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LIE BRACKET INVERTIBILITY

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Abstract In this paper, we investigate the resolution of the equation $[X, Y] = Z$ in the Lie algebra of vector fields on a manifold M, for X and Z both given. We give a local solution on a complex manifold M when X is a diagonal $\lambda-$resonant vector field, when X is a monomial $\lambda-$resonant vector field, and when X is a $\lambda-$resonant vector field.

1 Introduction

The problem we are interested in is the following : if X and Z are given vector fields on a manifold M, is it possible to find a vector field Y on M such that the Lie bracket $[X, Y] = Z$?

If X is a C^∞ vector field on a differential manifold M and x_0 a point of M with $X(x_0) \neq 0$, then there is a coordinate system $(x^1, ..., x^N)$ of M (where $N = \dim M$) on an open neighborhood U of x_0 such that $X = \frac{\partial}{\partial x^1}$ on U ([9] p.205). So if Z is a C^∞ vector field on M, there exists on U a vector field $Y = \sum\limits_{k=1}^{N} Y^k \frac{\partial}{\partial x^k}$ defined by (using locally the same notation),

$$Y^k(x_1, ..., x_N) = \int_{\alpha}^{x_1} Z^k(t, x_2, ..., x_N)dt$$

$\forall k = 1, ..., N$, where $Z = \sum\limits_{k=1}^{N} Z^k \frac{\partial}{\partial x^k}$ on U, such that $[X, Y] = Z$.

Therefore if x_0 is a regular point of X, then equation $[X, Y] = Z$ has a solution on a neighborhood of x_0.

Let (ϕ_t) be the flow generated by X on a neighborhood of a singular point and $(\phi_t)_* Y$ the transportation of Y along the flow (ϕ_t). Then locally

$$[X, Y] = \lim_{t \to 0} \frac{1}{t}(Y - (\phi_t)_* Y).$$

If we set $\gamma(t) = -(\phi_t)_* Y$, then

$$\frac{d}{dt}_{|t=0}\gamma(t) = \gamma'(0) = \lim_{t \to 0} \frac{\gamma(t) - \gamma(0)}{t} = [X, Y].$$

So we are looking for a vector field Y whose transportation $(\phi_t)_* Y$ along the flow (ϕ_t) generated by X fulfills

$$\frac{d}{dt}_{|t=0}(-(\phi_t)_* Y) = Z.$$

Among the works done to solve this equation, we have [11], [2].

In this note, we give a local solution of this equation on a complex manifold M when X is a diagonal λ-resonant vector field, when X is a monomial λ-resonant vector field and finally when X is a λ-resonant vector field.

Let M be a complex manifold of complex dimension N. A holomorphic vector field on M is a section $X : M \to (TM)^{1,0}$ of the holomorphic tangent bundle over M such that for any point $p \in M$, if $(z^1, ..., z^N)$ is a local holomorphic coordinate system of M on an open neighborhood U of p,

$$X_p = \sum_{k=1}^{N} X^k(p) \frac{\partial}{\partial z^k_{|p}}$$

where $X^1, ..., X^N : U \to \mathbb{C}$ are holomorphic functions.

The following definitions are in [1], [6], [8].

A vector $\lambda = (\lambda_1, ..., \lambda_N) \in \mathbb{C}^N$ such that $\lambda_k \neq 0$, $\forall k = 1, ..., N$ is said to be *resonant* if there exits $s \in \{1, ..., N\}$ and $p = (p_1, ..., p_N) \in \mathbb{N}^N \setminus \{0\}$ satisfying the relation

$$\lambda_s = \langle p, \lambda \rangle = \sum_{k=1}^{N} p_k \lambda_k$$

with $\sum_{k=1}^{N} p_k \geq 2$.

This relation is called an *additive resonance relation* of order $|p| = \sum_{k=1}^{N} p_k$.

Remark 1.1. For example the relation $\lambda_1 + \lambda_2 + 2\lambda_3 = 0$ gives $\lambda_2 = \lambda_1 + 2\lambda_2 + 2\lambda_3$ so is an additive resonance relation of order 5.
But the relation $2\lambda_1 + 7\lambda_2 = 5\lambda_3$ is not an additive resonance relation.

Let $\lambda = (\lambda_1, ..., \lambda_N) \in \mathbb{C}^N$ such that $\lambda_k \neq 0$, $\forall k = 1, ..., N$.

An *additive monomial $\lambda-$resonant vector field* on $\mathbb{C}^N$ is a holomorphic vector field of the form

$$a.z_1^{p_1} ... z_N^{p_N} \frac{\partial}{\partial z_s}$$

where $z_1, ..., z_N$ are coordinates in $\mathbb{C}^N$, $a \in \mathbb{C}$ and $p = (p_1, ..., p_N) \in \mathbb{N}^N \setminus \{0\}$ satisfying the *additive resonance relation*

$$\lambda_s = \langle p, \lambda \rangle = \sum_{k=1}^{N} p_k \lambda_k.$$

Notice that a multiplicative monomial $\lambda-$resonant vector field may be defined, but we will only deal with the additive case.

