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CONFORMAL SEMI-SLANT SUBMERSIONS FROM LOCALLY PRODUCT RIEMANNIAN MANIFOLD

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Abstract This article study the geometry of conformal semi-slant submersion from a locally product Riemannian manifold onto a Riemannian manifold. We provide non-trivial examples to make sure the existence of such submersions. We obtain the integrability conditions and study the leaves of the geometry of the distributions. Product theorems for the total manifold as well as for the fibres are also given. Moreover, we give the sufficient conditions for a conformal semi-slant submersions to be totally geodesic map.

1 Introduction

Let M and B be two Riemannian manifolds. By a Riemannian submersion we mean a C^{∞}-map $\pi : M \to B$ from M onto B such that π is of maximal rank and π_* preserves the length of horizontal vectors i.e., vectors orthogonal to the fibre $\pi^{-1}(q)$ for any $q \in B$.

The notion of Riemannian submersions between Riemannian manifolds was introduced by B. O'Neill [24]. In 1967, A. Gray also studied Riemannian submersions independently [15]. Riemannian submersions have many applications in Mathematics and Physics as well. (We refer to [9],[8],[22]). B. Watson considered Riemannian submersions between almost complex manifolds and called it almost Hermitian submersions [34]. He has shown that the horizontal and vertical distributions are invariant under the behavior of the almost complex structure of the total space.

B. Sahin studied Riemannian submersions from almost Hermitian manifolds to a Riemannian manifold under the name anti-invariant Riemannian submersion. He assumed that the fibres of these submersions are anti-invariant under the action of the almost complex structure of the total space, which in turn implies that the horizontal distribution is not invariant under the action of the almost complex struture.

It is noted that the geometry of anti-invariant Riemmanian submersions is quite different from the geometry of almost Hermitian submersions. Almost Hermition submersions are useful for describing the geometry of base manifolds, anti-invariant submersions are however serve to determine the geometry of the total manifold. Afterwards, several new submersions from almost Hermitian manifolds onto a Riemannian manifold are defined according to the conditions imposed on fibres of such submersions. For the details readers can go through [28], [29], [30].

On the other hand, the horizontally conformal submersions, which are in fact a natural generalization of Riemannian submersions, were introduced by B.Fuglede [13] and T. Ishihara [21], separately. The notion of conformal holomorphic submersions, a generalization of holomorphic submersions were defined by S. Gudmundsson and J. C. Wood [16]. B. Sahin and M. A. Akyol defined conformal anti-invariant submersions from almost Hermitian manifold onto a Riemannian manifold [2]. In the continuation, M. A. Akyol [6] defined conformal semi-slant submersion as a generalization of conformal anti-invariant, conformal semi-invariant and conformal slant submersions. For conformal submersions, see [3], [4], [5].

We now consider semi-slant submersion from locally product Riemannian manifold onto Riemannian manifold. Conformal semi-slant submersions are abbreviated as CSS submersion as well as locally product Riemannian manifold as l.p.R. manifold. Section 2 is primarily devoted to the summary of known results which will be used in the remaining portion of the paper.

Section 3 gives the integrability conditions of the distributions involved in the definition of CSS submersion. Section 4 deals with the geometry of leaves of the distribution and we also mention product theorems for the fibres of the submersion and for the total space as well. Finally sufficient conditions are given for CSS submersions to be totally geodesic map.

2 Conformal submersions

This section recalls the fundamental prerequisites of the locally product Riemannian (l.p.R.) manifold and summarizes known results of Riemannian submersions and horizontally conformal submersions which will be used in sequel. Generally, we will be using the terminology of B. O'Neill [24] which establishes a basic paper on the subject of submersion.

An m-dimensional manifold M equipped with a (1,1) tensor field $\mathcal{F}$ such that

$$\mathcal{F}^2 = I, (\mathcal{F} \neq I)$$

is called an almost product manifold with almost product structure $\mathcal{F}$. We set

$$\mathcal{P} = \frac{1}{2}(I + \mathcal{F}), \;\; \mathcal{Q} = \frac{1}{2}(I - \mathcal{F}).$$

Then

$$\mathcal{P} + \mathcal{Q} = I, \;\; \mathcal{P}^2 = \mathcal{P}, \;\; \mathcal{Q}^2 = Q, \;\; \mathcal{P}\mathcal{Q} = \mathcal{Q}\mathcal{P} = 0, \;\; \mathcal{F} = \mathcal{P} - \mathcal{Q}.$$

Thus, $\mathcal{P}$ and $\mathcal{Q}$ define two complementary distributions. We can easily see that $+1$ or -1 are the only eigenvalues of $\mathcal{F}$. An eigenvector corresponding to the eigenvalue $+1$ is in $\mathcal{P}$ and eigen vector corresponding to $-1$is in $\mathcal{Q}$. Thus, if $\mathcal{F}$ has eigenvalue $+1$ of multiplicity p and eigenvalue -1 of multiplicity q, then the dimension of $\mathcal{P}$ is p and that of $\mathcal{Q}$ is q. Conversely, if there exists two complementary distribution $\mathcal{P}$ and $\mathcal{Q}$ of dimension p and q respectively, in M where $p + q = m$ and $p, q \geq 1$. Then we define an almost product structure $\mathcal{F}$ on M by $\mathcal{F} = \mathcal{P} - \mathcal{Q}$. An almost product manifold M admitting a Riemannian metric g such that

$$g(\mathcal{F}E, \mathcal{F}F) = g(E, F), \tag{2.1}$$

for any vector fields E and F on M, is called an almost product Riemannian manifold $(M, g, \mathcal{F})$.

The manifold $(M, g, \mathcal{F})$ is called a l.p.R. manifold if $\mathcal{F}$ is parallel with respect to ∇, i.e.,

$$\nabla_E \mathcal{F} = 0, \;\; E \in \Gamma(TM), \tag{2.2}$$

where ∇ denotes the Levi-Civita connection on M with respect to g [35].

