$

Step 3. $N(x)$ maps bounded sets into equicontinuous sets of $C(J, \mathbb{R})$.

Let x be any element in B_r and $h \in N(x)$. Then there exists a function $v \in S_{F,x}$ such that for each $t \in J$, we have

$$h(t) = \frac{1}{\Gamma(\alpha)} \int_1^t \left(\log \frac{t}{s} \right)^{\alpha-1} v(s) \frac{ds}{s} + \frac{(\log t)^{\alpha-1}}{\Lambda} \left(\sum_{i=1}^k \frac{\theta_i}{\Gamma(\alpha+\lambda_i)} \left[\int_1^{\eta_i} \left(\log \frac{\eta_i}{s} \right)^{\alpha+\lambda_i-1} v(s) \frac{ds}{s} - \int_1^{\rho_i} \left(\log \frac{\rho_i}{s} \right)^{\alpha+\lambda_i-1} v(s) \frac{ds}{s} \right] - \frac{1}{\Gamma(\alpha-\beta)} \int_1^T \left(\log \frac{T}{s} \right)^{\alpha-\beta-1} v(s) \frac{ds}{s} \right).$$

Let $t_1, t_2 \in J$, $t_1 < t_2$. Then

$$\begin{aligned}
&|h(t_2) - h(t_1)| \\
&\leq \frac{1}{\Gamma(\alpha)} \int_1^{t_1} \left(\left(\log \frac{t_2}{s} \right)^{\alpha-1} - \left(\log \frac{t_1}{s} \right)^{\alpha-1} \right) |v(s)| \frac{ds}{s} + \frac{1}{\Gamma(\alpha)} \int_{t_1}^{t_2} \left(\log \frac{t_2}{s} \right)^{\alpha-1} |v(s)| \frac{ds}{s} \\
&+ \frac{\left((\log t_2)^{\alpha-1} - (\log t_1)^{\alpha-1} \right)}{|\Lambda|} \left(\sum_{i=1}^k \frac{|\theta_i|}{\Gamma(\alpha+\lambda_i)} \left[\int_1^{\eta_i} \left(\log \frac{\eta_i}{s} \right)^{\alpha+\lambda_i-1} |v(s)| \frac{ds}{s} \right.\right. \\
&\left.\left. + \int_1^{\rho_i} \left(\log \frac{\rho_i}{s} \right)^{\alpha+\lambda_i-1} |v(s)| \frac{ds}{s} \right] + \frac{1}{\Gamma(\alpha-\beta)} \int_1^T \left(\log \frac{T}{s} \right)^{\alpha-\beta-1} |v(s)| \frac{ds}{s} \right) \\
&\leq \frac{\|P\| Q(r)}{\Gamma(\alpha)} \int_1^{t_1} \left(\left(\log \frac{t_2}{s} \right)^{\alpha-1} - \left(\log \frac{t_1}{s} \right)^{\alpha-1} \right) \frac{ds}{s} + \frac{\|P\| Q(r)}{\Gamma(\alpha)} \int_{t_1}^{t_2} \left(\log \frac{t_2}{s} \right)^{\alpha-1} \frac{ds}{s} \\
&+ \frac{\|P\| Q(r) \left((\log t_2)^{\alpha-1} - (\log t_1)^{\alpha-1} \right)}{|\Lambda|} \left(2 \sum_{i=1}^k \frac{|\theta_i|}{\Gamma(\alpha+\lambda_i)} \int_1^T \left(\log \frac{T}{s} \right)^{\alpha+\lambda_i-1} \frac{ds}{s} \right. \\
&\left. + \frac{1}{\Gamma(\alpha-\beta)} \int_1^T \left(\log \frac{T}{s} \right)^{\alpha-\beta-1} \frac{ds}{s} \right) \\
&\leq \frac{\|P\| Q(r)}{\Gamma(\alpha+1)} \left((\log t_2)^{\alpha} - (\log t_1)^{\alpha} \right) \\
&+ \frac{\|P\| Q(r) \left((\log t_2)^{\alpha-1} - (\log t_1)^{\alpha-1} \right)}{|\Lambda|} \left(2 \sum_{i=1}^k \frac{|\theta_i| (\log T)^{\alpha+\lambda_i}}{\Gamma(\alpha+\lambda_i+1)} + \frac{(\log T)^{\alpha-\beta}}{\Gamma(\alpha-\beta+1)} \right).
\end{aligned}$$

The right hand side of the above inequality tends to zero independently of $x \in B_r$ as $t_1 \to t_2$. As a consequence of Steps 1–3 together with Arzela-Ascoli theorem, we conclude that $N : C(J, \mathbb{R}) \to \mathcal{P}(C(J, \mathbb{R}))$ is completely continuous.