A monomial $\lambda-$resonant vector field is said *diagonal* if it is of the form

$$a z_s \frac{\partial}{\partial z_s}$$

i.e. is associated to the trivial resonance relation $\lambda_s = \langle p, \lambda \rangle$ where $p = (p_1, ..., p_N)$ with $p_k = \delta_{ks}$ (Kronecker symbol), $\forall k = 1, ..., N$.

A *$\lambda-$resonant vector field* on $\mathbb{C}^N$ is a sum of monomial $\lambda-$resonant vector fields.

For instance the *diagonal vector field*

$$X_0 = \sum_{k=1}^{N} \lambda_k z_k \frac{\partial}{\partial z_k}$$

is $\lambda-$resonant.

2 Case of a diagonal $\lambda-$resonant vector field

Definition 2.1. [1] A vector $\lambda=(\lambda_1,...,\lambda_N)\in\mathbb{C}^N$ is said to be **of type** (A,δ) if there exists real constants $A>0$ and $\delta\geq 0$ such that for any $j=1,...,N$,

$$|\lambda_j-\langle m,\lambda\rangle|\geq\frac{A}{|m|^\delta}$$

for all $m=(m_1,...,m_N)\in\mathbb{N}^N$ with $|m|\geq 2$ where $|m|=\sum_{k=1}^{N}m_k$.

We then get the following result :

Theorem 2.2. *Let M be a complex manifold of complex dimension N. Let X be a holomorphic vector field on M.*

Suppose there is a point $x\in M$, a chart (U,ϕ) of M at x such that $X_{|U}$ is biholomorphically conjugated by ϕ, to a diagonal $\lambda-$ resonant vector field

$$X_U=\sum_{k=1}^{N}\lambda_k z_k\frac{\partial}{\partial z_k}$$

with $\lambda_k\neq 0$ for all $k=1,...,N$, and $\lambda=(\lambda_1,...,\lambda_N)\in\mathbb{C}^N$ is of type (A,δ).

Then for any holomorphic vector field Z on M such that $\phi_*(Z_{|U})$ is without linear part, there exists a holomorphic vector field Y on U such that $[X_{|U},Y]=Z_{|U}$.

Proof. Using the conjugation the equation $[X_{|U},Y]=Z_{|U}$ is equivalent to $[X_U,\phi_*(Y)]=\phi_*(Z_{|U})$.

Set $\phi_*(Y)=\sum_{j=1}^{N}Y^j\frac{\partial}{\partial z_j}$ and $\phi_*(Z_{|U})=\sum_{j=1}^{N}Z^j\frac{\partial}{\partial z_j}$. Then

$$\begin{aligned}
[X_U,\phi_*(Y)] &= \sum_{k=1}^{N}\lambda_k[z_k\tfrac{\partial}{\partial z_k},\phi_*(Y_{|U})]\\
&= \sum_{k=1}^{N}\lambda_k z_k[\tfrac{\partial}{\partial z_k},\phi_*(Y_{|U})]-\sum_{k=1}^{N}\lambda_k(\phi_*(Y_{|U}).z_k)\tfrac{\partial}{\partial z_k}\\
&= \sum_{k=1}^{N}\lambda_k z_k\sum_{j=1}^{N}[\tfrac{\partial}{\partial z_k},Y^j\tfrac{\partial}{\partial z_j}]-\sum_{k=1}^{N}\lambda_k Y^k\tfrac{\partial}{\partial z_k}\\
&= \sum_{k=1}^{N}\lambda_k z_k\sum_{j=1}^{N}\tfrac{\partial Y^j}{\partial z_k}\tfrac{\partial}{\partial z_j}-\sum_{k=1}^{N}\lambda_k Y^k\tfrac{\partial}{\partial z_k}.
\end{aligned}$$

So

$$[X_U,\phi_*(Y)]=\sum_{j=1}^{N}(\sum_{k=1}^{N}\lambda_k z_k\frac{\partial Y^j}{\partial z_k}-\lambda_j Y^j)\frac{\partial}{\partial z_j}.$$

Then equation $[X_U,\phi_*(Y)]=\phi_*(Z_{|U})$ is equivalent to the system

$$\sum_{k=1}^{N}\lambda_k z_k\frac{\partial Y^j}{\partial z_k}-\lambda_j Y^j=Z^j,\ \forall j=1,...,N$$

and therefore to the system $L_{X_U}(Y^j)-\lambda_j Y^j=Z^j$, $\forall j=1,...,N$, where L_{X_U} is the Lie derivative along the vector field X_U.