In the theory of submersion, S. Gudmundsson and J. C. Wood introduced horizontally conformal submersion which is defined as follows:

Consider a $C^\infty-$differential map $\pi : (M, g) \to (B, h)$, where (M, g) and (B, h) are two Riemannian manifolds. Then, the map π is called horizontally conformal at a point $p \in M$ if $(\pi_*)_p$ maps horizontal space $\mathcal{H}(T_p(M))$ conformally onto $T_{(\pi_*)_p}B$. That is, $(\pi_*)_p$ is surjective and for any horizontal vector fields Z, W on M

$$\lambda^2(p) g(Z, W) = h((\pi_*)_p Z, (\pi_*)_p W). \tag{2.3}$$

In (2.3), $\lambda(p)$ is called the dilation of the map π at p [15], [34]. Moreover, if (2.3) is satisfied for all $p \in M$, then the map π is called horizontally conformal on M.

Let $\pi : M \to B$ be a submersion. A vector field Z on M is basic if Z is π-related to Z' on B and in this case Z' has a unique horizontal lift on M.

The B.O'Neill's fundamental tensors of a Riemannian submersion, which are represented by $\mathcal{A}$ and $\mathcal{T}$ play an important role in submersion theory to that of the second fundamental form of an immersion. These tensors are defined by

$$\mathcal{A}_E F = \mathcal{V}\nabla_{\mathcal{H}E}\mathcal{H}F + \mathcal{H}\nabla_{\mathcal{H}E}\mathcal{V}F \tag{2.4}$$

$$\mathcal{T}_E F = \mathcal{H}\nabla_{\mathcal{V}E}\mathcal{V}F + \mathcal{V}\nabla_{\mathcal{V}E}\mathcal{H}F, \tag{2.5}$$

for any $E, F \in \Gamma(TM)$, where $\mathcal{V}E$ and $\mathcal{H}E$ represents the vertical and horizontal projections of E. $\mathcal{A}_E$ and $\mathcal{T}_E$ are skew symmetric operators on TM reversing the horizontal and vertical distributions. It is easily seen that $\mathcal{T}$ and $\mathcal{A}$ are vertical and horizontal, respectively and satisfy the following:

$$\mathcal{T}_\eta \xi = \mathcal{T}_\xi \eta \tag{2.6}$$

$$\mathcal{A}_Z W = -\mathcal{A}_W Z = \frac{1}{2}\mathcal{V}[Z, W] \tag{2.7}$$

for any $Z, W \in \Gamma(ker\pi_*)^\perp$ and $\eta, \xi \in \Gamma(ker\pi_*)$. Since $\mathcal{T}$ is skew symmetric operator, we note that π has totally geodesic fibres if and only if $\mathcal{T}$ vanishes identically i.e., $\mathcal{T} \equiv 0$.
Thus, from (2.4) and (2.5), we have

$$\nabla_\eta \xi = \mathcal{T}_\eta \xi + \hat{\nabla}_\eta \xi \tag{2.8}$$

$$\nabla_\eta Z = \mathcal{H}(\nabla_\eta Z) + \mathcal{T}_\eta Z \tag{2.9}$$

$$\nabla_Z \eta = \mathcal{A}_Z \eta + \mathcal{V}\nabla_Z \eta \tag{2.10}$$

$$\nabla_Z W = \mathcal{H}(\nabla_Z W) + \mathcal{A}_Z W, \tag{2.11}$$

for any $Z, W \in \Gamma(ker\pi_*)^\perp$ and $\eta, \xi \in \Gamma(ker\pi_*)$, where $\hat{\nabla}_\eta \xi = \mathcal{V}(\nabla_\eta \xi)$. Moreover, if Z is basic, then $\mathcal{H}(\nabla_\eta Z) = \mathcal{A}_Z \eta$.

For a smooth map $\pi : (M, g) \to (B, h)$ between two Riemannian manifolds (M, g) and (B, h), the second fundamental form of π is defined by

$$(\nabla \pi_*)(E, F) = \nabla^\pi_E \pi_*(F) - \pi_*(\nabla_E F) \tag{2.12}$$

for any $E, F \in \Gamma(TM)$, where ∇^π is the pullback connection. It is well known that the second fundamental form is symmetric. Here we mention the following lemma;

Lemma 2.1. *[33] Let (M, g) and (B, h) be Riemannian manifolds and suppose that $\varphi : M \to B$ is a smooth map between them.*

$$\nabla^\varphi_E \varphi_*(F) - \nabla^\varphi_F \varphi_*(E) - \varphi_*([E, F]) = 0 \tag{2.13}$$

for any $E, F \in \Gamma(TM)$.

We conclude this section by recalling the following lemma for horizontally conformal submersion;

Lemma 2.2. *[7] Let $\pi : M \to B$ be a horizontally conformal submersion. Then,*

(i) $(\nabla \pi_*)(Z, W) = Z(\ln \lambda)\pi_* W + W(\ln \lambda)\pi_* Z - g(Z, W)\pi_*(grad \ln \lambda)$*;*

(ii) $(\nabla \pi_*)(\eta, \xi) = -\pi_*(\mathcal{T}_\eta \xi)$*;*

(iii) $(\nabla \pi_*)(Z, \eta) = -\pi_*(\nabla^M_Z \eta) = -\pi_*(\mathcal{A}_Z \eta)$*,*

for any horizontal vector fields Z, W and vertical fields η, ξ on M.

3 Conformal semi-slant submersion

This section assesses the study of CSS submersions as the total space of the submersions are l.p.R. manifold. We define the CSS submersions and provide a non-trivial example to assure the existence of such submersion.

Definition 3.1. Let $(M, g, \mathcal{F})$ be a l.p.R. manifold with the product structure $\mathcal{F}$ and (B, h) be a Riemannian manifold. Consider a horizontally conformal submersion $\pi : (M, g, \mathcal{F}) \to (B, h)$. Then π is called CSS submersion if there is a distribution $\mathcal{D} \subseteq \ker \pi_*$ such that

$$ker\pi_* = \mathcal{D} \oplus \mathcal{D}^\theta, \quad J(\mathcal{D}) = \mathcal{D}, \tag{3.1}$$

where $\mathcal{D}^\theta$ is the orthogonal complement of $\mathcal{D}$ in $ker\pi_*$ and the angle $\theta = \theta(\xi)$ between $\mathcal{F}\xi$ and the space $(\mathcal{D}^\theta)_p$ is constant for non-zero $\xi \in (\mathcal{D}^\theta)_p$, $p \in M$. The angle θ is called the semi-slant angle of the submersion.