Since N is completely continuous, it is enough to show that it has a closed graph in view of Lemma 2.7, which will imply that N is u.s.c. This is done in the following step.

Step 4. N has a closed graph.

Let $x_n \to x_*$, $h_n \in N(x_n)$ and $h_n \to h_*$. Then we need to show that $h_* \in N(x_*)$. Observe that $h_n \in N(x_n)$ implies that there exists $v_n \in S_{F,x_n}$ such that for each $t \in J$,

$$h_n(t) = \frac{1}{\Gamma(\alpha)} \int_1^t \left(\log \frac{t}{s} \right)^{\alpha-1} v_n(s) \frac{ds}{s} + \frac{(\log t)^{\alpha-1}}{\Lambda} \left(\sum_{i=1}^k \frac{\theta_i}{\Gamma(\alpha+\lambda_i)} \left[\int_1^{\eta_i} \left(\log \frac{\eta_i}{s} \right)^{\alpha+\lambda_i-1} v_n(s) \frac{ds}{s} - \int_1^{\rho_i} \left(\log \frac{\rho_i}{s} \right)^{\alpha+\lambda_i-1} v_n(s) \frac{ds}{s} \right] - \frac{1}{\Gamma(\alpha-\beta)} \int_1^T \left(\log \frac{T}{s} \right)^{\alpha-\beta-1} v_n(s) \frac{ds}{s} \right).$$

Therefore, we must show that there exists $v_* \in S_{F,x_*}$ such that, for each $t \in J$,

$$h_*(t) = \frac{1}{\Gamma(\alpha)} \int_1^t \left(\log \frac{t}{s}\right)^{\alpha-1} v_*(s) \frac{ds}{s} + \frac{(\log t)^{\alpha-1}}{\Lambda} \left(\sum_{i=1}^k \frac{\theta_i}{\Gamma(\alpha+\lambda_i)} \left[\int_1^{\eta_i} \left(\log \frac{\eta_i}{s}\right)^{\alpha+\lambda_i-1} v_*(s) \frac{ds}{s} - \int_1^{\rho_i} \left(\log \frac{\rho_i}{s}\right)^{\alpha+\lambda_i-1} v_*(s) \frac{ds}{s} \right] - \frac{1}{\Gamma(\alpha-\beta)} \int_1^T \left(\log \frac{T}{s}\right)^{\alpha-\beta-1} v_*(s) \frac{ds}{s} \right).$$

Consider the continuous linear operator $\Theta : L^1(J, X) \to C(J, X)$ defined by

$$v \to \Theta(v)(t) = \frac{1}{\Gamma(\alpha)} \int_1^t \left(\log \frac{t}{s}\right)^{\alpha-1} v(s) \frac{ds}{s} + \frac{(\log t)^{\alpha-1}}{\Lambda} \left(\sum_{i=1}^k \frac{\theta_i}{\Gamma(\alpha+\lambda_i)} \left[\int_1^{\eta_i} \left(\log \frac{\eta_i}{s}\right)^{\alpha+\lambda_i-1} v(s) \frac{ds}{s} - \int_1^{\rho_i} \left(\log \frac{\rho_i}{s}\right)^{\alpha+\lambda_i-1} v(s) \frac{ds}{s} \right] - \frac{1}{\Gamma(\alpha-\beta)} \int_1^T \left(\log \frac{T}{s}\right)^{\alpha-\beta-1} v(s) \frac{ds}{s} \right).$$