Let's use the power series expansion of Y^j and Z^j on some neighbourhood of 0 :

$$Y^j=\sum_{|m|=0}^{+\infty}a_m^j z^m \text{ and } Z^j=\sum_{|m|=0}^{+\infty}b_m^j z^m$$

with $m=(m_1,...,m_N)\in\mathbb{N}^N$, $z^m=z_1^{m_1}...z_N^{m_N}$, and a_m^j , $b_m^j\in\mathbb{C}$.

Then

$$L_{X_U}(Y^j)=\sum_{|m|=0}^{+\infty}(\sum_{k=1}^{N}m_k\lambda_k)a_m^jz^m=\sum_{|m|=0}^{+\infty}\langle m,\lambda\rangle\, a_m^jz^m.$$

Therefore equation $L_{X_U}(Y^j)-\lambda_jY^j=Z^j$ is equivalent to

$$(\lambda_j-\langle m,\lambda\rangle)a_m^j=-b_m^j.$$

If $m=0$, then

$$a_0^j=\frac{-b_0^j}{\lambda_j}.$$

So

$$\left|a_0^j\right|\leq\frac{\left|b_0^j\right|}{\min\limits_{1\leq k\leq N}|\lambda_k|}.$$

If $|m|=1$, then $b_m^j=0$ so we can take $a_m^j=0$.

If $\lambda=(\lambda_1,...,\lambda_N)$ is of type (A,δ), then for all $j=1,...,N$,

$$|\lambda_j-\langle m,\lambda\rangle|\geq\frac{A}{|m|^\delta}$$

for any $m\in\mathbb{N}^N$ with $|m|\geq 2$, where $A>0$ and $\delta\geq 0$. Then

$$\left|a_m^j\right|=\frac{\left|b_m^j\right|}{|\lambda_j-\langle m,\lambda\rangle|}\leq\frac{|m|^\delta\left|b_m^j\right|}{A}.$$

Since the function Z^j is holomorphic, the power series $\sum\limits_{|m|=0}^{+\infty}b_m^jz^m$ is absolutely convergent on an open polydisk with center 0. So the power series $\sum\limits_{|m|=0}^{+\infty}a_m^jz^m$ is absolutely convergent on this open polydisk where it defines a unique holomorphic function Y^j.

Finally we can conclude that there exists a unique holomorphic vector field $Y^U=\sum\limits_{j=1}^{N}Y^j\frac{\partial}{\partial z_j}$ on $\phi(U)$ such that $[X_U,Y^U]=\phi_*(Z_{|U})$. So there exists a unique holomorphic vector field Y on U such that $[X_{|U},Y]=Z_{|U}$. □

The following theorem gives a sufficient condition for a holomorphic vector field to be conjugated to a diagonal vector field.

Theorem 2.3. *(Siegel's theorem* [1] *p.187) : Let* $X=\sum a_m^sz^m\frac{\partial}{\partial z_s}$ *be a holomorphic vector field on a neighbourhood of* 0 *in* $\mathbb{C}^N$*, an isolated singular point.*

Let $\lambda_1,...,\lambda_N$ *be the eighenvalues of the matrix* (a_i^s) *of the linear part of* X*.*

If $\lambda=(\lambda_1,...,\lambda_N)\in\mathbb{C}^N$ *is of type* (A,δ)*, then there exists a biholomorphism* h *on a neighborhood of* 0 *in* $\mathbb{C}^N$ *such that*

$$h_*X=\sum_{k=1}^{N}\lambda_kz_k\frac{\partial}{\partial z_k}$$

i.e. X *is biholomorphically conjugated to the diagonal* $\lambda-$ *resonant vector field*

$$\sum_{k=1}^{N}\lambda_kz_k\frac{\partial}{\partial z_k}.$$

Using Siegel's theorem, we get on a complex manifold :

Proposition 2.4. *Let M be a complex manifold of complex dimension N. Let X be a holomorphic vector field on M.*

Suppose there is a point $x \in M$, a chart (U, ϕ) of M centered at x such that $X_{|U}$ is biholomorphically conjugated by ϕ, to a holomorphic vector field

$$X_U = \sum a_m^s z^m \frac{\partial}{\partial z_s}$$

on $\phi(U)$ a neighborhood of $0 \in \mathbb{C}^N$, an isolated singular point.

Let $\lambda_1, ..., \lambda_N$ be the eighenvalues of the matrix (a_i^s) of the linear part of X_U.

Suppose $\lambda_k \neq 0$ for all $k = 1, ..., N$, and $\lambda = (\lambda_1, ..., \lambda_N) \in \mathbb{C}^N$ is of type (A, δ).

Then for any holomorphic vector field Z on M such that $\phi_* \left(Z_{|U}\right)$ is without linear part, there exists a holomorphic vector field Y on U such that $\left[X_{|U}, Y\right] = Z_{|U}$.