It can be easily seen that fibers $\pi^{-1}(q)$, $q \in B$, of the submersion are semi-slant submanifold of M. (Readers can go through [26] for semi- slant submersions).

Let π be a CSS submersion from a l.p.R. manifold $(M, g, \mathcal{F})$ onto a Riemannian manifold (B, h). We can easily decompose any $\eta \in \Gamma(\ker \pi_*)$ in the following manner;

$$\eta = \mathcal{P}\eta + \mathcal{Q}\eta, \tag{3.2}$$

where $\mathcal{P}\eta \in \Gamma(\mathcal{D})$ and $\mathcal{Q}\eta \in \Gamma(\mathcal{D}^\theta)$ and also

$$\mathcal{F}\eta = \alpha\eta + \beta\eta, \tag{3.3}$$

where $\alpha\eta \in \Gamma(ker\pi_*)$ and $\beta\eta \in \Gamma((\ker \pi_*)^\perp)$. Similarly,

$$\mathcal{F}Z = BZ + CZ, \tag{3.4}$$

for any $Z \in (ker\pi_*)^\perp$, where $BZ \in \Gamma(ker\pi_*)$ and $CZ \in \Gamma(\ker \pi_*)^\perp$. Then, $(ker\pi_*)^\perp$ is decomposed as

$$(\ker \pi_*)^\perp = \beta\mathcal{D}^\theta \oplus \nu, \tag{3.5}$$

where ν is the orthogonal complement of $\beta\mathcal{D}^\theta$ in $(\ker \pi_*)^\perp$ and invariant under the almost product manifold $\mathcal{F}$.

Example 3.2. [2] Every conformal semi-invariant submersion whose total space is l.p.R. manifold is a CSS submersion with the semi-slant angle $\theta = \frac{\pi}{2}$.

Consider an Euclidean space $\mathbb{R}^{2m}$ with coordinates $(u_1, u_2, \ldots, u_{2m})$. We can canonically choose an almost product structure $\mathcal{F}$ on $\mathbb{R}^{2m}$ as follows:

$$\begin{aligned}\mathcal{F}\left(a_1\frac{\partial}{\partial u_1} + a_2\frac{\partial}{\partial u_2} + \ldots + a_{2m-1}\frac{\partial}{\partial u_{2m-1}} + a_{2m}\frac{\partial}{\partial u_{2m}}\right) &= a_2\frac{\partial}{\partial u_1} + a_1\frac{\partial}{\partial u_2} + a_4\frac{\partial}{\partial u_3} \\ &+ a_3\frac{\partial}{\partial u_4} + \ldots + a_{2m}\frac{\partial}{\partial u_{2m-1}} + a_{2m-1}\frac{\partial}{\partial u_{2m}},\end{aligned} \tag{3.6}$$

where $a_1, a_2, \ldots, a_{2m}$ are real valued $\mathbb{C}^\infty-$functions defined on $\mathbb{R}^{2m}$.

Example 3.3. Consider a map $\pi : \mathbb{R}^8 \to \mathbb{R}^2$ such that

$$\pi(u_1, u_2, \ldots, u_8) = e^{17}\left(\frac{u_1 - u_3}{\sqrt{2}}, u_4\right).$$

Then it follows that

$$(ker\pi_*) = \left\langle V_1 = \frac{\partial}{\partial u_2}, V_2 = \frac{1}{\sqrt{2}}\left(\frac{\partial}{\partial u_1} + \frac{\partial}{\partial u_3}\right), V_3 = \frac{\partial}{\partial u_5}, V_4 = \frac{\partial}{\partial u_6}, V_5 = \frac{\partial}{\partial u_7}, V_6 = \frac{\partial}{\partial u_8}\right\rangle$$

and $(ker\pi_*)^\perp = \left\langle H_1 = \frac{1}{\sqrt{2}}\left(\frac{\partial}{\partial u_1} - \frac{\partial}{\partial u_3}\right), H_2 = \frac{\partial}{\partial u_4}\right\rangle$.
Thus, π is a CSS submerion from a l.p.R. manifold $(R^8, g_{\mathbb{R}^8}, \mathcal{F})$ to a Riemannian manifold $(\mathbb{R}^2, g_{\mathbb{R}^2})$ with the slant angle $\theta = \frac{\pi}{4}$ and $\lambda = e^{17}$, where the distribution $\mathcal{D}$ and $\mathcal{D}_\theta$ are $\mathcal{D} = \left\langle V_3 = \frac{\partial}{\partial u_5}, V_4 = \frac{\partial}{\partial u_6}, V_5 = \frac{\partial}{\partial u_7}, V_7 = \frac{\partial}{\partial u_8}\right\rangle$ and $\mathcal{D}_\theta = \left\langle V_1 = \frac{\partial}{\partial u_2}, V_2 = \frac{\partial}{\partial u_1} + \frac{\partial}{\partial u_3}\right\rangle$, respectively.