Observe that

$$\begin{aligned} &\|h_n - h_*\| \\ &= \Bigg\| \frac{1}{\Gamma(\alpha)} \int_1^t \left(\log \frac{t}{s}\right)^{\alpha-1} (v_n(s) - v_*(s)) \frac{ds}{s} \\ &+ \frac{(\log(t))^{\alpha-1}}{\Lambda} \left(\sum_{i=1}^k \frac{\theta_i}{\Gamma(\alpha+\lambda_i)} \left[\int_1^{\eta_i} \left(\log \frac{\eta_i}{s}\right)^{\alpha+\lambda_i-1} (v_n(s) - v_*(s)) \frac{ds}{s} \right.\right. \\ &- \left. \int_1^{\rho_i} \left(\log \frac{\rho_i}{s}\right)^{\alpha+\lambda_i-1} (v_n(s) - v_*(s)) \frac{ds}{s} \right] \\ &- \left. \frac{1}{\Gamma(\alpha-\beta)} \int_1^T \left(\log \frac{T}{s}\right)^{\alpha-\beta-1} (v_n(s) - v_*(s)) \frac{ds}{s} \right) \Bigg\| \to 0, \end{aligned}$$

as $n \to \infty$. So it follows from Lemma 2.8, that $\Theta \circ S_{F,x}$ is a closed graph operator. Moreover, we have

$$h_n \in \Theta(S_{F,x_n}).$$

Since $x_n \to x_*$, Lemma 2.8 implies that

$$h_*(t) = \frac{1}{\Gamma(\alpha)} \int_1^t \left(\log \frac{t}{s}\right)^{\alpha-1} v_*(s) \frac{ds}{s} + \frac{(\log t)^{\alpha-1}}{\Lambda} \left(\sum_{i=1}^k \frac{\theta_i}{\Gamma(\alpha+\lambda_i)} \left[\int_1^{\eta_i} \left(\log \frac{\eta_i}{s}\right)^{\alpha+\lambda_i-1} v_*(s) \frac{ds}{s} - \int_1^{\rho_i} \left(\log \frac{\rho_i}{s}\right)^{\alpha+\lambda_i-1} v_*(s) \frac{ds}{s} \right] - \frac{1}{\Gamma(\alpha-\beta)} \int_1^T \left(\log \frac{T}{s}\right)^{\alpha-\beta-1} v_*(s) \frac{ds}{s} \right).$$

for some $v_* \in S_{F,x_*}$.

Thus, the operator N satisfies assumptions of Theorem 2.11. So, it yields that either condition (i) N has a fixed-point in $\overline{\mathcal{U}}$ or (ii) there exists a $x \in \partial\mathcal{U}$ and $\mu \in (0,1)$ with $x \in \mu N(x)$. We show that conclusion (ii) is not possible. If $x \in \mu N(x)$ for $\mu \in (0,1)$, then there is $v \in L^1(J, \mathbb{R})$

with $v \in S_{F,x}$ such that, for $t \in J$, we have

$$x(t) = \frac{\mu}{\Gamma(\alpha)} \int_1^t \left(\log \frac{t}{s}\right)^{\alpha-1} v(s) \frac{ds}{s} + \frac{\mu (\log(t))^{\alpha-1}}{\Lambda} \left(\sum_{i=1}^k \frac{\theta_i}{\Gamma(\alpha+\lambda_i)} \left[\int_1^{\eta_i} \left(\log \frac{\eta_i}{s}\right)^{\alpha+\lambda_i-1} v(s) \frac{ds}{s} - \int_1^{\rho_i} \left(\log \frac{\rho_i}{s}\right)^{\alpha+\lambda_i-1} v(s) \frac{ds}{s} \right] - \frac{1}{\Gamma(\alpha-\beta)} \int_1^T \left(\log \frac{T}{s}\right)^{\alpha-\beta-1} v(s) \frac{ds}{s} \right).$$

Using the method of computation employed in Step 2, for each $t \in J$, we get

$$|x(t)| \leq \Lambda_2 \|P\| Q(\|x\|),$$

which can alternatively be written as

$$\frac{\|x\|}{\Lambda_2 \|P\| Q(\|x\|)} \leq 1. \tag{4.4}$$

Now. In view of (A3), there exists $M > 0$ such that

$$\frac{M}{\Lambda_2 \|P\| Q(M)} > 1. \tag{4.5}$$

Let us set

$$\mathcal{U} = \{x \in C(J, \mathbb{R}) : \|x\| < M\}.$$

Suppose that condition (ii) of Theorem (2.11) is hold, then there is $x \in \partial\mathcal{U}$ and $\mu \in (0,1)$ with $x \in \mu N(x)$. Then $\|x\| = M$, satisfies (4.4), which contradicts (4.5). So, the condition (ii) in Theorem (2.11) does not hold, and consequently N has a fixed point $x \in \overline{\mathcal{U}}$ which is a solution of the boundary value problem (1.2). This completes the proof. □

4.2 The Lipschitz case

Now we prove the existence of solutions for the boundary value problem (1.2) with nonconvex-valued right hand side by applying a fixed point theorem for multivalued map due to Covitz and Nadler [14].