Proof. By Siegel's theorem, if $\lambda = (\lambda_1, ..., \lambda_N) \in \mathbb{C}^N$ is of type (A, δ), then a biholomorphism h on a neighbourhood of 0 in $\mathbb{C}^N$ such that $h_* (X_U) = \sum\limits_{k=1}^{N} \lambda_k z_k \frac{\partial}{\partial z_k}$ exists.

Set $\psi = h \circ \phi$. Then $X_{|U}$ is biholomorphically conjugated by ψ to the diagonal $\lambda-$resonant vector field $X^U = \sum\limits_{k=1}^{N} \lambda_k z_k \frac{\partial}{\partial z_k}$ with $\lambda_k \neq 0$ for all $k = 1, ..., N$, and $\lambda = (\lambda_1, ..., \lambda_N) \in \mathbb{C}^N$ is of type (A, δ).

Therefore, since $\psi_* \left[X_{|U}, Y\right] = \left[\psi_* \left(X_{|U}\right), \psi_* (Y)\right]$, by theorem 2.2, for any holomorphic vector field Z on M with $\psi_* \left(Z_{|U}\right)$ without linear part, we can find a holomorphic vector field Y on U such that $\left[X_{|U}, Y\right] = Z_{|U}$. □

3 Case of a monomial $\lambda-$resonant vector field

On $\mathbb{C}^N$, we consider the monomial $\lambda-$resonant vector field

$$X = a.z_1^{p_1}...z_N^{p_N} \frac{\partial}{\partial z_s} = a z^p \frac{\partial}{\partial z_s}$$

where $a \in \mathbb{C}^*$, $p = (p_1, ..., p_N) \in \mathbb{N}^N \setminus \{0\}$ and $\lambda = (\lambda_1, ..., \lambda_N) \in \mathbb{C}^N$ with $\lambda_k \neq 0$, $\forall k = 1, ..., N$ fulfilling the additive resonance relation

$$\lambda_s = \langle p, \lambda \rangle = \sum_{k=1}^{N} p_k \lambda_k.$$

In [10] the following result has been established :

Theorem 3.1. *[10] Let $\lambda = (\lambda_1, ..., \lambda_N) \in \mathbb{C}^N$ such that $\lambda_k \neq 0, \forall k = 1, ..., N$. Let $X = a.z_1^{p_1}...z_N^{p_N} \frac{\partial}{\partial z_s}$ be a monomial $\lambda-$ resonant vector field on $\mathbb{C}^N$, where $a \in \mathbb{C}^*$, $p = (p_1, ..., p_N) \in \mathbb{N}^N \setminus \{0\}$ fulfilling the additive resonance relation*

$$\lambda_s = \langle p, \lambda \rangle = \sum_{k=1}^{N} p_k \lambda_k.$$

If g is a holomorphic function on $\mathbb{C}^N$ satisfying the condition

$$\frac{1}{k_1!...k_N!} \frac{\partial^{k_1+...+k_N} g}{\partial z_1^{k_1}...\partial z_N^{k_N}} (0, ..., 0) = 0$$

for any $(k_1, ..., k_N) \in \mathbb{N}^N$ with $k_i < p_i$ for some i, then there exists a holomorphic function f on $\mathbb{C}^N$, (unique up to the consequence of the necessary condition), such that $L_X f = g$.

From this we deduce :

Theorem 3.2. *Let M be a complex manifold of complex dimension N. Let X be a holomorphic vector field on M.*

Suppose there is a point $x \in M$, a chart (U, ϕ) of M at x such that $X_{|U}$ is biholomorphically conjugated by ϕ, to a monomial $\lambda-$ resonant vector field

$$X_U = a.z_1^{p_1}...z_N^{p_N}\frac{\partial}{\partial z_s} = az^p\frac{\partial}{\partial z_s}$$

where $a \in \mathbb{C}^*$, $p = (p_1, ..., p_N) \in \mathbb{N}^N$ and $\lambda = (\lambda_1, ..., \lambda_N) \in \mathbb{C}^N$ with $\lambda_k \neq 0$, $\forall k = 1, ..., N$ such that $\lambda_s = \langle p, \lambda \rangle$.

If Z is a holomorphic vector field on M such that $\phi_*(Z_{|U}) = Z_U = \sum\limits_{j=1}^{N} Z_U^j \frac{\partial}{\partial z_j}$ satisfies the condition : $\forall j = 1, ..., N$

$$\frac{1}{k_1!...k_N!}\frac{\partial^{k_1+...+k_N} Z_U^j}{\partial z_1^{k_1}...\partial z_N^{k_N}}(0, ..., 0) = 0$$

for any $(k_1, ..., k_N) \in \mathbb{N}^N$ with $k_i < p_i$ for some i, then, in the sub-Lie algebra of holomorphic vector fields Y on U such that $L_{\phi_*(Y)}(z^p) = 0$, there exists a holomorphic vector field Y such that $\left[X_{|U}, Y\right] = Z_{|U}$.