Example 3.4. Consider a map $\pi : \mathbb{R}^{10} \to \mathbb{R}^4$ such that

$$\pi(u_1, u_2, \ldots, u_{10}) = e^8\left(\frac{u_4 - u_6}{\sqrt{2}}, u_9, \frac{u_5 - u_7}{\sqrt{2}}, u_{10}\right).$$

Then the map π is a CSS submersion with the distributions,

$$\mathcal{D} = \left\langle V_1 = \frac{\partial}{\partial u_1}, V_2 = \frac{\partial}{\partial u_2}\right\rangle$$

$$\mathcal{D}_\theta = \left\langle V_3 = \frac{\partial}{\partial u_3}, V_4 = \frac{1}{\sqrt{2}}\left(\frac{\partial}{\partial u_4} + \frac{\partial}{\partial u_6}\right), V_5 = \frac{\partial}{\partial u_8}, V_6 = \frac{1}{\sqrt{2}}\left(\frac{\partial}{\partial u_5} - \frac{\partial}{\partial u_7}\right)\right\rangle$$

and $(ker\pi_*)^\perp = \left\langle H_1 = \frac{1}{\sqrt{2}}\left(\frac{\partial}{\partial u_4} - \frac{\partial}{\partial u_6}\right) H_2 = \frac{\partial}{\partial u_9}, H_3 = \frac{1}{\sqrt{2}}\left(\frac{\partial}{\partial u_5} - \frac{\partial}{\partial u_7}\right), H_4 = \frac{\partial}{\partial u_{10}}\right\rangle$,
with the slant angle $\theta = \frac{\pi}{4}$ and $\lambda = e^8$.

We start with the preliminary results of CSS submersions which will be of future use.

Proposition 3.5. *Let $\pi : (M, g, \mathcal{F}) \to (B, h)$ be a CSS submersion from a l.p.R. manifold $(M, g, \mathcal{F})$ onto a Riemannian manifold (B, h). Then*

$(i)\ \alpha\mathcal{D} = \mathcal{D}$, $(ii)\ \beta\mathcal{D} = 0$, $(iii)\ \alpha\mathcal{D}^\theta \subset \mathcal{D}^\theta$, $(iv)\ B(\ker\pi_*)^\perp = \mathcal{D}^\theta$,

$(v)\ \alpha^2 + B\beta = -id$, $(vi)\ C^2 + \beta B = -id$, $(vii)\ \beta\alpha + C\beta = 0$, $(viii)\ BC + \alpha B = 0$.

Proof. These can easily be obtained with the help of (3.3), (3.4) and (3.5). □

For any $\eta, \ \xi \in \Gamma(ker\pi_*)$, using (2.8), (2.9), (3.3) and (3.4), we have the covariant derivative of α and β as follows;

$$(\nabla_\eta\alpha)\xi = B\mathcal{T}_\eta\xi - \mathcal{T}_\eta\beta\xi \tag{3.7}$$

$$(\nabla_\eta\beta)\xi = C\mathcal{T}_\eta\xi - \mathcal{T}_\eta\alpha\xi \tag{3.8}$$

$$(\nabla_\eta\alpha)\xi = \hat{\nabla}_\eta\alpha\xi - \alpha\hat{\nabla}_\eta\xi \tag{3.9}$$

$$(\nabla_\eta\beta)\xi = \mathcal{A}_{\beta\xi}\eta - \beta\hat{\nabla}_\eta\xi, \tag{3.10}$$

(3.11)

In view of (2.8)-(2.11), (3.3) and (3.4), we have

Lemma 3.6. *Let $\pi : (M, g, \mathcal{F}) \to (B, h)$ be a CSS submersion from a l.p.R. manifold $(M, g, \mathcal{F})$ onto a Riemannian manifold (B, h). Then*

(a) $\mathcal{A}_Z BW + \mathcal{H}\nabla_Z CW = C\mathcal{H}\nabla_Z W + \beta\mathcal{A}_Z W$
$\mathcal{V}\nabla_Z BW + \mathcal{A}_Z CW = B\mathcal{H}\nabla_Z W + \alpha\mathcal{A}_Z W,$

(b) $\mathcal{T}_\eta\alpha\xi + \mathcal{A}_{\beta\xi}\eta = C\mathcal{T}_\eta\xi + \beta\hat{\nabla}_\eta\xi$
$\hat{\nabla}_\eta\alpha\xi + \mathcal{T}_\eta\beta\xi = B\mathcal{T}_{W\eta}\xi + \alpha\hat{\nabla}_\eta\xi,$

(c) $\mathcal{A}_Z\alpha\eta + \mathcal{H}\nabla_\eta\beta\xi = C\mathcal{A}_Z\eta + \beta\mathcal{V}\nabla_Z\eta$
$\mathcal{V}\nabla_Z\alpha\eta + \mathcal{A}_Z\beta\eta = B\mathcal{A}_Z\eta + \alpha\mathcal{V}\nabla_Z\eta,$

for any $\eta, \xi \in \Gamma(\ker\pi_*)$ *and* $Z, W \in \Gamma(\ker\pi_*)^\perp$.

The next results give some important relations for semi-slant angle θ. These kind of results seem to have originated from B. Y. Chen [11].

Proposition 3.7. *Let* π *be a CSS submersion from a l.p.R. manifold* $(M, g, \mathcal{F})$ *onto a Riemannian manifold* (B, h)*. Then*

$$\alpha^2\xi = \cos^2\theta\xi, \quad \xi \in \Gamma(\mathcal{D}^\theta),$$

where θ *denotes the semi-slant angle of* $\mathcal{D}^\theta$.

Proof. Since, $\cos\theta = \dfrac{g(\mathcal{F}\xi, \alpha\xi)}{|\mathcal{F}\xi||\alpha\xi|}$. By using (3.6) and (3.3), we get

$$cos\theta = \frac{g\left(\xi, \alpha^2\xi\right)}{|\xi||\alpha\xi|} \tag{3.12}$$

Also,

$$\cos\theta = \frac{|\alpha\xi|}{|\mathcal{F}\xi|}, \tag{3.13}$$

By using (3.12) and (3.13), we get

$$\cos^2\theta = \frac{g\left(\xi, \alpha^2\xi\right)}{|\xi|^2}, \text{ for any } \xi \in \Gamma(\mathcal{D}^\theta).$$

Hence, the result follows. □

Corollary 3.8. *Let* π *be a CSS submersion from a l.p.R. manifold* $(M, g, \mathcal{F})$ *onto a Riemannian manofold* (B, h)*. Then*

$$g(\alpha\xi_1, \alpha\xi_2) = \cos^2\theta g(\xi_1, \xi_2)$$
$$g(\beta\xi_1, \beta\xi_2) = \sin^2\theta g(\xi_1, \xi_2),$$

for any $\xi_1, \xi_2 \in \mathcal{D}^\theta$.

