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SOME INTEGRALS INVOLVING GENERALIZED M-SERIES USING HADAMARD PRODUCT

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Abstract. In this paper, we establish some theorems in order to evaluate certain single and double integral formulas associated with the generalized M-series. The results are generalizations and unification of formulas for a fairly general class of special functions with a focus on the Mittag-Leffler function and hypergeometric function. By specializing the parameters, some new or known integrals are also investigated as corollaries in terms of generalized hypergeometric function. Further, applying the Hadamard product of two analytic functions, we have represented all our main findings in the Hadamard product of two known functions.

1 Introduction and Preliminaries

Special functions play a vital role in solving various problems of mathematics. The generalized M-series has newly emerged in connection with special functions, which is important due to its particular cases being followed by the Mittag-Leffler function and hypergeometric function, and these functions play a key role in solving various problems in applied sciences. The generalized M-series [1] is defined as

$$\begin{aligned} {}_pM_q^{\alpha,\beta}(z) &= {}_pM_q^{\alpha,\beta}(c_1,\cdots,c_p;d_1,\cdots,d_q;z), \\ &= \sum_{k=0}^{\infty}\frac{(c_1)_k\cdots(c_p)_k}{(d_1)_k\cdots(d_q)_k}\frac{z^k}{\Gamma(\beta+k\alpha)}, \end{aligned} \tag{1.1}$$

where $\alpha,\beta\in\mathbb{C}$, $z\in\mathbb{C}$, $\Re(\alpha)>0$; $(c_\tau)_k$ $(\tau=\overline{1,p})$ and $(d_\ell)_k$ $(\ell=\overline{1,q})$ are Pochhammer symbols. The series (1.1) is defined when none of the parameters (d_ℓ) $(\ell=\overline{1,q})$ is a negative integer or zero; if any numerator parameter c_τ is a negative integer or zero, then series terminates to a polynomial in z. The series (1.1) is convergent for all z if $p\leq q$; it is convergent for $|z|<\delta=\alpha^\alpha$ if $p=q+1$ and divergent if $p>q+1$. When $p=q+1$ and $|z|=\delta$, the series is convergent on conditions depending on the parameters.The detailed account of the M-series can be found in the paper [1]. By specializing the parameters of (1.1), we obtain various trigonometrical functions and classical special functions such as exponential function, binomial series, sine function, cosine function, Mittag-Leffler function [2], generalized Mittag-Leffler function [3] (for current research of generalized Mittag-Leffler function, see [4]), generalized hypergeometric function [5] as particular cases of M-series. For instance,

$${}_0M_0^{1,1}(-;-;z)=\sum_{k=0}^{\infty}\frac{z^k}{k!}=e^z, \tag{1.2}$$

$${}_1M_0^{1,1}(c;-;z)=\sum_{k=0}^{\infty}\frac{(c)_kz^k}{k!}=(1-z)^{-c}, \tag{1.3}$$

$$z\,{}_0M_1^{1,1}(-;3/2;-z^2/4)=\sum_{k=0}^{\infty}\frac{(-1)^kz^{2k+1}}{(2k+1)!}=\sin z, \tag{1.4}$$

$$
{}_0M_1^{1,1}(-;1/2;-z^2/4) = \sum_{k=0}^{\infty} \frac{(-1)^k z^{2k}}{(2k)!} = \cos z, \tag{1.5}
$$

$$
{}_0M_0^{\alpha,\beta}(-;-;z) = \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(\beta + k\alpha)} = E_{\alpha,\beta}(z), \tag{1.6}
$$

$$
{}_1M_1^{\alpha,\beta}(\rho;1;z) = \sum_{k=0}^{\infty} \frac{(\rho)_k}{(1)_k} \frac{z^k}{\Gamma(\beta + k\alpha)} = E_{\alpha,\beta}^{\rho}(z), \tag{1.7}
$$

$$
\begin{aligned}
{}_pM_q^{1,1}(c_1,\cdots,c_p;d_1,\cdots,d_q;z) &= \sum_{k=0}^{\infty} \frac{(c_1)_k \cdots (c_p)_k}{(d_1)_k \cdots (d_q)_k} \frac{z^k}{k!} \\
&= {}_pF_q(c_1,\cdots,c_p;d_1,\cdots,d_q;z).
\end{aligned} \tag{1.8}
$$

Fractional calculus is a fast-growing area of research. A lot of work has been done related to this theory. For example, see Jarad and Abdeljawad [6], Shah et al. [7]. Many authors have studied M-series in connection with fractional calculus theory, such as Gehlot [8], [9] obtained integral representation and certain properties associated with fractional calculus. Kumar and Saxena [10] investigated generalized fractional calculus formulas involving F_3 hypergeometric function, Sachan et al. [11] established fractional calculus formulas for the product of generalized M-series and I-function of two variables using Saigo-Maeda operators.

Recently, Sachan and Jaloree [12] obtained various integral transforms like Laplace transform, Beta transform, Hankel transform and many more, including fractional Fourier transform of (1.1). For more properties and application of (1.1), see Chouhan and Saraswat [13], Khan et al. [14], Singh [15], Chouhan and Khan [16], Saxena [17]. In this paper, our main concern is to find certain integrals related to M-series using Hadamard product.

Now we recall the idea of Hadamard product[18][19] or convolution of two power series, which is very useful to find a new type of results of our main findings. The Hadamard product of two power series is an entire function if one of the power series is an entire function involved in the product. This concept can help us to write a newly emerged function into two known functions. Let two power series whose radius of convergence are denoted by R_f and R_g,

$$
f(z) = \sum_{k=0}^{\infty} u_k z^k, \ (|z| < R_f),
$$

$$
g(z) = \sum_{k=0}^{\infty} v_k z^k, \ (|z| < R_g),
$$

then the Hadamard product of two power series is defined by

$$
(f * g)(z) = \sum_{k=0}^{\infty} u_k v_k z^k = (g * f)(z), \ (|z| < R), \tag{1.9}
$$

where

$$
R = \lim_{k\to\infty} \left| \frac{u_k v_k}{u_{k+1} v_{k+1}} \right| = \left(\lim_{k\to\infty} \left| \frac{u_k}{u_{k+1}} \right| \right) . \left(\lim_{k\to\infty} \left| \frac{v_k}{v_{k+1}} \right| \right) = R_f . R_g .
$$

Therefore, in general we have $R \geq R_f . R_g$.

2 Required Results

We consider following important formulas for our investigation. From [5], we have

$$
(a)_{mk} = m^{mk} \left(\frac{a}{m}\right)_k \left(\frac{a+1}{m}\right)_k \ldots \left(\frac{a+m-1}{m}\right)_k, \tag{2.1}
$$

where $(a)_{mk}$ denotes a product of mk factors, a is neither zero nor a negative integer, m is a positive integer and k is a non-negative integer.
The Lavoie-Trottier integral formula [20].

$$\int_0^1 x^{\mu-1}(1-x)^{2\lambda-1}\left(1-\frac{x}{3}\right)^{2\mu-1}\left(1-\frac{x}{4}\right)^{\lambda-1} \mathrm{d}x = \left(\frac{2}{3}\right)^{2\mu} \frac{\Gamma(\mu)\Gamma(\lambda)}{\Gamma(\mu+\lambda)}, \tag{2.2}$$

provided $\Re(\mu) > 0$ and $\Re(\lambda) > 0$.
Edward double integral formula [21].

$$\int_0^1\int_0^1 y^{\mu}(1-x)^{\mu-1}(1-y)^{\lambda-1}(1-xy)^{1-\mu-\lambda}\,\mathrm{d}x\,\mathrm{d}y = \frac{\Gamma(\mu)\Gamma(\lambda)}{\Gamma(\mu+\lambda)}, \tag{2.3}$$

provided $\Re(\mu) > 0$ and $\Re(\lambda) > 0$.