Let (X, d) be a metric pace induced from the normed space $(X, \|.\|)$. Consider $H_d : \mathcal{P}(X) \times \mathcal{P}(X) \to \mathbb{R}^+ \cup \{\infty\}$ given by

$$H_d(A, B) = \max \left\{ \sup_{a \in A} d(a, B), \sup_{b \in B} d(A, b) \right\},$$

where $d(A, b) = \inf_{a \in A} d(a, b)$ and $d(a, B) = \inf_{b \in B} d(a, b)$. Then $(\mathcal{P}_{b,cl}(X), H_d)$ is a metric space (see [21]).

Definition 4.3. A multivalued operator $N : X \to \mathcal{P}_{cl}(X)$ is called

(a) γ-Lipschitz if and only if there exists $\gamma > 0$ such that

$$H_d(N(x), N(y)) \leq \gamma d(x, y) \text{ for each } x,\ y \in X,$$

(b) a contraction if and only if it is γ-Lipschitz with $\gamma < 1$.

Theorem 4.4. *Let (2.1) holds. Assume that the following conditions hold*

(A4) $\mathrm{F} : J \times \mathbb{R} \to \mathcal{P}_{cp}(\mathbb{R})$ *is such that* $\mathrm{F}(., x) : J \to \mathcal{P}_{cp}(\mathbb{R})$ *is measurable for each* $x \in \mathbb{R}$,

(A5) $H_d(\mathrm{F}(t, x), \mathrm{F}(t, \overline{x})) \leq m(t) |x - \overline{x}|$ *for almost all* $t \in J$ *and* $x, \overline{x} \in \mathbb{R}$ *with* $m \in C(J, \mathbb{R}^+)$ *and* $d(0, \mathrm{F}(t, 0)) \leq m(t)$ *for almost all* $t \in J$.

Then the boundary value problem (1.2) has at least one solution on J *if*

$$\Lambda_2 \|m\| < 1,$$

where Λ_2 *is given by (4.1).*

Proof. Observe that the set $S_{F,x}$ is nonempty for each $x \in C(J,\mathbb{R})$ by assumption (A4), so F has a measurable selection (see [13], Theorem III.6). Now we show that the operator $N : C(J,\mathbb{R}) \to \mathcal{P}(C(J,\mathbb{R}))$ defined in (4.3) satisfies the assumptions of Theorem 2.12. To show that $N(x)$ is closed for each $x \in C(J,\mathbb{R})$. Let $\{u_n\}_{n\geq 0} \in N(x)$ be such that $u_n \to u$ $(n \to \infty)$ in $C(J,\mathbb{R})$. Then $u \in C(J,\mathbb{R})$ and there exists $v_n \in S_{F,x_n}$ such that, for each $t \in J$,

$$\begin{aligned} u_n(t) &= \frac{1}{\Gamma(\alpha)} \int_1^t \left(\log \frac{t}{s}\right)^{\alpha-1} v_n(s) \frac{ds}{s} \\ &+ \frac{(\log t)^{\alpha-1}}{\Lambda} \left(\sum_{i=1}^{k} \frac{\theta_i}{\Gamma(\alpha+\lambda_i)} \left[\int_1^{\eta_i} \left(\log \frac{\eta_i}{s}\right)^{\alpha+\lambda_i-1} v_n(s) \frac{ds}{s} \right.\right. \\ &\left.\left. - \int_1^{\rho_i} \left(\log \frac{\rho_i}{s}\right)^{\alpha+\lambda_i-1} v_n(s) \frac{ds}{s} \right] - \frac{1}{\Gamma(\alpha-\beta)} \int_1^T \left(\log \frac{T}{s}\right)^{\alpha-\beta-1} v_n(s) \frac{ds}{s} \right). \end{aligned}$$