Proof. Applying (3.6), (3.3) and Proposition 3.7, result follows. □

Proposition 3.5 and Proposition 3.7 allow us to state;

Corollary 3.9. *Let* $\pi : (M, g, \mathcal{F}) \to (B, h)$ *be a CSS submersion from a l.p.R. manifold* $(M, g, \mathcal{F})$ *onto a Riemannian manifold* (B, h)*. Then, there exists a constant* $k \in [0, 1]$ *such that* $B\beta = kI$*, where* $k = sin^2\theta$ *and* θ *is the semi-slant angle of* $\mathcal{D}^\theta$.

Following lemma is crucial for our work.

Lemma 3.10. *Let* $(M, g, \mathcal{F})$ *be a l.p.R. manifold and* (B, h)*, a Riemannian manifold and let* $\pi : (M, g, \mathcal{F}) \to (B, h)$ *be a CSS submersion. Then*

$$\mathcal{T}_{\alpha\xi_1}\alpha\xi_1 = \cos^2\theta\mathcal{T}_{\xi_1}\xi_1, \textit{ for any } \xi_1 \in \Gamma(\mathcal{D}^\theta),$$

if β *is parallel with respect to* ∇ *on* $\mathcal{D}^\theta$.

Proof. In (3.8), if β is parallel, we get $C\mathcal{T}_{\xi_1}\xi_2 = \mathcal{T}_{\xi_1}\alpha\xi_2$, for $\xi_1, \xi_2 \in \Gamma(\mathcal{D}^\theta)$. By interchanging ξ_1 and ξ_2, we get $C\mathcal{T}_{\xi_2}\xi_1 = \mathcal{T}_{\xi_2}\alpha\xi_1$. Therefore,

$$C\mathcal{T}_{\xi_1}\xi_2 - C\mathcal{T}_{\xi_2}\xi_1 = \mathcal{T}_{\xi_1}\alpha\xi_2 - \mathcal{T}_{\xi_2}\alpha\xi_1.$$

Using (2.6), we get

$$\mathcal{T}_{\xi_1}\alpha\xi_2 = \mathcal{T}_{\xi_2}\alpha\xi_1. \tag{3.14}$$

By replacing ξ_2 by $\alpha\xi_1$, (3.14) changes to $\mathcal{T}_{\xi_1}\alpha^2\xi_1 = \mathcal{T}_{\alpha\xi_1}\alpha\xi_1$. Immediate from Proposition (3.7), result follows. □

We proceed now to the main result of this section where we prove the equivalent conditions for the integrability of the distributions $\mathcal{D}$ and $\mathcal{D}^\theta$.

Theorem 3.11. *Let $\pi : (M, g, \mathcal{F}) \to (B, h)$ be a CSS submersion from almost product manifold $(M, g, \mathcal{F})$ onto a Riemannian manifold (B, h). Then the following conditions are equivalent to each other;*

(a) The distribution $\mathcal{D}$ is integrable.

(b) $h((\nabla\pi_*)(\eta_1, \alpha\eta_2) - (\nabla\pi_*)(\eta_2, \alpha\eta_1), \pi_*\beta\xi) = \lambda^2 g(\alpha(\hat{\nabla}_{\eta_1}\alpha\eta_2 - \hat{\nabla}_{\eta_2}\alpha\eta_1), \xi)$,

(c) β is parallel and $(\hat{\nabla}_{\eta_1}\alpha\eta_2 - \hat{\nabla}_{\eta_2}\alpha\eta_1) \in \Gamma(\mathcal{D})$,

for any $\eta_1, \eta_2 \in \Gamma(\mathcal{D})$ and $\xi \in \Gamma(\mathcal{D}^\theta)$.

Proof. We note that $\mathcal{D}$ is integrable if and only if $g([\eta_1, \eta_2], \xi) = 0$, for any $\eta_1, \eta_2 \in \Gamma(\mathcal{D})$, $\xi \in \Gamma(\mathcal{D}^{\theta)}$ as well as $g([\eta_1, \eta_2], W) = 0$, for any $W \in \Gamma(\ker\pi_*)^\perp$. Since $\ker\pi_*$ is always integrable, we can easily obtain $g([\eta_1, \eta_2], W) = 0$. Furthermore, applying (3.6), (2.2), (2.8) and (3.3)

$$g([\eta_1, \eta_2], \xi) = g(\mathcal{H}\nabla_{\eta_1}\alpha\eta_2, \beta\xi) + g(\hat{\nabla}_{\eta_1}\alpha\eta_2, \alpha\xi) - g(\mathcal{H}\nabla_{\eta_2}\alpha\eta_1, \beta\xi) - g(\hat{\nabla}_{\eta_2}\alpha\eta_1, \alpha\xi).$$

Meanwhile, Using (3.3) and Lemma 2.2, we get

$$\begin{aligned} g([\eta_1, \eta_2], \xi) &= \lambda^{-2} h(-(\nabla\pi_*)(\eta_1, \alpha\eta_2) + \nabla^\pi_\eta \pi_*\alpha\eta_2, \pi_*\beta\xi) - \lambda^{-2} h(-(\nabla\pi_*)(\eta_2, \alpha\eta_1) \\ &\quad + \nabla^\pi_\xi \pi_*\alpha\eta_1, \pi_*\beta\xi) + g(\alpha(\hat{\nabla}_{\eta_1}\alpha\eta_2 - \hat{\nabla}_{\eta_2}\alpha\eta_1), \xi) \\ &= \lambda^{-2} h((\nabla\pi_*)(\eta_2, \alpha\eta_1) - (\nabla\pi_*)(\eta_1, \alpha\eta_2), \pi_*\beta\xi) + g(\alpha(\hat{\nabla}_{\eta_1}\alpha\eta_2 - \hat{\nabla}_{\eta_2}\alpha\eta_1), \xi). \end{aligned} \tag{3.15}$$

Hence, $(a) \Leftrightarrow (b)$ follows.