3 Main Integrals

This section investigates some integral formulas associated with generalized M-series. Some new or known results are also investigated as corollaries in terms of hypergeometric function.

Theorem 3.1. *If* $\alpha, \beta, \delta \in \mathbb{C}, z \in \mathbb{C}, c_\tau, d_\ell \in \mathbb{C}, (\tau = \overline{1,p}; \ell = \overline{1,q}), \Re(\alpha) > 0, \Re(\beta) > 0, \Re(\delta) > 0$, *and no* d_ℓ *is negative or zero, then*

$$\frac{1}{\Gamma(\delta)}\int_0^1 t^{\beta-1}(1-t)^{\delta-1}\,{}_pM_q^{\alpha,\beta}(zt^\alpha)\,\mathrm{d}t = {}_pM_q^{\alpha,\beta+\delta}(z). \tag{3.1}$$

Proof. Let $\mathcal{L}$ be the left-handed member of (3.1), term by term integration gives us the following expression

$$\begin{aligned}
\mathcal{L} = \frac{1}{\Gamma(\delta)}\int_0^1 t^{\beta-1}(1-t)^{\delta-1}\sum_{k=0}^{\infty}\frac{(c_1)_k\cdots(c_p)_k}{(d_1)_k\cdots(d_q)_k}\frac{z^k t^{\alpha k}}{\Gamma(\beta+k\alpha)}\,\mathrm{d}t,\\
= \frac{1}{\Gamma(\delta)}\sum_{k=0}^{\infty}\frac{(c_1)_k\cdots(c_p)_k}{(d_1)_k\cdots(d_q)_k}\frac{z^k}{\Gamma(\beta+k\alpha)}\int_0^1 t^{\beta+k\alpha-1}(1-t)^{\delta-1}\,\mathrm{d}t,\\
= \frac{1}{\Gamma(\delta)}\sum_{k=0}^{\infty}\frac{(c_1)_k\cdots(c_p)_k}{(d_1)_k\cdots(d_q)_k}\frac{z^k}{\Gamma(\beta+k\alpha)}B(\beta+k\alpha,\delta),\\
= \sum_{k=0}^{\infty}\frac{(c_1)_k\cdots(c_p)_k}{(d_1)_k\cdots(d_q)_k}\frac{z^k}{\Gamma(\beta+\delta+k\alpha)},\\
= {}_pM_q^{\alpha,\beta+\delta}(z).
\end{aligned}$$

This is the completion of Theorem 3.1. □

Corollary 3.2. *If* $z \in \mathbb{C}, c_\tau, d_\ell \in \mathbb{C}, (\tau = \overline{1,p}; \ell = \overline{1,q}), \Re(\delta) > 0$ *and no* d_ℓ *is negative or zero, then*

$$\frac{1}{\Gamma(\delta)}\int_0^1 (1-t)^{\delta-1}\,{}_pF_q(zt)\,\mathrm{d}t = \frac{1}{\Gamma(\delta+1)}\,{}_{p+1}F_{q+1}(1, c_1\cdots, c_p; 1+\delta, d_1, \cdots, d_q; z). \tag{3.2}$$

Theorem 3.3. *If* $\alpha, \beta, \delta, \lambda \in \mathbb{C}, z \in \mathbb{C}, c_\tau, d_\ell \in \mathbb{C}, (\tau = \overline{1,p}; \ell = \overline{1,q}), \Re(\alpha) > 0, \Re(\beta) > 0, \Re(\delta) > 0$, *and no* d_ℓ *is negative or zero, then*

$$\frac{1}{\Gamma(\delta)}\int_t^z (z-s)^{\delta-1}(s-t)^{\beta-1}\,{}_pM_q^{\alpha,\beta}(\lambda(s-t)^\alpha)\,\mathrm{d}s = (z-t)^{\beta+\delta-1}\,{}_pM_q^{\alpha,\beta+\delta}(\lambda(z-t)^\alpha). \tag{3.3}$$

Proof. Let $\mathcal{L}$ be the left-handed member of (3.3), term by term integration gives us the following expression

$$\mathcal{L} = \frac{1}{\Gamma(\delta)} \int_t^z (z-s)^{\delta-1}(s-t)^{\beta-1} \sum_{k=0}^{\infty} \frac{(c_1)_k \cdots (c_p)_k}{(d_1)_k \cdots (d_q)_k} \frac{\lambda^k (s-t)^{\alpha k}}{\Gamma(\beta+k\alpha)} \, \mathrm{d}s,$$

$$= \frac{1}{\Gamma(\delta)} \sum_{k=0}^{\infty} \frac{(c_1)_k \cdots (c_p)_k}{(d_1)_k \cdots (d_q)_k} \frac{\lambda^k}{\Gamma(\beta+k\alpha)} \int_t^z (z-s)^{\delta-1}(s-t)^{\alpha k+\beta-1} \, \mathrm{d}s,$$

by changing the variable $s = t + \omega(z-t)$

$$\mathcal{L} = \frac{1}{\Gamma(\delta)} \sum_{k=0}^{\infty} \frac{(c_1)_k \cdots (c_p)_k}{(d_1)_k \cdots (d_q)_k} \frac{\lambda^k (z-t)^{\alpha k+\beta+\delta-1}}{\Gamma(\beta+k\alpha)} \int_0^1 \omega^{\alpha k+\beta-1}(1-\omega)^{\delta-1} \mathrm{d}\omega,$$

$$= \frac{1}{\Gamma(\delta)} \sum_{k=0}^{\infty} \frac{(c_1)_k \cdots (c_p)_k}{(d_1)_k \cdots (d_q)_k} \frac{\lambda^k (z-t)^{\alpha k+\beta+\delta-1}}{\Gamma(\beta+k\alpha)} B(\beta+k\alpha, \delta),$$

$$= (z-t)^{\beta+\delta-1} \sum_{k=0}^{\infty} \frac{(c_1)_k \cdots (c_p)_k}{(d_1)_k \cdots (d_q)_k} \frac{\lambda^k (z-t)^{\alpha k}}{\Gamma(\beta+\delta+k\alpha)},$$

$$= (z-t)^{\beta+\delta-1} {}_pM_q^{\alpha,\beta+\delta}(\lambda(z-t)^{\alpha}).$$

This is the completion of Theorem 3.3. □

Corollary 3.4. *If $\delta, \lambda \in \mathbb{C}, z \in \mathbb{C}, c_\tau, d_\ell \in \mathbb{C}, (\tau = \overline{1,p}; \ell = \overline{1,q}), \Re(\delta) > 0$ and no d_ℓ is negative or zero, then*

$$\frac{1}{\Gamma(\delta)} \int_t^z (z-s)^{\delta-1} {}_pF_q(\lambda(s-t)) \, \mathrm{d}s = \frac{(z-t)^{\delta-1}}{\Gamma(\delta+1)} {}_{p+1}F_{q+1}(1, c_1 \cdots, c_p; 1+\delta, d_1, \cdots, d_q; \lambda(z-t)). \tag{3.4}$$

Theorem 3.5. *If $\alpha, \beta \in \mathbb{C}, z \in \mathbb{C}, c_\tau, d_\ell \in \mathbb{C}, (\tau = \overline{1,p}; \ell = \overline{1,q}), \Re(\alpha) > 0, \Re(\beta) > 0$ and no d_ℓ is negative or zero, then*

$$\frac{1}{\Gamma(\alpha)} \int_0^z t^{\beta-1}(z-t)^{\alpha-1} {}_pM_q^{\alpha,\beta}(\lambda t^{\alpha}) \, \mathrm{d}t = z^{\alpha+\beta-1} {}_pM_q^{\alpha,\alpha+\beta}(\lambda z^{\alpha}). \tag{3.5}$$