Proof. Using the conjugation the equation $\left[X_{|U}, Y\right] = Z_{|U}$ is equivalent to

$$[X_U, \phi_*(Y)] = \phi_*\left(Z_{|U}\right).$$

Set $\phi_*(Y) = \sum\limits_{j=1}^{N} Y^j \frac{\partial}{\partial z_j}$ and $\phi_*\left(Z_{|U}\right) = \sum\limits_{j=1}^{N} Z^j \frac{\partial}{\partial z_j}$.

Then

$$\begin{aligned}
[X_U, \phi_*(Y)] &= \left[az^p\frac{\partial}{\partial z_s}, \phi_*(Y)\right] \\
&= az^p\left[\frac{\partial}{\partial z_s}, \phi_*(Y)\right] - a\left(\left(\phi_*(Y)\right).z^p\right)\frac{\partial}{\partial z_s} \\
&= az^p\sum_{j=1}^{N}\left[\frac{\partial}{\partial z_s}, Y^j\frac{\partial}{\partial z_j}\right] - a\left(\left(\phi_*\left(Y_{|U_i}\right)\right).z^p\right)\frac{\partial}{\partial z_s} \\
&= \sum_{j=1}^{N} az^p\frac{\partial Y^j}{\partial z_s}\frac{\partial}{\partial z_j} - a\left(\left(\phi_*(Y)\right).z^p\right)\frac{\partial}{\partial z_s}
\end{aligned}$$

So equation $[X_U, \phi_*(Y)] = \phi_*\left(Z_{|U}\right)$ is equivalent to the system

$$\begin{cases} az^p\frac{\partial Y^j}{\partial z_s} = Z^j, \forall j \neq s \\ az^p\frac{\partial Y^s}{\partial z_s} - a\left(\left(\phi_{i*}\left(Y_{|U_i}\right)\right).z^p\right) = Z^s \end{cases}$$

i.e. to the system

$$(S4)\begin{cases} L_{X_{U_i}}\left(Y^j\right) = Z^j, \forall j \neq s \\ L_{X_{U_i}}\left(Y^s\right) - aL_{\phi_{i*}\left(Y_{|U_i}\right)}\left(z^p\right) = Z^s \end{cases}$$

If $L_{\phi_{i*}(Y)}\left(z^p\right) = 0$, then system $(S4)$ becomes the system of continuous cohomological equations $L_{X_U}\left(Y^j\right) = Z^j, \forall j = 1, ..., N$.

By theorem 3.1, there exists a holomorphic function Y^j on $\phi(U)$ such that $L_{X_U}\left(Y^j\right) = Z^j$.

Finally we can conclude that there exists a holomorphic vector field $Y^U = \sum\limits_{j=1}^{N} Y^j\frac{\partial}{\partial z_j}$ on $\phi(U)$ such that $\left[X_U, Y^U\right] = \phi_*\left(Z_{|U}\right)$. So there exists a holomorphic vector field Y on U such that $\left[X_{|U}, Y\right] = Z_{|U}$. □

More generally, we have :

Theorem 3.3. *Let M be a complex manifold of complex dimension N. Let X be a holomorphic vector field on M.*

Suppose there is a point $x \in M$, a chart (U, ϕ) of M at x such that $X_{|U}$ is biholomorphically conjugated by ϕ, to a monomial $\lambda-$resonant vector field

$$X_U = a.z_1^{p_1}...z_N^{p_N}\frac{\partial}{\partial z_s} = az^p\frac{\partial}{\partial z_s}$$

where $a \in \mathbb{C}^*$, $p = (p_1, ..., p_N) \in \mathbb{N}^N$ with $p_s \geq 1$ and $\lambda = (\lambda_1, ..., \lambda_N) \in \mathbb{C}^N$ with $\lambda_k \neq 0$, $\forall k = 1, ..., N$ such that $\lambda_s = \langle p, \lambda \rangle$.

If Z is a holomorphic vector field on M such that, $\phi_*(Z_{|U}) = Z_U = \sum_{j=1}^{N} Z_U^j \frac{\partial}{\partial z_j}$ satisfies the condition : $\forall j = 1, ..., N$

$$\frac{1}{k_1!...k_N!}\frac{\partial^{k_1+...+k_N} Z_U^j}{\partial z_1^{k_1}...\partial z_N^{k_N}}(0, ..., 0) = 0$$

for any $(k_1, ..., k_N) \in \mathbb{N}^N$ with $k_i < p_i$ for some i, and $k_s < p_s + 1$, then, there exists a holomorphic vector field Y such that $\left[X_{|U}, Y\right] = Z_{|U}$.