In addition with Lemma 2.2 and (2.3), (3.15) reduces to

$$g([\eta_1, \eta_2], \xi) = -g(\mathcal{T}_{\eta_2}\alpha\eta_1 - \mathcal{T}_{\eta_1}\alpha\eta_2, \beta\xi) + g(\alpha(\hat{\nabla}_{\eta_1}\alpha\eta_2 - \hat{\nabla}_{\eta_2}\alpha\eta_1), \xi)$$

In effect of (3.8) and Proposition 3.5, we easily obtain $(a) \Leftrightarrow (b)$.

□

Theorem 3.12. *Let $(M, g, \mathcal{F})$ be a l.p.R. manifold and (B, h) a Riemannian manifold and $\pi : (M, g, \mathcal{F}) \to (B, h)$ be a CSS submersion. Then the following conditions are equivalent to each other;*

$$(a) \mathcal{D}^\theta is integrable.$$

$$(b) h((\nabla\pi_*)(\xi_1, \alpha\xi_2) + \nabla\pi_*)(\xi_2, \alpha\xi_1), \eta) = -\lambda^2 g(\mathcal{T}_{\xi_1}\beta\xi_2 + \mathcal{T}_{\xi_2}\beta\xi_1, \eta),$$

$$(c) \mathcal{T}_{\xi_1}\beta\xi_2 - \mathcal{T}_{\xi_2}\beta\xi_1 + \alpha(\mathcal{T}_{\xi_1}\beta\alpha\xi_2 - \mathcal{T}_{\xi_2}\beta\alpha\xi_1) \in \Gamma(\mathcal{D}^\theta)$$

for any $\eta \in \Gamma(\mathcal{D})$ and $\xi_1, \xi_2 \in \Gamma(\mathcal{D}^\theta)$.

Proof. For the distribution $\mathcal{D}^\theta$ to be integrable, it is necessary and sufficient that $g([\xi_1, \xi_2], \mathcal{F}\eta) = 0$, as well as $g([\xi_1, \xi_2], W) = 0$, for any $\xi_1, \xi_2 \in \Gamma(\mathcal{D}^{\theta)}$, $\eta \in \Gamma(\mathcal{D})$ and $W \in \Gamma(\ker\pi_*)^\perp$. Since the distribution $\ker\pi_*$ is integrable, then we can easily get $g([\xi_1, \xi_2], W) = 0$.

In addition, using (3.6), (2.2), (3.3) and (2.12), we obtain

$$\begin{aligned} g([\xi_1, \xi_2], \mathcal{F}\eta) &= g(\nabla_{\xi_1}\alpha\xi_2, \eta) + g(\nabla_{\xi_1}\beta\xi_2, \eta) + g(\nabla_{\xi_2}\alpha\xi_1, \eta) + g(\nabla_{\xi_2}\beta\xi_1, \eta). \\ &= \lambda^{-2} h((\nabla\pi_*)(\xi_1, \alpha\xi_2) + (\nabla\pi_*)(\xi_2, \alpha\xi_1), \eta) + g(\mathcal{T}_{\xi_1}\beta\xi_2 + \mathcal{T}_{\xi_2}\beta\xi_1, \eta), \end{aligned}$$

which gives $(a) \Leftrightarrow (b)$.

Alternatively, using (3.6), (2.2) and (3.3) we arrive

$$g([\xi_1,\xi_2],\mathcal{F}\eta) = g(\nabla_{\xi_1}\alpha\xi_2,\eta) + g(\nabla_{\xi_1}\beta\xi_2,\eta) + g(\nabla_{\xi_2}\alpha\xi_1,\eta) + g(\nabla_{\xi_2}\beta\xi_1,\eta).$$

Again by using (3.3), we get

$$\begin{aligned} g([\xi_1,\xi_2],\mathcal{F}\eta) &= g(\nabla_{\xi_1}\alpha^2\xi_2,\mathcal{F}\eta) + g(\nabla_{\xi 1}\beta\alpha\xi_2,\mathcal{F}\eta) + g(\nabla_{\xi_1}\beta\xi_2,\eta) \\ &\quad - g(\nabla_{\xi_2}\alpha^2\xi_1,\mathcal{F}\eta) - g(\nabla_{\xi_2}\beta\alpha\xi_1,\mathcal{F}\eta) - g(\nabla_{\xi_2}\beta\xi_1,\eta). \end{aligned}$$

From (2.9) and Proposition (3.7), we get

$$g([\xi_1,\xi_2],\mathcal{F}\eta) = \cos^2\theta g([\xi_1,\xi_2],\mathcal{F}\eta) + g(-\mathcal{T}_{\xi_2}\beta\xi_1 + \mathcal{T}_{\xi_1}\beta\xi_2,\eta) + g(\alpha(\mathcal{T}_{\xi_1}\beta\alpha\xi_2 - \mathcal{T}_{\xi_2}\beta\alpha\xi_1),\eta),$$

which implies that

$$\sin^2\theta g([\xi_1,\xi_2],\mathcal{F}\eta) = g(-\mathcal{T}_{\xi_2}\beta\xi_1 + \mathcal{T}_{\xi_1}\beta\xi_2,\eta) + g(\alpha(\mathcal{T}_{\xi_1}\beta\alpha\xi_2 - \mathcal{T}_{\xi_2}\beta\alpha\xi_1),\eta).$$

which gives $(a) \Leftrightarrow (b)$ as $sin^2\theta$ is non-zero for all $\theta \in \left(0, \frac{\pi}{2}\right)$. □

Meanwhile, we study the integrability of the horizontal distribution $(ker\pi_*)^\perp$, whereas it known that vertical distribution $\Gamma(ker\pi_*)$ is always integrable.