Proof. Let $\mathcal{L}$ be the left-handed member of (3.5), term by term integration gives us the following expression

$$\mathcal{L} = \frac{1}{\Gamma(\alpha)} \int_0^z t^{\beta-1}(z-t)^{\alpha-1} \sum_{k=0}^{\infty} \frac{(c_1)_k \cdots (c_p)_k}{(d_1)_k \cdots (d_q)_k} \frac{\lambda^k t^{\alpha k}}{\Gamma(\beta+k\alpha)} \, \mathrm{d}t,$$

$$= \frac{1}{\Gamma(\alpha)} \sum_{k=0}^{\infty} \frac{(c_1)_k \cdots (c_p)_k}{(d_1)_k \cdots (d_q)_k} \frac{\lambda^k}{\Gamma(\beta+k\alpha)} \int_0^z t^{\alpha k+\beta-1}(z-t)^{\alpha-1} \, \mathrm{d}t,$$

substituting $t = zv$

$$\mathcal{L} = \frac{1}{\Gamma(\alpha)} \sum_{k=0}^{\infty} \frac{(c_1)_k \cdots (c_p)_k}{(d_1)_k \cdots (d_q)_k} \frac{\lambda^k z^{\alpha k+\alpha+\beta-1}}{\Gamma(\beta+k\alpha)} \int_0^1 v^{\alpha k+\beta-1}(1-v)^{\alpha-1} \, \mathrm{d}v,$$

$$= \frac{1}{\Gamma(\alpha)} z^{\alpha+\beta-1} \sum_{k=0}^{\infty} \frac{(c_1)_k \cdots (c_p)_k}{(d_1)_k \cdots (d_q)_k} \frac{\lambda^k z^{\alpha k}}{\Gamma(\beta+k\alpha)} B(\beta+k\alpha, \alpha),$$

$$= z^{\alpha+\beta-1} \sum_{k=0}^{\infty} \frac{(c_1)_k \cdots (c_p)_k}{(d_1)_k \cdots (d_q)_k} \frac{\lambda^k z^{\alpha k}}{\Gamma(\alpha+\beta+k\alpha)},$$

$$= z^{\alpha+\beta-1} {}_pM_q^{\alpha,\alpha+\beta}(\lambda z^{\alpha}).$$

This is the completion of Theorem 3.5. □

Corollary 3.6. *If* $z \in \mathbb{C}, c_\tau, d_\ell \in \mathbb{C}, (\tau = \overline{1,p}; \ell = \overline{1,q})$ *and no* d_ℓ *is negative or zero, then*

$$\int_0^z {}_pF_q(\lambda t)\,\mathrm{d}t = z_{p+1}F_{q+1}(1, c_1 \cdots, c_p; 2, d_1, \cdots, d_q; \lambda z). \tag{3.6}$$

Theorem 3.7. *If* $\alpha, \beta, \mu, \lambda \in \mathbb{C}, z \in \mathbb{C}, c_\tau, d_\ell \in \mathbb{C}, (\tau = \overline{1,p}; \ell = \overline{1,q}), \Re(\alpha) > 0, \Re(\beta) > 0, \Re(\mu) > 0, \Re(\lambda) > 0$ *and no* d_ℓ *is negative or zero, then*

$$\int_0^t z^{\mu-1}(t-z)^{\lambda-1}{}_pM_q^{\alpha,\beta}(cz^m)\,\mathrm{d}z = B(\mu,\lambda)t^{\mu+\lambda-1}{}_{p+m}M_{q+m}^{\alpha,\beta}(P_1; Q_1; ct^m), \tag{3.7}$$

where

$$P_1 = c_1, \ldots\ldots\ldots, c_p, \frac{\mu}{m}, \frac{\mu+1}{m}, \ldots\ldots\ldots\ldots, , \frac{\mu+m-1}{m},$$

$$Q_1 = d_1, \ldots, d_q, \frac{\mu+\lambda}{m}, \frac{\mu+\lambda+1}{m}, \ldots, \frac{\mu+\lambda+m-1}{m}.$$

Proof. Let $\mathcal{L}$ be the left-handed member of (3.7), term by term integration gives us the following expression

$$\mathcal{L} = \int_0^t z^{\mu-1}(t-z)^{\lambda-1}\sum_{k=0}^{\infty}\frac{(c_1)_k\cdots(c_p)_k}{(d_1)_k\cdots(d_q)_k}\frac{c^k z^{mk}}{\Gamma(\beta+k\alpha)}\,\mathrm{d}z,$$

$$= \sum_{k=0}^{\infty}\frac{(c_1)_k\cdots(c_p)_k}{(d_1)_k\cdots(d_q)_k}\frac{c^k}{\Gamma(\beta+k\alpha)}\int_0^t z^{mk+\mu-1}(t-z)^{\lambda-1}\,\mathrm{d}z,$$

substituting $z = tv$, which gives

$$\mathcal{L} = \sum_{k=0}^{\infty}\frac{(c_1)_k\cdots(c_p)_k}{(d_1)_k\cdots(d_q)_k}\frac{c^k t^{mk+\mu+\lambda-1}}{\Gamma(\beta+k\alpha)}\int_0^1 v^{mk+\mu-1}(1-v)^{\lambda-1}\,\mathrm{d}v,$$

$$= t^{\mu+\lambda-1}\sum_{k=0}^{\infty}\frac{(c_1)_k\cdots(c_p)_k}{(d_1)_k\cdots(d_q)_k}\frac{c^k t^{mk}}{\Gamma(\beta+k\alpha)}B(mk+\mu,\lambda),$$

$$= t^{\mu+\lambda-1}\sum_{k=0}^{\infty}\frac{(c_1)_k\cdots(c_p)_k}{(d_1)_k\cdots(d_q)_k}\frac{c^k t^{mk}}{\Gamma(\beta+k\alpha)}\frac{\Gamma(mk+\mu)\Gamma(\lambda)}{\Gamma(mk+\mu+\lambda)},$$

$$= B(\mu,\lambda)t^{\mu+\lambda-1}\sum_{k=0}^{\infty}\frac{(c_1)_k\cdots(c_p)_k}{(d_1)_k\cdots(d_q)_k}\frac{c^k t^{mk}}{\Gamma(\beta+k\alpha)}\frac{(\mu)_{mk}}{(\mu+\lambda)_{mk}},$$

using formula (2.1), above expression can be written as

$$\mathcal{L} = B(\mu,\lambda)t^{\mu+\lambda-1}\sum_{k=0}^{\infty}\frac{(c_1)_k\cdots(c_p)_k}{(d_1)_k\cdots(d_q)_k}\frac{\prod_{i=1}^{m}\left(\frac{\mu+i-1}{m}\right)_k}{\prod_{j=1}^{m}\left(\frac{\mu+\lambda+j-1}{m}\right)_k}\frac{c^k t^{mk}}{\Gamma(\beta+k\alpha)},$$

$$= B(\mu,\lambda)t^{\mu+\lambda-1}{}_{p+m}M_{q+m}^{\alpha,\beta}(P_1; Q_1; ct^m).$$

This is the completion of Theorem 3.7. □

Corollary 3.8. *If* $\mu, \lambda \in \mathbb{C}, z \in \mathbb{C}, c_\tau, d_\ell \in \mathbb{C}, (\tau = \overline{1,p}; \ell = \overline{1,q}), \Re(\mu) > 0, \Re(\lambda) > 0$ *and no* d_ℓ *is negative or zero, then there hold the following known formula [5].*

$$\int_0^t z^{\mu-1}(t-z)^{\lambda-1} {}_pF_q(cz^m)\,\mathrm{d}z = B(\mu,\lambda)t^{\mu+\lambda-1} {}_{p+m}F_{q+m}(P_1; Q_1; ct^m), \tag{3.8}$$

where P_1 and Q_1 are same as given in Theorem 3.7.