Proof. Using the conjugation the equation $\left[X_{|U}, Y\right] = Z_{|U}$ is equivalent to

$$[X_U, \phi_*(Y)] = \phi_*\left(Z_{|U}\right).$$

As in the proof of theorem 3.2, if we set
$\phi_*(Y) = \sum_{j=1}^{N} Y^j \frac{\partial}{\partial z_j}$ and $\phi_*\left(Z_{|U}\right) = \sum_{j=1}^{N} Z^j \frac{\partial}{\partial z_j}$, then we get the system

$$(S4)\quad \left\{\begin{array}{lcl} L_{X_{U_i}}(Y^j) & = & Z^j, \forall j \neq s \\ L_{X_{U_i}}(Y^s) - aL_{\phi_{i*}\left(Y_{|U_i}\right)}(z^p) & = & Z^s \end{array}\right.$$

By theorem 3.1, for any $j \neq s$ we can find a holomorphic function Y^j on $\phi(U)$ such that $L_{X_U}(Y^j) = Z^j$.

If $j = s$, we have the equation $L_{X_U}(Y^s) - aL_{\phi_{i*}(Y)}(z^p) = Z^s$. Wich is equivalent to

$$az^p\frac{\partial Y^s}{\partial z_s} - a\sum_{j=1}^{N} Y^j\frac{\partial z^p}{\partial z_j} = Z^s$$

So

$$(S5)\quad z^p\frac{\partial Y^s}{\partial z_s} - \frac{\partial z^p}{\partial z_s}Y^s = \sum_{j=1, j\neq s}^{N} Y^j\frac{\partial z^p}{\partial z_j} + \frac{Z^s}{a}$$

As $j \neq s$, if we set Γ^s the right side of this equation, it is a holomophic function already known.

Let's use the power series expansion of Y^s and Γ^s on some neighbourhood of 0 :

$$Y^s = \sum_{|m|=0}^{+\infty} a_m^s z^m \text{ and } \Gamma^s = \sum_{|m|=0}^{+\infty} \gamma_m^s z^m$$

with $m = (m_1, ..., m_N) \in \mathbb{N}^N$, $z^m = z_1^{m_1}...z_N^{m_N}$, and a_m^s , $\gamma_m^s \in \mathbb{C}$.

Then the product of equation $(S5)$ by z_s gives :

$$z^p\sum_{m_s\geq 1, |m|\geq 1} m_s a_m^s z^m - p_s z^p\sum_{|m|=0}^{+\infty} a_m^s z^m = z_s\sum_{|m|=0}^{+\infty}\gamma_m^s z^m$$

So

$$\sum_{m_s\geq 1,|m|\geq 1} m_s a_m^s z^{m+p} - p_s \sum_{|m|=0}^{+\infty} a_m^s z^{m+p} = \sum_{|m|=0}^{+\infty} \gamma_m^s z_1^{m_1}...z_{s-1}^{m_{s-1}} z_s^{m_s+1} z_{s+1}^{m_{s+1}}...z_N^{m_N}$$

Therefore

$$\begin{aligned}\sum_{m_s\geq p_s+1, m_i\geq p_i, i\neq s} (m_s - p_s)\, a_{m-p}^s z^m - p_s \sum_{m_i\geq p_i}^{+\infty} a_{m-p}^s z^m \\ = \sum_{m_s\geq 1, m_i\geq 0, i\neq s}^{+\infty} \gamma_{m_s-1,m_i}^s z^m\end{aligned}$$

So we have the necessary condition $\gamma_{m_s-1,m_i}^s = 0$ if $m_s < p_s + 1$ and $m_i < p_i$, for $i \neq s$. When $m_s \geq p_s + 1$ and $m_i \geq p_i, i \neq s$

$$(m_s - 2p_s)\, a_{m-p}^s = \gamma_{m_s-1,m_i}^s.$$

Which gives for $m_s \geq 1$ and $m_i \geq 0, i \neq s$

$$(m_s - p_s)\, a_m^s = \gamma_{m_s+p_s+1,m_i+p_i}^s.$$

Therefore, if $m_s > p_s$ then

$$a_m^s = \frac{1}{m_s - p_s}\gamma_{m_s+p_s+1,m_i+p_i}^s.$$

So

$$|a_m^s| \leq \left|\gamma_{m_s+p_s+1,m_i+p_i}^s\right|.$$

Since the function Γ^s is holomorphic, the power series $\sum\limits_{|m|=0}^{+\infty} \gamma_m^s z^m$ is absolutely convergent on an open polydisk with center 0. So the power series $\sum\limits_{|m|=0}^{+\infty} a_m^s z^m$ is absolutely convergent on this open polydisk where it defines a unique holomorphic function Y^s.

Therefore there exists a holomorphic function Y^s on $\phi(U)$ solution of the equation $L_{X_U}(Y^s) - aL_{\phi_*(Y_{|U})}(z^p) = Z^s$.