As F has compact values, we pass onto a subsequence to obtain that v_n converges to v in $L^1(J,\mathbb{R})$. Thus $v \in S_{F,x}$ and for each $t \in J$, we have

$$\begin{aligned} u_n(t) \to u(t) &= \frac{1}{\Gamma(\alpha)} \int_1^t \left(\log \frac{t}{s}\right)^{\alpha-1} v(s) \frac{ds}{s} \\ &+ \frac{(\log t)^{\alpha-1}}{\Lambda} \left(\sum_{i=1}^{k} \frac{\theta_i}{\Gamma(\alpha+\lambda_i)} \left[\int_1^{\eta_i} \left(\log \frac{\eta_i}{s}\right)^{\alpha+\lambda_i-1} v(s) \frac{ds}{s} \right.\right. \\ &\left.\left. - \int_1^{\rho_i} \left(\log \frac{\rho_i}{s}\right)^{\alpha+\lambda_i-1} v(s) \frac{ds}{s} \right] - \frac{1}{\Gamma(\alpha-\beta)} \int_1^T \left(\log \frac{T}{s}\right)^{\alpha-\beta-1} v(s) \frac{ds}{s} \right). \end{aligned}$$

Hence $u \in N(x)$.

Next we show that there exists $0 < \tau < 1$, $(\tau = \Lambda_2 \|m\|)$ such that

$$H_d(N(x), N(\overline{x})) \leq \tau \|x - \overline{x}\| \text{ for each } x, \overline{x} \in C(J,\mathbb{R}).$$

Let $x, \overline{x} \in C(J,\mathbb{R})$ and $h_1 \in N(x)$. Then there exists $v_1(t) \in F(t, x(t))$ such that, for each $t \in J$,

$$\begin{aligned} h_1(t) &= \frac{1}{\Gamma(\alpha)} \int_1^t \left(\log \frac{t}{s}\right)^{\alpha-1} v_1(s) \frac{ds}{s} \\ &+ \frac{(\log t)^{\alpha-1}}{\Lambda} \left(\sum_{i=1}^{k} \frac{\theta_i}{\Gamma(\alpha+\lambda_i)} \left[\int_1^{\eta_i} \left(\log \frac{\eta_i}{s}\right)^{\alpha+\lambda_i-1} v_1(s) \frac{ds}{s} \right.\right. \\ &\left.\left. - \int_1^{\rho_i} \left(\log \frac{\rho_i}{s}\right)^{\alpha+\lambda_i-1} v_1(s) \frac{ds}{s} \right] - \frac{1}{\Gamma(\alpha-\beta)} \int_1^T \left(\log \frac{T}{s}\right)^{\alpha-\beta-1} v_1(s) \frac{ds}{s} \right). \end{aligned}$$

By (A5), we have

$$H_d(F(t,x), F(t,\overline{x})) \leq m(t) |x(t) - \overline{x}(t)|.$$

Therefore, there exists $w \in F(t, \overline{x}(t))$ such that

$$|v_1(t) - w| \leq m(t) |x(t) - \overline{x}(t)|, \; t \in J.$$

Define $\mathcal{U} : J \to \mathcal{P}(\mathbb{R})$ by

$$\mathcal{U}(t) = \{w \in \mathbb{R} : |v_1(t) - w| \leq m(t) |x(t) - \overline{x}(t)|\}.$$

Since the multivalued operator $\mathcal{U}(t) \cap F(t, \overline{x}(t))$ is measurable (see [13], Proposition III.4), there exists a function v_2 which is a measurable selection for $\mathcal{U}$. So $v_2(t) \in F(t, \overline{x}(t))$ and or each $t \in J$, we have $|v_1(t) - v_2(t)| \leq m(t) |x(t) - \overline{x}(t)|$.