Theorem 3.9. *If* $\alpha, \beta, \mu, \lambda \in \mathbb{C}, z \in \mathbb{C}, c_\tau, d_\ell \in \mathbb{C}, (\tau = \overline{1,p}; \ell = \overline{1,q}), \Re(\alpha) > 0, \Re(\beta) > 0, \Re(\mu) > 0, \Re(\lambda) > 0$ *and no* d_ℓ *is negative or zero, then*

$$\int_0^t z^{\mu-1}(t-z)^{\lambda-1} {}_pM_q^{\alpha,\beta}(c(t-z)^s)\,\mathrm{d}z = B(\mu,\lambda)t^{\mu+\lambda-1} {}_{p+s}M_{q+s}^{\alpha,\beta}(P_2; Q_2; ct^s), \tag{3.9}$$

where

$$P_2 = c_1, \ldots\ldots\ldots\ldots, c_p, \frac{\lambda}{s}, \frac{\lambda+1}{s}, \ldots\ldots\ldots\ldots, , \frac{\lambda+s-1}{s},$$

$$Q_2 = d_1, \ldots, d_q, \frac{\mu+\lambda}{s}, \frac{\mu+\lambda+1}{s}, \ldots, \frac{\mu+\lambda+s-1}{s}.$$

Proof. Let $\mathcal{L}$ be the left-handed member of (3.9), term by term integration gives us the following expression

$$\mathcal{L} = \int_0^t z^{\mu-1}(t-z)^{\lambda-1} \sum_{k=0}^{\infty} \frac{(c_1)_k \cdots (c_p)_k}{(d_1)_k \cdots (d_q)_k} \frac{c^k (t-z)^{sk}}{\Gamma(\beta+k\alpha)}\,\mathrm{d}z,$$

$$= \sum_{k=0}^{\infty} \frac{(c_1)_k \cdots (c_p)_k}{(d_1)_k \cdots (d_q)_k} \frac{c^k}{\Gamma(\beta+k\alpha)} \int_0^t z^{\mu-1}(t-z)^{sk+\lambda-1}\,\mathrm{d}z,$$

substituting $z = tv$, which gives

$$\mathcal{L} = \sum_{k=0}^{\infty} \frac{(c_1)_k \cdots (c_p)_k}{(d_1)_k \cdots (d_q)_k} \frac{c^k t^{sk+\mu+\lambda-1}}{\Gamma(\beta+k\alpha)} \int_0^1 v^{\mu-1}(1-v)^{sk+\lambda-1}\,\mathrm{d}v,$$

$$= t^{\mu+\lambda-1} \sum_{k=0}^{\infty} \frac{(c_1)_k \cdots (c_p)_k}{(d_1)_k \cdots (d_q)_k} \frac{c^k t^{sk}}{\Gamma(\beta+k\alpha)} B(\mu, sk+\lambda),$$

$$= t^{\mu+\lambda-1} \sum_{k=0}^{\infty} \frac{(c_1)_k \cdots (c_p)_k}{(d_1)_k \cdots (d_q)_k} \frac{c^k t^{sk}}{\Gamma(\beta+k\alpha)} \frac{\Gamma(\mu)\Gamma(sk+\lambda)}{\Gamma(sk+\mu+\lambda)},$$

$$= B(\mu,\lambda)t^{\mu+\lambda-1} \sum_{k=0}^{\infty} \frac{(c_1)_k \cdots (c_p)_k}{(d_1)_k \cdots (d_q)_k} \frac{c^k t^{sk}}{\Gamma(\beta+k\alpha)} \frac{(\lambda)_{sk}}{(\mu+\lambda)_{sk}},$$

using formula (2.1), above expression can be written as

$$\mathcal{L} = B(\mu,\lambda)t^{\mu+\lambda-1} \sum_{k=0}^{\infty} \frac{(c_1)_k \cdots (c_p)_k}{(d_1)_k \cdots (d_q)_k} \frac{\prod_{i=1}^{s} \left(\frac{\lambda+i-1}{s}\right)_k}{\prod_{j=1}^{s} \left(\frac{\mu+\lambda+j-1}{s}\right)_k} \frac{c^k t^{sk}}{\Gamma(\beta+k\alpha)},$$

$$= B(\mu,\lambda)t^{\mu+\lambda-1} {}_{p+s}M_{q+s}^{\alpha,\beta}(P_2; Q_2; ct^s).$$

This is the completion of Theorem 3.9. □

Corollary 3.10. *If $\mu, \lambda \in \mathbb{C}, z \in \mathbb{C}, c_\tau, d_\ell \in \mathbb{C}, (\tau = \overline{1,p}; \ell = \overline{1,q}), \Re(\mu) > 0, \Re(\lambda) > 0$ and no d_ℓ is negative or zero, then there hold the following known formula [22].*

$$\int_0^t z^{\mu-1}(t-z)^{\lambda-1}{}_pF_q(c(t-z)^s)\,\mathrm{d}z = B(\mu,\lambda)t^{\mu+\lambda-1}{}_{p+s}F_{q+s}(P_2;Q_2;ct^s), \tag{3.10}$$

where P_2 and Q_2 are same as given in Theorem 3.9.

Theorem 3.11. *If $\alpha, \beta, \mu, \lambda \in \mathbb{C}, z \in \mathbb{C}, c_\tau, d_\ell \in \mathbb{C}, (\tau = \overline{1,p}; \ell = \overline{1,q}), \Re(\alpha) > 0, \Re(\beta) > 0, \Re(\mu) > 0, \Re(\lambda) > 0$ and no d_ℓ is negative or zero, then*

$$\begin{aligned} &\int_0^t z^{\mu-1}(t-z)^{\lambda-1}{}_pM_q^{\alpha,\beta}\Big(cz^m(t-z)^s\Big)\,\mathrm{d}z \\ &= B(\mu,\lambda)t^{\mu+\lambda-1}{}_{p+m+s}M_{q+m+s}^{\alpha,\beta}\left(P_3;Q_3;\frac{m^m s^s ct^{m+s}}{(m+s)^{m+s}}\right), \end{aligned} \tag{3.11}$$

where

$$P_3 = c_1,...,c_p, \frac{\mu}{m}, \frac{\mu+1}{m}, ..., , \frac{\mu+m-1}{m}, \frac{\lambda}{s}, \frac{\lambda+1}{s}, ..., \frac{\lambda+s-1}{s},$$
$$Q_3 = d_1,...,d_q, \frac{\mu+\lambda}{m+s}, \frac{\mu+\lambda+1}{m+s}, ..., \frac{\mu+\lambda+m+s-1}{m+s}.$$