Finally we can conclude that there exists a holomorphic vector field $Y^U = \sum\limits_{j=1}^{N} Y^j \frac{\partial}{\partial z_j}$ on $\phi(U)$ such that $\left[X_U, Y^U\right] = \phi_*\left(Z_{|U}\right)$. So there exists a holomorphic vector field Y on U such that $\left[X_{|U}, Y\right] = Z_{|U}$. □

4 Case of a $\lambda-$resonant vector field

Definition 4.1. Let $\lambda = (\lambda_1, ..., \lambda_N) \in \mathbb{C}^N$ such that $\lambda_k \neq 0, \forall k = 1, ..., N$. A $\lambda-$resonant vector field on $\mathbb{C}^N$ is a vector field of the form

$$X = \sum a_p^s.z_1^{p_1}...z_N^{p_N}\frac{\partial}{\partial z_s} = \sum a_p^s.z^p\frac{\partial}{\partial z_s}$$

where $a_p^s \in \mathbb{C}$ and $\lambda_s = \langle p, \lambda\rangle$ with $p = (p_1, ..., p_N) \in \mathbb{N}^N$. The sum runs over the sequences (s, p) such that $\lambda_s = \langle p, \lambda\rangle$.

For a $\lambda-$resonant vector field we get :

Theorem 4.2. *Let M be a complex manifold of complex dimension N. Let X be a holomorphic vector field on M.*

Suppose there is a centered chart (U,ϕ) of M such that $X_{|U}$ is biholomorphically conjugated by ϕ, to a $\lambda-$ resonant vector field

$$X_U = \sum a_p^s.z_1^{p_1}...z_N^{p_N}\frac{\partial}{\partial z_s} = \sum a_p^s.z^p\frac{\partial}{\partial z_s}$$

where $p_i \geq 1$, $\forall i = 1,...N$ and $\left|m_j a_p^j - (m_s - p_s)a_p^s\right| \leq 1$ for $j \neq s$, $(m_1,...,m_N) \in \mathbb{N}^N$.

If Z is a holomorphic vector field on M such that $\phi_*(Z_{|U}) = Z_U = \sum\limits_{j=1}^{N} Z_U^j \frac{\partial}{\partial z_j}$ satisfies the condition : $\forall j = 1,...,N$

$$\frac{1}{k_1!...k_N!}\frac{\partial^{k_1+...+k_N} Z_U^j}{\partial z_1^{k_1}...\partial z_N^{k_N}}(0,...,0) = 0$$

for any $(k_1,...,k_N) \in \mathbb{N}^N$ with $k_i < p_i - 1$ for some i, then there exists a holomorphic vector field Y on U such that $\left[X_{|U}, Y\right] = Z_{|U}$.

Proof. Equation $\left[X_{|U}, Y\right] = Z_{|U}$ is equivalent to $[X_U, \phi_*(Y)] = \phi_*\left(Z_{|U}\right)$.

Set $\phi_*(Y) = \sum\limits_{j=1}^{N} Y^j \frac{\partial}{\partial z_j}$.

As $s \in \{1,...,N\}$, we can assume that $a_p^s = 0$ when there is no p such that $\lambda_s = \langle p, \lambda\rangle$.

Then

$$\begin{aligned}
[X_U, \phi_*(Y)] &= \sum_{s=1}^{N} a_p^s \left[z^p \frac{\partial}{\partial z_s}, Y\right] \\
&= \sum_{s=1}^{N} a_p^s z^p \left[\frac{\partial}{\partial z_s}, Y\right] - \sum_{s=1}^{N} a_p^s (Y.z^p)\frac{\partial}{\partial z_s} \\
&= \sum_{s=1}^{N} a_p^s z^p \sum_{j=1}^{N}\left[\frac{\partial}{\partial z_s}, Y^j\frac{\partial}{\partial z_j}\right] - \sum_{s=1}^{N} a_p^s (Y.z^p)\frac{\partial}{\partial z_s} \\
&= \sum_{s=1}^{N} a_p^s z^p \sum_{j=1}^{N}\frac{\partial Y^j}{\partial z_s}\frac{\partial}{\partial z_j} - \sum_{s=1}^{N} a_p^s (L_Y(z^p))\frac{\partial}{\partial z_s} \\
&= \sum_{s=1}^{N}\left(\sum_{j=1}^{N} a_p^j z^p \frac{\partial Y^s}{\partial z_j}\right)\frac{\partial}{\partial z_s} - \sum_{s=1}^{N} a_p^s (L_Y(z^p))\frac{\partial}{\partial z_s}
\end{aligned}$$