Theorem 3.13. *Let $\pi : (M,g,\mathcal{F}) \to (B,h)$ be a CSS submersion from a l.p.R. manifold $(M,g,\mathcal{F})$ onto a Riemannian manifold (B,h). Then the distribution $(ker\pi_*)^\perp$ is integrable if and only if*

$$\begin{aligned} \lambda^{-2}h(\nabla^\pi_W\pi_*(CZ) - \nabla^\pi_Z\pi_*(CW),\pi_*\beta\xi) &= g(\mathcal{A}_W BZ - \mathcal{A}_Z BW - CW(\ln\lambda)Z \\ &\quad + CZ(\ln\lambda)W + 2g(Z,CW)grad(\ln\lambda),\beta\xi) \\ &\quad - g(\alpha(\mathcal{V}\nabla_W BZ - \mathcal{V}\nabla_Z BW + \mathcal{A}_W CZ - \mathcal{A}_Z CW),\xi) \end{aligned}$$

for any $\eta \in \Gamma(\mathcal{D})$, $\xi \in \Gamma(\mathcal{D}^\theta)$ and $Z,W \in \Gamma(ker\pi_*)^\perp$.

Proof. For any $Z,W \in \Gamma(ker\pi_*)^\perp$ and $\xi \in \Gamma(ker\pi_*)$, applying (3.6) and (2.2) and (3.4), we have

$$g([Z,W],\xi) = g(\nabla_Z BW,\mathcal{F}\xi) + g(\nabla_Z CW,\mathcal{F}\xi) - g(\nabla_W BZ,\mathcal{F}\xi) - g(\nabla_W CZ,\mathcal{F}\xi).$$

In view of (2.10), (2.11) and (3.3), preceding equation yields

$$\begin{aligned} g([Z,W],\xi) &= g(\mathcal{V}\nabla_Z BW - \mathcal{V}\nabla_W BZ + \mathcal{A}_Z CW - \mathcal{A}_W CZ,\alpha\xi) + g(\mathcal{H}\nabla_Z BW,\beta\xi) \\ &\quad - g(\mathcal{H}\nabla_W BZ,\beta\xi) + g(\mathcal{H}\nabla_Z CW,\beta\xi) - g(\mathcal{H}\nabla_W CZ,\beta\xi). \end{aligned}$$

Again appealing to (2.12), (2.3), (2.10) and Lemma 2.2, we have

$$\begin{aligned} g([Z,W],\xi) &= g(\mathcal{V}\nabla_Z BW - \mathcal{V}\nabla_W BZ + \mathcal{A}_Z CW - \mathcal{A}_W CZ,\alpha\xi) \\ &\quad - g(\mathcal{A}_Z BW,\beta\xi) + g(\mathcal{A}_W BZ,\beta\xi) \\ &\quad - \lambda^{-2}h(grad\ln\lambda,Z)h(\pi_*CW,\pi_*\beta\xi) - \lambda^{-2}g(grad\ln\lambda,CW)h(\pi_*Z,\pi_*\beta\xi) \\ &\quad + \lambda^{-2}g(Z,CW)h(\pi_*(grad\ln\lambda),\pi_*\beta\xi) + \lambda^{-2}h(\nabla^\pi_Z\pi_*CW,\pi_*\beta\xi) \\ &\quad + \lambda^{-2}g(grad\ln\lambda,W)h(\pi_*CZ,\pi_*\beta\xi) + \lambda^{-2}g(grad\ln\lambda,CZ)h(\pi_*W,\pi_*\beta\xi) \\ &\quad - \lambda^{-2}g(W,CZ)h(\pi_*(grad\ln\lambda),\pi_*\beta\xi) - \lambda^{-2}h(\nabla^\pi_W\pi_*CZ,\pi_*\beta\xi). \end{aligned}$$

Moreover, we take into account that π is CSS submersion we get

$$\begin{aligned} g([Z,W],\xi) &= g(\mathcal{A}_W BZ - \mathcal{A}_Z BW - CW(\ln\lambda)Z + CZ(\ln\lambda)W + 2g(Z,CW)grad\ln\lambda,\beta\xi) \\ &\quad + g(\xi\nabla_Z BW - \mathcal{V}\nabla_W BZ + \mathcal{A}_Z CW - \mathcal{A}_W CZ,\alpha\xi) \\ &\quad - \lambda^{-2}h(\nabla^\pi_W\pi_*CZ - \nabla^\pi_Z\pi_*CW,\pi_*\beta\xi), \end{aligned} \tag{3.16}$$

which proves our assertion.

□

We call a horizontally conformal submersion is homethetic if the gradient of its dilation λ is vertical, i.e.,

$$\mathcal{H}(grad(\lambda) = 0.$$

Now, we assume $(ker\pi_*)^{\perp}$ is integrable and in addition

$$h(\nabla^{\pi}_{W}\pi_* CZ - \nabla^{\pi}_{Z}\pi_* CW, \pi_*\beta\xi) = \lambda^2\{g(\mathcal{A}_W BZ - \mathcal{A}_Z BW, \beta\xi) + g(\mathcal{V}\nabla_Z BW - \mathcal{V}\nabla_W BZ + \mathcal{A}_Z CW - \mathcal{A}_W CZ, \alpha\xi)\}$$

for any $\xi \in \Gamma(\ker\pi_*)$ and $Z, W \in \Gamma(\ker\pi_*)^{\perp}$. Consequently, (3.16) changes to

$$g(-g(grad\ln\lambda, CW)Z + g(grad\ln\lambda, CZ)W + 2g(Z, CW)grad\ln\lambda, \beta\xi) = 0. \qquad (3.17)$$

Since $\beta\xi \in \Gamma(ker\pi_*)^{\perp}$ for any $\xi \in \Gamma(\mathcal{D}^{\theta})$, we take $W = \beta\xi$ in (3.17) and obtain

$$g(grad\ln\lambda, CZ)g(\beta\xi, \beta\xi) = 0.$$

As a result we get that λ is a constant on $\Gamma(\nu)$. However, if we assume $W = CZ$ in (3.17) for $Z \in \Gamma(\nu)$ it yields

$$2g(Z, C^2Z)g(grad\ln\lambda, \beta\xi) = 2g(Z, Z)g(grad\ln\lambda, \beta\xi) = 0,$$

which means λ is a constant on $\Gamma(\beta\mathcal{D}^{\theta})$. Converse can easily be shown on the same lines. On the whole we state the following result;