Proof. Let $\mathcal{L}$ be the left-handed member of (3.11), term by term integration gives us the following expression

$$\begin{aligned} \mathcal{L} &= \int_0^t z^{\mu-1}(t-z)^{\lambda-1}\sum_{k=0}^{\infty}\frac{(c_1)_k\cdots(c_p)_k}{(d_1)_k\cdots(d_q)_k}\frac{c^k z^{mk}(t-z)^{sk}}{\Gamma(\beta+k\alpha)}\,\mathrm{d}z, \\ &= \sum_{k=0}^{\infty}\frac{(c_1)_k\cdots(c_p)_k}{(d_1)_k\cdots(d_q)_k}\frac{c^k}{\Gamma(\beta+k\alpha)}\int_0^t z^{mk+\mu-1}(t-z)^{sk+\lambda-1}\,\mathrm{d}z, \end{aligned}$$

substituting $z = tv$, which gives

$$\begin{aligned} \mathcal{L} &= \sum_{k=0}^{\infty}\frac{(c_1)_k\cdots(c_p)_k}{(d_1)_k\cdots(d_q)_k}\frac{c^k t^{mk+sk+\mu+\lambda-1}}{\Gamma(\beta+k\alpha)}\int_0^1 v^{mk+\mu-1}(1-v)^{sk+\lambda-1}\,\mathrm{d}v, \\ &= t^{\mu+\lambda-1}\sum_{k=0}^{\infty}\frac{(c_1)_k\cdots(c_p)_k}{(d_1)_k\cdots(d_q)_k}\frac{c^k t^{(m+s)k}}{\Gamma(\beta+k\alpha)}B(mk+\mu, sk+\lambda), \\ &= t^{\mu+\lambda-1}\sum_{k=0}^{\infty}\frac{(c_1)_k\cdots(c_p)_k}{(d_1)_k\cdots(d_q)_k}\frac{c^k t^{(m+s)k}}{\Gamma(\beta+k\alpha)}\frac{\Gamma(mk+\mu)\Gamma(sk+\lambda)}{\Gamma((m+s)k+\mu+\lambda)}, \\ &= B(\mu,\lambda)t^{\mu+\lambda-1}\sum_{k=0}^{\infty}\frac{(c_1)_k\cdots(c_p)_k}{(d_1)_k\cdots(d_q)_k}\frac{c^k t^{(m+s)k}}{\Gamma(\beta+k\alpha)}\frac{(\mu)_{mk}(\lambda)_{sk}}{(\mu+\lambda)_{(m+s)k}}, \end{aligned}$$

using formula (2.1), above expression can be written as

$$\begin{aligned} \mathcal{L} &= B(\mu,\lambda)t^{\mu+\lambda-1}\sum_{k=0}^{\infty}\frac{(c_1)_k\cdots(c_p)_k}{(d_1)_k\cdots(d_q)_k}\frac{c^k t^{(m+s)k}}{\Gamma(\beta+k\alpha)}, \\ &\times \frac{m^{mk}\prod_{i=1}^{m}\left(\frac{\mu+i-1}{m}\right)_k s^{sk}\prod_{j=1}^{s}\left(\frac{\lambda+j-1}{s}\right)_k}{(m+s)^{(m+s)k}\prod_{\omega=1}^{m+s}\left(\frac{\mu+\lambda+\omega-1}{m+s}\right)_k}, \end{aligned}$$

$$= B(\mu,\lambda)t^{\mu+\lambda-1}{}_{p+m+s}M^{\alpha,\beta}_{q+m+s}\left(P_3;Q_3;\frac{m^m s^s c t^{m+s}}{(m+s)^{m+s}}\right).$$

This is the completion of Theorem 3.11. □

Corollary 3.12. *If* $\mu,\lambda\in\mathbb{C}, z\in\mathbb{C}, c_\tau, d_\ell\in\mathbb{C}, (\tau=\overline{1,p};\ell=\overline{1,q}), \Re(\mu)>0, \Re(\lambda)>0$ *and no* d_ℓ *is negative or zero, then there hold the following known formula [5].*

$$\int_0^t z^{\mu-1}(t-z)^{\lambda-1}{}_pF_q\left(cz^m(t-z)^s\right)\mathrm{d}z = B(\mu,\lambda)t^{\mu+\lambda-1}{}_{p+m+s}F_{q+m+s}\left(P_3;Q_3;\frac{m^m s^s c t^{m+s}}{(m+s)^{m+s}}\right), \tag{3.12}$$

where P_3 and Q_3 are same as given in Theorem 3.11.

Theorem 3.13. *If* $\alpha,\beta,\mu,\lambda\in\mathbb{C}, z\in\mathbb{C}, c_\tau, d_\ell\in\mathbb{C}, (\tau=\overline{1,p};\ell=\overline{1,q}), \Re(\alpha)>0, \Re(\beta)>0, \Re(\mu)>0, \Re(\lambda)>0$ *and no* d_ℓ *is negative or zero, then*

$$\begin{aligned}&\int_0^1 x^{\mu-1}(1-x)^{2(\mu+\lambda)-1}\left(1-\frac{x}{3}\right)^{2\mu-1}\left(1-\frac{x}{4}\right)^{(\mu+\lambda)-1}{}_pM^{\alpha,\beta}_q\left(zx\left(1-\frac{x}{3}\right)^2\right)\mathrm{d}x\\&=\left(\frac{2}{3}\right)^{2\mu}B(\mu,\mu+\lambda)\,{}_{p+1}M^{\alpha,\beta}_{q+1}\left(c_1,...,c_p,\mu;d_1,...,d_q,2\mu+\lambda;\frac{4z}{9}\right).\end{aligned} \tag{3.13}$$

Proof. Let $\mathcal{L}$ be the left-handed member of (3.13), by applying (1.1) in the integrand of (3.13) and interchanging the order of integral and summation, we get

$$\begin{aligned}\mathcal{L}&=\int_0^1 x^{\mu-1}(1-x)^{2(\mu+\lambda)-1}\left(1-\frac{x}{3}\right)^{2\mu-1}\left(1-\frac{x}{4}\right)^{(\mu+\lambda)-1}{}_pM^{\alpha,\beta}_q\left(zx\left(1-\frac{x}{3}\right)^2\right)\mathrm{d}x,\\&=\sum_{k=0}^{\infty}\frac{(c_1)_k\cdots(c_p)_k}{(d_1)_k\cdots(d_q)_k}\frac{z^k}{\Gamma(\beta+k\alpha)}\\&\times\int_0^1 x^{\mu+k-1}(1-x)^{2(\mu+\lambda)-1}\left(1-\frac{x}{3}\right)^{2(\mu+k)-1}\left(1-\frac{x}{4}\right)^{(\mu+\lambda)-1}\mathrm{d}x,\end{aligned}$$

applying integral formula (2.2), we obtain

$$\begin{aligned}\mathcal{L}&=\sum_{k=0}^{\infty}\frac{(c_1)_k\cdots(c_p)_k}{(d_1)_k\cdots(d_q)_k}\frac{z^k}{\Gamma(\beta+k\alpha)}\left(\frac{2}{3}\right)^{2(\mu+k)}\frac{\Gamma(\mu+k)\Gamma(\mu+\lambda)}{\Gamma(2\mu+\lambda+k)},\\&=\left(\frac{2}{3}\right)^{2\mu}\frac{\Gamma(\mu)\Gamma(\mu+\lambda)}{\Gamma(2\mu+\lambda)}\sum_{k=0}^{\infty}\frac{(c_1)_k\cdots(c_p)_k}{(d_1)_k\cdots(d_q)_k}\frac{(\mu)_k}{(2\mu+\lambda)_k}\frac{(\frac{4z}{9})^k}{\Gamma(\beta+k\alpha)},\\&=\left(\frac{2}{3}\right)^{2\mu}B(\mu,\mu+\lambda)\,{}_{p+1}M^{\alpha,\beta}_{q+1}\left(c_1,...,c_p,\mu;d_1,...,d_q,2\mu+\lambda;\frac{4z}{9}\right).\end{aligned}$$

This is the completion of Theorem 3.13. □

Corollary 3.14. *If* $\mu,\lambda\in\mathbb{C}, z\in\mathbb{C}, c_\tau, d_\ell\in\mathbb{C}, (\tau=\overline{1,p};\ell=\overline{1,q}), \Re(\mu)>0, \Re(\lambda)>0$ *and no* d_ℓ *is negative or zero,then*

$$\begin{aligned}&\int_0^1 x^{\mu-1}(1-x)^{2(\mu+\lambda)-1}\left(1-\frac{x}{3}\right)^{2\mu-1}\left(1-\frac{x}{4}\right)^{(\mu+\lambda)-1}{}_pF_q\left(zx\left(1-\frac{x}{3}\right)^2\right)\mathrm{d}x\\&=\left(\frac{2}{3}\right)^{2\mu}B(\mu,\mu+\lambda)\,{}_{p+1}F_{q+1}\left(c_1,...,c_p,\mu;d_1,...,d_q,2\mu+\lambda;\frac{4z}{9}\right).\end{aligned} \tag{3.14}$$