For each $t \in J$, let us define

$$h_2(t) = \frac{1}{\Gamma(\alpha)} \int_1^t \left(\log \frac{t}{s} \right)^{\alpha-1} v_2(s) \frac{ds}{s}$$
$$+ \frac{(\log t)^{\alpha-1}}{\Lambda} \left(\sum_{i=1}^{k} \frac{\theta_i}{\Gamma(\alpha+\lambda_i)} \left[\int_1^{\eta_i} \left(\log \frac{\eta_i}{s} \right)^{\alpha+\lambda_i-1} v_2(s) \frac{ds}{s} \right. \right.$$
$$\left. \left. - \int_1^{\rho_i} \left(\log \frac{\rho_i}{s} \right)^{\alpha+\lambda_i-1} v_2(s) \frac{ds}{s} \right] - \frac{1}{\Gamma(\alpha-\beta)} \int_1^T \left(\log \frac{T}{s} \right)^{\alpha-\beta-1} v_2(s) \frac{ds}{s} \right).$$

In consequence, we get

$$|h_1(t) - h_2(t)|$$
$$\leq \frac{1}{\Gamma(\alpha)} \int_1^t \left(\log \frac{t}{s} \right)^{\alpha-1} |v_1(s) - v_2(s)| \frac{ds}{s}$$
$$+ \frac{(\log t)^{\alpha-1}}{|\Lambda|} \left(\sum_{i=1}^{k} \frac{|\theta_i|}{\Gamma(\alpha+\lambda_i)} \left[\int_1^{\eta_i} \left(\log \frac{\eta_i}{s} \right)^{\alpha+\lambda_i-1} |v_1(s) - v_2(s)| \frac{ds}{s} \right. \right.$$
$$\left. + \int_1^{\rho_i} \left(\log \frac{\rho_i}{s} \right)^{\alpha+\lambda_i-1} |v_1(s) - v_2(s)| \frac{ds}{s} \right]$$
$$\left. + \frac{1}{\Gamma(\alpha-\beta)} \int_1^T \left(\log \frac{T}{s} \right)^{\alpha-\beta-1} |v_1(s) - v_2(s)| \frac{ds}{s} \right)$$
$$\leq \left(\frac{\|m\| (\log T)^{\alpha}}{\Gamma(\alpha+1)} + \frac{\|m\| (\log T)^{\alpha-1}}{|\Lambda|} \left(2 \sum_{i=1}^{k} \frac{|\theta_i| (\log T)^{\alpha+\lambda_i}}{\Gamma(\alpha+\lambda_i+1)} + \frac{(\log T)^{\alpha-\beta}}{\Gamma(\alpha-\beta+1)} \right) \right) \|x - \overline{x}\|,$$

thus

$$|h_1(t) - h_2(t)|$$
$$\leq \left(\frac{(\log T)^{\alpha}}{\Gamma(\alpha+1)} + \frac{(\log T)^{\alpha-1}}{|\Lambda|} \left(2 \sum_{i=1}^{k} \frac{|\theta_i| (\log T)^{\alpha+\lambda_i}}{\Gamma(\alpha+\lambda_i+1)} + \frac{(\log T)^{\alpha-\beta}}{\Gamma(\alpha-\beta+1)} \right) \right) \|m\| \|x - \overline{x}\|.$$

Hence

$$\|h_1 - h_2\| \leq \Lambda_2 \|m\| \|x - \overline{x}\|.$$

Analogously, interchanging the roles of x and $\overline{x}$, we obtain

$$H_d(N(x), N(\overline{x})) \leq \Lambda_2 \|m\| \|x - \overline{x}\|.$$

Since N is a contraction, it follows by Theorem 2.12 that N has a fixed point x which is a solution of (1.2). This completes the proof. □

5 Example

As an application of our results, we consider the following boundary value problems of a fractional differential equation or inclusion

$$\begin{cases} {}^H\mathfrak{D}^{\frac{7}{2}} x(t) = f(t, x(t)) \text{ or } \in F(t, x(t)), \ t \in [1, e], \\ x(1) = 0, \ x'(1) = 0, \ x''(1) = 0, \ x'''(1) = 0, \\ {}^H\mathfrak{D}^{\frac{5}{2}} x(e) = \sum_{i=1}^{2} \theta_i \left[{}^H\mathfrak{J}^{\lambda_i} x(\eta_i) - {}^H\mathfrak{J}^{\lambda_i} x(\rho_i) \right]. \end{cases} \tag{5.1}$$

Here $\alpha = \frac{7}{2}$, $\beta = \frac{5}{2}$, $k = 2$, $\lambda_1 = \frac{1}{2}$, $\lambda_2 = \frac{1}{4}$, $\theta_1 = \frac{1}{6}$, $\theta_2 = \frac{1}{8}$, $\eta_1 = \frac{3}{2}$, $\rho_1 = \frac{4}{3}$, $\eta_2 = \frac{5}{2}$, $\rho_2 = 2$, $n = 4$ and $T = e$. With these data we find $\Delta = 3.3216 \neq 0$.