Set $\phi_*\left(Z_{|U}\right) = \sum\limits_{s=1}^{N} Z^s \frac{\partial}{\partial z_s}$. Then equation $[X_U, \phi_*(Y)] = \phi_*\left(Z_{|U}\right)$ is equivalent to

$$\sum_{j=1}^{N} a_p^j z^p \frac{\partial Y^s}{\partial z_j} - a_p^s \sum_{j=1}^{N} Y^j \frac{\partial(z^p)}{\partial z_j} = Z^s$$

Let's use the power series expansion of Y^s and Z^s on some neighbourhood of 0 :

$$Y^s = \sum_{|m|=0}^{+\infty} b_m^s z^m \text{ and } Z^s = \sum_{|m|=0}^{+\infty} c_m^s z^m$$

with $m = (m_1,...,m_N) \in \mathbb{N}^N$, $z^m = z_1^{m_1}...z_N^{m_N}$, and b_m^s , $c_m^s \in \mathbb{C}$.

Then

$$\begin{aligned}
&\sum_{j=1}^{N} a_p^j z^p \sum_{|m|=0}^{+\infty} m_j b_m^s z_1^{m_1}...z_{j-1}^{m_{j-1}} z_j^{m_j-1} z_{j+1}^{m_{j+1}}...z_N^{m_N} \\
&-a_p^s \sum_{j=1}^{N} p_j z_1^{p_1}...z_{j-1}^{p_{j-1}} z_j^{p_j-1} z_{j+1}^{p_{j+1}}...z_N^{p_N} \sum_{|m|=0}^{+\infty} b_m^j z^m \\
&= \sum_{|m|=0}^{+\infty} c_m^s z^m
\end{aligned}$$

So

$$\sum_{|m|=0}^{+\infty} z^m \sum_{j=1}^{N} (m_j a_p^j b_m^s - p_j a_p^s b_m^j) z_1^{p_1} ... z_{j-1}^{p_{j-1}} z_j^{p_j - 1} z_{j+1}^{p_{j+1}} ... z_N^{p_N} = \sum_{|m|=0}^{+\infty} c_m^s z^m$$

Therefore

$$\sum_{|m|=0}^{+\infty} \sum_{j=1}^{N} (m_j a_p^j b_m^s - p_j a_p^s b_m^j) z_1^{p_1+m_1} ... z_{j-1}^{p_{j-1}+m_{j-1}} z_j^{p_j+m_j-1} z_{j+1}^{p_{j+1}+m_{j+1}} ... z_N^{p_N+m_N}$$
$$= \sum_{|k|=0}^{+\infty} c_k^s z^k$$

If $k \neq (p_1 + m_1, ..., p_{j-1} + m_{j-1}, p_j + m_j - 1, p_{j+1} + m_{j+1}, ..., p_N + m_N)$, then $c_k^s = 0$.

If $k = (p_1 + m_1, ..., p_{j-1} + m_{j-1}, p_j + m_j - 1, p_{j+1} + m_{j+1}, ..., p_N + m_N)$, then

$$m_j a_p^j b_m^s - p_j a_p^s b_m^j = c_k^s.$$

When $j = s$, then $(m_s - p_s) a_p^s b_m^s = c_k^s$.
As $a_p^s \neq 0$, if $m_s = p_s$, then $c_k^s = 0$, and we may take $b_m^s = 0$.
If $m_s \neq p_s$, then

$$b_m^s = \frac{c_k^s}{(m_s - p_s) a_p^s}.$$

So $|b_m^s| \leq \frac{|c_k^s|}{A}$ with $A = \min |a_p^s|$.
When $j \neq s$, then $p_j a_p^s b_m^j = m_j a_p^j b_m^s - c_k^s$.
As $p_j \geq 1$ and $a_p^s \neq 0$, then

$$b_m^j = \frac{m_j a_p^j b_m^s - c_k^s}{p_j a_p^s} = \frac{\left(m_j a_p^j - (m_s - p_s) a_p^s\right) c_k^s}{(m_s - p_s) p_j \left(a_p^s\right)^2}.$$

So if $\left|m_j a_p^j - (m_s - p_s) a_p^s\right| \leq 1$, then $\left|b_m^j\right| \leq \frac{|c_k^s|}{A^2}$.

Since the function Z^s is holomorphic, the power series $\sum_{|m|=0}^{+\infty} c_m^s z^m$ is absolutely convergent on an open polydisk with center 0. So the power series $\sum_{|m|=0}^{+\infty} b_m^s z^m$ is absolutely convergent on this open polydisk where it defines a unique holomorphic function Y^s.

Finally we can conclude that there exists a holomorphic vector field $Y^U = \sum_{j=1}^{N} Y^j \frac{\partial}{\partial z_j}$ on $\phi(U)$ such that $\left[X_U, Y^U\right] = \phi_* \left(Z_{|U}\right)$. So there exists a holomorphic vector field Y on U such that $\left[X_{|U}, Y\right] = Z_{|U}$. □

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