Theorem 3.15. *If* $\alpha, \beta, \mu, \lambda \in \mathbb{C}, z \in \mathbb{C}, c_\tau, d_\ell \in \mathbb{C}, (\tau = \overline{1,p}; \ell = \overline{1,q}), \Re(\alpha) > 0, \Re(\beta) > 0, \Re(\mu) > 0, \Re(\lambda) > 0$ *and no* d_ℓ *is negative or zero, then*

$$\int_0^1 x^{\mu+\lambda-1}(1-x)^{2\mu-1}\left(1-\frac{x}{3}\right)^{2(\mu+\lambda)-1}\left(1-\frac{x}{4}\right)^{\mu-1} {}_pM_q^{\alpha,\beta}\left(z\left(1-\frac{x}{4}\right)(1-x)^2\right)\mathrm{d}x = \left(\frac{2}{3}\right)^{2(\mu+\lambda)} B(\mu,\mu+\lambda)\, {}_{p+1}M_{q+1}^{\alpha,\beta}(c_1,...,c_p,\mu;d_1,...,d_q,2\mu+\lambda;z). \tag{3.15}$$

Proof. Proof is similar as of Theorem 3.13. □

Corollary 3.16. *If* $\mu, \lambda \in \mathbb{C}, z \in \mathbb{C}, c_\tau, d_\ell \in \mathbb{C}, (\tau = \overline{1,p}; \ell = \overline{1,q}), \Re(\mu) > 0, \Re(\lambda) > 0$ *and no* d_ℓ *is negative or zero, then*

$$\int_0^1 x^{\mu+\lambda-1}(1-x)^{2\mu-1}\left(1-\frac{x}{3}\right)^{2(\mu+\lambda)-1}\left(1-\frac{x}{4}\right)^{\mu-1} {}_pF_q\left(z\left(1-\frac{x}{4}\right)(1-x)^2\right)\mathrm{d}x = \left(\frac{2}{3}\right)^{2(\mu+\lambda)} B(\mu,\mu+\lambda)\, {}_{p+1}F_{q+1}(c_1,...,c_p,\mu;d_1,...,d_q,2\mu+\lambda;z). \tag{3.16}$$

Theorem 3.17. *If* $\alpha, \beta, \mu, \lambda \in \mathbb{C}, z \in \mathbb{C}, c_\tau, d_\ell \in \mathbb{C}, (\tau = \overline{1,p}; \ell = \overline{1,q}), \Re(\alpha) > 0, \Re(\beta) > 0, \Re(\mu) > 0, \Re(\lambda) > 0$ *and no* d_ℓ *is negative or zero, then*

$$\int_0^1\int_0^1 y^\mu(1-x)^{\mu-1}(1-y)^{\lambda-1}(1-xy)^{1-\mu-\lambda}\, {}_pM_q^{\alpha,\beta}\left(\frac{z(1-y)}{1-xy}\right)\mathrm{d}x\,\mathrm{d}y = B(\mu,\lambda)_{p+1}M_{q+1}^{\alpha,\beta}(c_1,...,c_p,\lambda;d_1,...,d_q,\mu+\lambda;z). \tag{3.17}$$

Proof. Let $\mathcal{L}$ be the left-handed member of (3.17), by applying (1.1) in the integrand of (3.17) and interchanging the order of integral and summation, we get

$$\mathcal{L} = \int_0^1\int_0^1 y^\mu(1-x)^{\mu-1}(1-y)^{\lambda-1}(1-xy)^{1-\mu-\lambda}\, {}_pM_q^{\alpha,\beta}\left(\frac{z(1-y)}{1-xy}\right)\mathrm{d}x\,\mathrm{d}y,$$

$$= \sum_{k=0}^{\infty}\frac{(c_1)_k\cdots(c_p)_k}{(d_1)_k\cdots(d_q)_k}\frac{z^k}{\Gamma(\beta+k\alpha)}\int_0^1\int_0^1 y^\mu(1-x)^{\mu-1}(1-y)^{\lambda+k-1}(1-xy)^{1-\mu-\lambda-k}\,\mathrm{d}x\,\mathrm{d}y,$$

applying integral formula (2.3)

$$\mathcal{L} = \sum_{k=0}^{\infty}\frac{(c_1)_k\cdots(c_p)_k}{(d_1)_k\cdots(d_q)_k}\frac{z^k}{\Gamma(\beta+k\alpha)}\frac{\Gamma(\mu)\Gamma(\lambda+k)}{\Gamma(\mu+\lambda+k)},$$

$$= B(\mu,\lambda)\sum_{k=0}^{\infty}\frac{(c_1)_k\cdots(c_p)_k}{(d_1)_k\cdots(d_q)_k}\frac{z^k}{\Gamma(\beta+k\alpha)}\frac{(\lambda)_k}{(\mu+\lambda)_k},$$

$$= B(\mu,\lambda)_{p+1}M_{q+1}^{\alpha,\beta}(c_1,...,c_p,\lambda;d_1,...,d_q,\mu+\lambda;z).$$

This is the completion of Theorem 3.17. □

Corollary 3.18. *If* $\mu, \lambda \in \mathbb{C}, z \in \mathbb{C}, c_\tau, d_\ell \in \mathbb{C}, (\tau = \overline{1,p}; \ell = \overline{1,q}), \Re(\mu) > 0, \Re(\lambda) > 0$ *and no* d_ℓ *is negative or zero, then*

$$\int_0^1\int_0^1 y^\mu(1-x)^{\mu-1}(1-y)^{\lambda-1}(1-xy)^{1-\mu-\lambda}\, {}_pF_q\left(\frac{z(1-y)}{1-xy}\right)\mathrm{d}x\,\mathrm{d}y = B(\mu,\lambda)_{p+1}F_{q+1}(c_1,...,c_p,\lambda;d_1,...,d_q,\mu+\lambda;z). \tag{3.18}$$

Theorem 3.19. *If* $\alpha, \beta, \mu, \lambda \in \mathbb{C}, z \in \mathbb{C}, c_\tau, d_\ell \in \mathbb{C}, (\tau = \overline{1,p}; \ell = \overline{1,q}), \Re(\alpha) > 0, \Re(\beta) > 0, \Re(\mu) > 0, \Re(\lambda) > 0$ *and no* d_ℓ *is negative or zero, then*

$$\int_0^1 \int_0^1 y^\mu (1-x)^{\mu-1}(1-y)^{\lambda-1}(1-xy)^{1-\mu-\lambda} {}_pM_q^{\alpha,\beta}\left(\frac{zy(1-x)}{1-xy}\right) \mathrm{d}x \, \mathrm{d}y = B(\mu,\lambda)_{p+1}M_{q+1}^{\alpha,\beta}(c_1, ..., c_p, \mu; d_1, ..., d_q, \mu+\lambda; z). \tag{3.19}$$

Proof. Proof is similar as of Theorem 3.17. □

Corollary 3.20. *If* $\mu, \lambda \in \mathbb{C}, z \in \mathbb{C}, c_\tau, d_\ell \in \mathbb{C}, (\tau = \overline{1,p}; \ell = \overline{1,q}), \Re(\mu) > 0, \Re(\lambda) > 0$ *and no* d_ℓ *is negative or zero, then*

$$\int_0^1 \int_0^1 y^\mu (1-x)^{\mu-1}(1-y)^{\lambda-1}(1-xy)^{1-\mu-\lambda} {}_pF_q\left(\frac{zy(1-x)}{1-xy}\right) \mathrm{d}x \, \mathrm{d}y = B(\mu,\lambda)_{p+1}F_{q+1}(c_1, ..., c_p, \mu; d_1, ..., d_q, \mu+\lambda; z). \tag{3.20}$$