5.1 Single-valued case

(i) Let

$$f(t,x)=\frac{\sin(t)}{\exp(t^2-1)+5}\left(\frac{|x|}{|x|+1}\right). \tag{5.2}$$

For $x,y\in\mathbb{R}$, we have

$$\begin{aligned}|f(t,x)-f(t,y)| &= \left|\frac{\sin(t)}{\exp(t^2-1)+5}\left(\frac{|x|}{|x|+1}-\frac{|y|}{|y|+1}\right)\right| \\ &\le \frac{|x-y|}{(\exp(t^2-1)+5)(1+|x|)(1+|y|)} \\ &\le \frac{1}{6}|x-y|,\end{aligned}$$

thus, the assumption (H1) is satisfied with $L_f=\frac{1}{6}$ and

$$\begin{aligned}\kappa &= \frac{L_f(\log T)^{\alpha}}{\Gamma(\alpha+1)}+\frac{L_f(\log T)^{\alpha-1}}{|\Lambda|}\left(2\sum_{i=1}^{k}\frac{|\theta_i|(\log T)^{\alpha+\lambda_i}}{\Gamma(\alpha+\lambda_i+1)}+\frac{(\log T)^{\alpha-\beta}}{\Gamma(\alpha-\beta+1)}\right)\\ &\simeq 0.07<1.\end{aligned}$$

Hence, all conditions of Theorem 3.1 are satisfied. We deduce that the boundary value problem (5.1), with f given by (5.2), has a unique solution on $[1,e]$.

(ii) With the function f given by (5.2), we remark that

$$|f(t,x)|\le\frac{1}{(\exp(t^2-1)+5)}=\Psi(t).$$

Hence, by Theorem 3.2, the boundary value problem (5.1) has at least one solution on $[1,e]$.

5.2 Multivalued case

(i) Consider the multivalued map $F:[1,e]\times\mathbb{R}\to\mathcal{P}(\mathbb{R})$ given by

$$x\to F(t,x)=\left[\frac{1}{(\exp(t^2)+4)}\frac{x^2}{(x^2+1)},\frac{1}{2\sqrt{\log t+1)}}\frac{|x|}{|x|+1}\right].$$

Clearly the multivalued map F satisfies condition (A1) and that

$$\|F(t,x)\|_{\mathcal{P}}=\sup\{|y|:y\in F(t,x)\}\le\frac{1}{2\sqrt{\log t+1}}=P(t)\,Q(|x|),$$

which yields $\|P\|=\frac{1}{2}$ and $Q(|x|)=1$. Therefore, the condition (A2) is fulfilled. By the condition (A3), it found that $M>0.19788$. Hence all assumptions of Theorem 4.2 hold. So there exists at least one solution of the problem (5.1) on $[1,e]$.

(ii) Let the multivalued map $F:[1,e]\times\mathbb{R}\to\mathcal{P}(\mathbb{R})$ defined by

$$x\to F(t,x)=\left[0,\frac{2\sin(x)}{(\log t)+6}+\frac{1}{10}\right].$$

Clearly $H_d(F(t,x),F(t,\overline{x}))\le m(t)|x-\overline{x}|$, where $m(t)=\frac{2}{(\log t)+6}$. Also $d(0,F(t,0))=\frac{1}{10}\le m(t)$ for almost all $t\in[1,e]$. In addition, we get $\|m\|=\frac{1}{3}$ which leads to $\Lambda_2\|m\|\approx 0.13<1$. As the hypotheses of Theorem 4.4 are satisfied, therefore we conclude that the multivalued problem (5.1) has at least one solution on $[1,e]$.

Conclusion

In this article, we have studied the existence and uniqueness of solutions for higher order fractional differential equations and inclusions with multi-strip Hadamard type fractional integral boundary conditions. We applied the fixed point theorems for single-valued and multivalued maps to obtain the desired results for the given problems. The obtained outcomes are well explained through with relevant illustrative example.

In the future studies, we will try to extend the problems presented in this article to a general structure with the Mittag-Leffler power law [6] and for ψ-Hilfer fractional operator [30].

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Received: 2022-09-15
Accepted: 2023-04-13