Theorem 3.21. *If* $\alpha, \beta, \mu, \lambda \in \mathbb{C}, z \in \mathbb{C}, c_\tau, d_\ell \in \mathbb{C}, (\tau = \overline{1,p}; \ell = \overline{1,q}), \Re(\alpha) > 0, \Re(\beta) > 0, \Re(\mu) > 0, \Re(\lambda) > 0$ *and no* d_ℓ *is negative or zero, then*

$$\int_0^1 \int_0^1 y^\mu (1-x)^{\mu-1}(1-y)^{\lambda-1}(1-xy)^{1-\mu-\lambda} {}_pM_q^{\alpha,\beta}\left(\frac{zy(1-x)(1-y)}{(1-xy)^2}\right) \mathrm{d}x \, \mathrm{d}y = B(\mu,\lambda)_{p+2}M_{q+2}^{\alpha,\beta}\left(c_1, ..., c_p, \mu, \lambda; d_1, ..., d_q, \frac{\mu+\lambda}{2}, \frac{\mu+\lambda+1}{2}; \frac{z}{4}\right). \tag{3.21}$$

Proof. Following similar lines as of Theorem 3.17, applying formula (2.3), leads the proof of Theorem. □

Corollary 3.22. *If* $\mu, \lambda \in \mathbb{C}, z \in \mathbb{C}, c_\tau, d_j \in \mathbb{C}, (\tau = \overline{1,p}; j = \overline{1,q}), \Re(\mu) > 0, \Re(\lambda) > 0$ *and no* d_j *is negative or zero, then*

$$\int_0^1 \int_0^1 y^\mu (1-x)^{\mu-1}(1-y)^{\lambda-1}(1-xy)^{1-\mu-\lambda} {}_pF_q\left(\frac{zy(1-x)(1-y)}{(1-xy)^2}\right) \mathrm{d}x \, \mathrm{d}y = B(\mu,\lambda)_{p+2}F_{q+2}\left(c_1, ..., c_p, \mu, \lambda; d_1, ..., d_q, \frac{\mu+\lambda}{2}, \frac{\mu+\lambda+1}{2}; \frac{z}{4}\right). \tag{3.22}$$

4 Representations in Hadamard product

This section is devoted to establishing our main integrals in terms of the product of two functions applying Hadamard product concept of power series defined in (1.9).

Theorem 4.1. *If* $\alpha, \beta, \delta \in \mathbb{C}, z \in \mathbb{C}, c_\tau, d_\ell \in \mathbb{C}, (\tau = \overline{1,p}; \ell = \overline{1,q}), \Re(\alpha) > 0, \Re(\beta) > 0, \Re(\delta) > 0,$ *and no* d_ℓ *is negative or zero, then*

$$\frac{1}{\Gamma(\delta)} \int_0^1 t^{\beta-1}(1-t)^{\delta-1} {}_pM_q^{\alpha,\beta}(zt^\alpha) \, \mathrm{d}t = {}_{p+1}F_q(1, c_1, \cdots, c_p; d_1, \cdots, d_q; z) * E_{\alpha,\beta+\delta}(z). \tag{4.1}$$

Proof. Applying (1.9) and in view of (1.6) and (1.8), we have

$${}_pM_q^{\alpha,\beta+\delta}(z) = {}_{p+1}F_q(1, c_1, \cdots, c_p; d_1, \cdots, d_q; z) * E_{\alpha,\beta+\delta}(z), \tag{4.2}$$

use of (4.2) in Theorem 3.1, leads to proof of the Theorem 4.1. □

Following similar lines as of Theorem 4.1, Theorems 3.3 to 3.21 can easily be represented in the Hadamard product of two power series as:

Theorem 4.2. *If* $\alpha, \beta, \delta, \lambda \in \mathbb{C}, z \in \mathbb{C}, c_\tau, d_\ell \in \mathbb{C}, (\tau = \overline{1,p}; \ell = \overline{1,q}), \Re(\alpha) > 0, \Re(\beta) > 0, \Re(\delta) > 0$, *and no* d_ℓ *is negative or zero, then*

$$\frac{1}{\Gamma(\delta)} \int_t^z (z-s)^{\delta-1}(s-t)^{\beta-1} {}_pM_q^{\alpha,\beta}(\lambda(s-t)^\alpha)\,\mathrm{d}s \\ = (z-t)^{\beta+\delta-1} {}_{p+1}F_q(1, c_1, \cdots, c_p; d_1, \cdots, d_q; \lambda(z-t)^\alpha) * E_{\alpha,\beta+\delta}(\lambda(z-t)^\alpha). \tag{4.3}$$

Theorem 4.3. *If* $\alpha, \beta \in \mathbb{C}, z \in \mathbb{C}, c_\tau, d_\ell \in \mathbb{C}, (\tau = \overline{1,p}; \ell = \overline{1,q}), \Re(\alpha) > 0, \Re(\beta) > 0$ *and no* d_ℓ *is negative or zero, then*

$$\frac{1}{\Gamma(\alpha)} \int_0^z t^{\beta-1}(z-t)^{\alpha-1} {}_pM_q^{\alpha,\beta}(\lambda t^\alpha)\,\mathrm{d}t \\ = z^{\alpha+\beta-1} {}_{p+1}F_q(1, c_1, \cdots, c_p; d_1, \cdots, d_q; \lambda z^\alpha) * E_{\alpha,\beta+\delta}(\lambda z^\alpha). \tag{4.4}$$

Theorem 4.4. *If* $\alpha, \beta, \mu, \lambda \in \mathbb{C}, z \in \mathbb{C}, c_\tau, d_\ell \in \mathbb{C}, (\tau = \overline{1,p}; \ell = \overline{1,q}), \Re(\alpha) > 0, \Re(\beta) > 0, \Re(\mu) > 0, \Re(\lambda) > 0$ *and no* d_ℓ *is negative or zero, then*

$$\int_0^t z^{\mu-1}(t-z)^{\lambda-1} {}_pM_q^{\alpha,\beta}(cz^m)\,\mathrm{d}z = B(\mu,\lambda)t^{\mu+\lambda-1} {}_pM_q^{\alpha,\beta}(ct^m) * {}_{m+1}F_m(P'_1; Q'_1; ct^m), \tag{4.5}$$

where

$$P'_1 = 1, \frac{\mu}{m}, \frac{\mu+1}{m}, \ldots\ldots\ldots\ldots, \frac{\mu+m-1}{m},$$
$$Q'_1 = \frac{\mu+\lambda}{m}, \frac{\mu+\lambda+1}{m}, \ldots, \frac{\mu+\lambda+m-1}{m}.$$

Theorem 4.5. *If* $\alpha, \beta, \mu, \lambda \in \mathbb{C}, z \in \mathbb{C}, c_\tau, d_\ell \in \mathbb{C}, (\tau = \overline{1,p}; \ell = \overline{1,q}), \Re(\alpha) > 0, \Re(\beta) > 0, \Re(\mu) > 0, \Re(\lambda) > 0$ *and no* d_ℓ *is negative or zero, then*

$$\int_0^t z^{\mu-1}(t-z)^{\lambda-1} {}_pM_q^{\alpha,\beta}(c(t-z)^s)\,\mathrm{d}z = B(\mu,\lambda)t^{\mu+\lambda-1} {}_pM_q^{\alpha,\beta}(ct^s) * {}_{s+1}F_s(P'_2; Q'_2; ct^s), \tag{4.6}$$

where

$$P'_2 = 1, \frac{\lambda}{s}, \frac{\lambda+1}{s}, \ldots\ldots\ldots\ldots, \frac{\lambda+s-1}{s},$$
$$Q'_2 = \frac{\mu+\lambda}{s}, \frac{\mu+\lambda+1}{s}, \ldots, \frac{\mu+\lambda+s-1}{s}.$$

Theorem 4.6. *If* $\alpha, \beta, \mu, \lambda \in \mathbb{C}, z \in \mathbb{C}, c_\tau, d_\ell \in \mathbb{C}, (\tau = \overline{1,p}; \ell = \overline{1,q}), \Re(\alpha) > 0, \Re(\beta) > 0, \Re(\mu) > 0, \Re(\lambda) > 0$ *and no* d_ℓ *is negative or zero, then*

$$\int_0^t z^{\mu-1}(t-z)^{\lambda-1} {}_pM_q^{\alpha,\beta}\left(cz^m(t-z)^s\right)\mathrm{d}z \\ = B(\mu,\lambda)t^{\mu+\lambda-1} {}_pM_q^{\alpha,\beta}\left(\frac{m^m s^s ct^{m+s}}{(m+s)^{m+s}}\right) * {}_{m+s+1}F_{m+s}\left(P'_3; Q'_3; \frac{m^m s^s ct^{m+s}}{(m+s)^{m+s}}\right), \tag{4.7}$$

where

$$P'_3 = 1, \frac{\mu}{m}, \frac{\mu+1}{m}, \ldots, \frac{\mu+m-1}{m}, \frac{\lambda}{s}, \frac{\lambda+1}{s}, \ldots, \frac{\lambda+s-1}{s},$$
$$Q'_3 = \frac{\mu+\lambda}{m+s}, \frac{\mu+\lambda+1}{m+s}, \ldots\ldots\ldots\ldots\ldots, \frac{\mu+\lambda+m+s-1}{m+s}.$$

Theorem 4.7. *If* $\alpha, \beta, \mu, \lambda \in \mathbb{C}, z \in \mathbb{C}, c_\tau, d_\ell \in \mathbb{C}, (\tau = \overline{1,p}; \ell = \overline{1,q}), \Re(\alpha) > 0, \Re(\beta) > 0, \Re(\mu) > 0, \Re(\lambda) > 0$ *and no* d_ℓ *is negative or zero, then*

$$\int_0^1 x^{\mu-1}(1-x)^{2(\mu+\lambda)-1}\left(1-\frac{x}{3}\right)^{2\mu-1}\left(1-\frac{x}{4}\right)^{(\mu+\lambda)-1} {}_pM_q^{\alpha,\beta}\left(zx\left(1-\frac{x}{3}\right)^2\right)\mathrm{d}x$$
$$= \left(\frac{2}{3}\right)^{2\mu} B(\mu, \mu+\lambda)\, {}_{p+2}F_{q+1}\left(1, c_1, ..., c_p, \mu; d_1, ..., d_q, 2\mu+\lambda; \frac{4z}{9}\right) * E_{\alpha,\beta}\left(\frac{4z}{9}\right). \tag{4.8}$$

Theorem 4.8. *If* $\alpha, \beta, \mu, \lambda \in \mathbb{C}, z \in \mathbb{C}, c_\tau, d_\ell \in \mathbb{C}, (\tau = \overline{1,p}; \ell = \overline{1,q}), \Re(\alpha) > 0, \Re(\beta) > 0, \Re(\mu) > 0, \Re(\lambda) > 0$ *and no* d_ℓ *is negative or zero, then*

$$\int_0^1 x^{\mu+\lambda-1}(1-x)^{2\mu-1}\left(1-\frac{x}{3}\right)^{2(\mu+\lambda)-1}\left(1-\frac{x}{4}\right)^{\mu-1} {}_pM_q^{\alpha,\beta}\left(z\left(1-\frac{x}{4}\right)(1-x)^2\right)\mathrm{d}x$$
$$= \left(\frac{2}{3}\right)^{2(\mu+\lambda)} B(\mu, \mu+\lambda)\, {}_{p+2}F_{q+1}(1, c_1, ..., c_p, \mu; d_1, ..., d_q, 2\mu+\lambda; z) * E_{\alpha,\beta}(z). \tag{4.9}$$

Theorem 4.9. *If* $\alpha, \beta, \mu, \lambda \in \mathbb{C}, z \in \mathbb{C}, c_\tau, d_\ell \in \mathbb{C}, (\tau = \overline{1,p}; \ell = \overline{1,q}), \Re(\alpha) > 0, \Re(\beta) > 0, \Re(\mu) > 0, \Re(\lambda) > 0$ *and no* d_ℓ *is negative or zero, then*

$$\int_0^1\int_0^1 y^{\mu}(1-x)^{\mu-1}(1-y)^{\lambda-1}(1-xy)^{1-\mu-\lambda} {}_pM_q^{\alpha,\beta}\left(\frac{z(1-y)}{1-xy}\right)\mathrm{d}x\,\mathrm{d}y$$
$$= B(\mu, \lambda)_{p+2}F_{q+1}(1, c_1, ..., c_p, \lambda; d_1, ..., d_q, \mu+\lambda; z) * E_{\alpha,\beta}(z). \tag{4.10}$$

Theorem 4.10. *If* $\alpha, \beta, \mu, \lambda \in \mathbb{C}, z \in \mathbb{C}, c_\tau, d_\ell \in \mathbb{C}, (\tau = \overline{1,p}; \ell = \overline{1,q}), \Re(\alpha) > 0, \Re(\beta) > 0, \Re(\mu) > 0, \Re(\lambda) > 0$ *and no* d_ℓ *is negative or zero, then*

$$\int_0^1\int_0^1 y^{\mu}(1-x)^{\mu-1}(1-y)^{\lambda-1}(1-xy)^{1-\mu-\lambda} {}_pM_q^{\alpha,\beta}\left(\frac{zy(1-x)}{1-xy}\right)\mathrm{d}x\,\mathrm{d}y$$
$$= B(\mu, \lambda)_{p+2}F_{q+1}(1, c_1, ..., c_p, \mu; d_1, ..., d_q, \mu+\lambda; z) * E_{\alpha,\beta}(z). \tag{4.11}$$

Theorem 4.11. *If* $\alpha, \beta, \mu, \lambda \in \mathbb{C}, z \in \mathbb{C}, c_\tau, d_\ell \in \mathbb{C}, (\tau = \overline{1,p}; \ell = \overline{1,q}), \Re(\alpha) > 0, \Re(\beta) > 0, \Re(\mu) > 0, \Re(\lambda) > 0$ *and no* d_ℓ *is negative or zero, then*

$$\int_0^1\int_0^1 y^{\mu}(1-x)^{\mu-1}(1-y)^{\lambda-1}(1-xy)^{1-\mu-\lambda} {}_pM_q^{\alpha,\beta}\left(\frac{zy(1-x)(1-y)}{(1-xy)^2}\right)\mathrm{d}x\,\mathrm{d}y$$
$$= B(\mu, \lambda)_pM_q^{\alpha,\beta}\left(\frac{z}{4}\right) * {}_3F_2\left(1, \mu, \lambda; \frac{\mu+\lambda}{2}, \frac{\mu+\lambda+1}{2}; \frac{z}{4}\right). \tag{4.12}$$

5 Conclusion

In this study, we have developed various integral formulas pertaining to generalized M-series, which are expressed in terms of M-series itself. Due to the general nature of M-series, the results obtained in this article can be easily converted in terms of various classical special functions (such as the Mittag-Leffler function) and trigonometrical functions after suitable parametric replacements. Further, We have represented integrals in the Hadamard product of two known power series. The formulas obtained in this paper may be of some interest to researchers in related areas.

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