Theorem 3.14. *Let π be a CSS submersion from a l.p.R. manifold $(M, g, \mathcal{F})$ onto a Riemannian manifold (B, h). Then any two conditions below imply the third:*

(i) $ker\pi_*$ *is integrable.*

(ii) π *is a horizontally homothetic map.*

(iii) $h(\nabla^{\pi}_{W}\pi_* CZ - \nabla^{\pi}_{Z}\pi_* CW, \pi_*\beta\xi) = \lambda^2\{g(\mathcal{A}_W BZ - \mathcal{A}_Z BW, \beta\xi) + g(\mathcal{V}\nabla_Z BW - \mathcal{V}\nabla_W BZ + \mathcal{A}_Z CW - \mathcal{A}_W CZ, \alpha\xi)\}$

for any $\xi \in \Gamma(\ker\pi_*)$ and $Z, W \in \Gamma(\ker\pi_*)^{\perp}$.

4 Study of leaves of the distributions

This section deals with the geometry of the leaves of the distributions. Also we give some decomposition theorems for the fibres of the submersion as well as the total manifold. We begin with the distribtuion $\mathcal{D}$;

Theorem 4.1. *Let π be a CSS submersion from a l.p.R. manifold $(M, g, \mathcal{F})$ onto a Riemannian manifold (B, h). Then, the following three assertions are equivalent to each other;*

(a) *The distribution $\mathcal{D}$ defines a totally geodesic foliation on M.*

(b) $g(\hat{\nabla}_{\eta_1}\alpha\eta_2, \alpha\xi_1) + g(\mathcal{T}_{\eta_1}\alpha\eta_2, \beta\xi_1) = 0,$
$g(\nabla_{\eta_1}\mathcal{F}BZ, \eta_2) - g(\mathcal{T}_{\eta_1}\mathcal{F}\eta_2, CZ) = 0.$

(c) $h((\nabla\pi_*)(\eta_1, \alpha\eta_2), \pi_*\beta\xi_1) = \lambda^2 g(\hat{\nabla}_{\eta_1}\alpha\eta_2, \alpha\xi_1)$ *and*
$h((\nabla\pi_*)(\eta_1, \alpha\eta_2), \pi_* CZ) = -\lambda^2 g(\hat{\nabla}_{\eta_1}\alpha BZ + \mathcal{T}_{\eta_1}\beta BZ, \eta_2),$

for any $\eta_1, \eta_2 \in \Gamma(\mathcal{D})$, $\xi_1 \in \Gamma(\mathcal{D}^{\theta})$ and $Z \in \Gamma(ker\pi_*)^{\perp}$.

Proof. The distribution $\mathcal{D}$ defines a totally geodesic foliation on M if and only if $g(\nabla_{\eta_1}\eta_2, \xi_1) = 0$ and $g(\nabla_{\eta_1}\eta_2, Z) = 0$ for any $\eta_1, \eta_2 \in \Gamma(\mathcal{D})$, $\xi_1 \in \Gamma(\mathcal{D}^{\theta})$ and $Z \in \Gamma((ker\pi_*)^{\perp})$. In view of (3.6) and (2.2) and Proposition (3.5), we have

$$\begin{aligned} g(\nabla_{\eta_1}\eta_2, \xi_1) &= g(\nabla_{\eta_1}\mathcal{F}\eta_2, \alpha\xi_1) + g(\nabla_{\eta_1}\mathcal{F}\eta_2, \beta\xi_1) \\ &= g(\hat{\nabla}_{\eta_1}\alpha\eta_2, \alpha\xi_1) + g(\mathcal{H}\nabla_{\eta_1}\alpha\eta_2, \beta\xi_1). \end{aligned} \tag{4.1}$$

Furthermore, using (2.12), we get

$$g(\nabla_{\eta_1}\eta_2, \xi_1) = g(\hat{\nabla}_{\eta_1}\alpha\eta_2, \alpha\xi_1) - \lambda^{-2}h((\nabla\pi_*)(\eta_1, \alpha\eta_2), \pi_*\beta\xi_1). \tag{4.2}$$

However, using (3.6), (2.2), (2.8) and (3.3), we derive

$$\begin{aligned} g(\nabla_{\eta_1}\eta_2, Z) &= g(\nabla_{\eta_1}\mathcal{F}\eta_2, BZ) + g(\nabla_{\eta_1}\mathcal{F}\eta_2, CZ) \\ &= -g(\eta_2, \nabla_{\eta_1}\mathcal{F}BZ) + g(\mathcal{H}\nabla_{\eta_1}\mathcal{F}\eta_2, CZ). \end{aligned} \tag{4.3}$$

By appealing to the Definition (3.1) with (2.9), (2.12), we get

$$\begin{aligned} g(\nabla_{\eta_1}\eta_2, Z) &= -g(\eta_2, \hat{\nabla}_{\eta_1}\alpha BZ) - g(\eta_2, \mathcal{T}_{\eta_1}\beta BZ) \\ &\quad - \lambda^{-2}h((\nabla\pi_*)(\eta_1, \alpha\eta_2), \pi_*CZ). \end{aligned} \tag{4.4}$$

Therefore, by the virtue of (4.1 and (4.3) we arrive at $(a) \Leftrightarrow (b)$ and with the help of (4.2) and (4.4), $(a) \Leftrightarrow (c)$ follows. □

In the same manner, we have the following theorem for the distribution $\mathcal{D}^{\theta}$;

Theorem 4.2. *Let $(M, g, \mathcal{F})$ be a l.p.R. manifold and (B, h), a Riemannian manifold. Let $\pi : (M, g, \mathcal{F}) \to (B, h)$ be a CSS submersion. Then, the following three assertions are equivalent to each other;*

(a)*The distribution $\mathcal{D}^{\theta}$ defines a totally geodesic foliation on M.*

(b) $g(\hat{\nabla}_{\xi_1}\alpha\xi_2, \eta_1) + g(\mathcal{A}_{\beta\xi_2}\xi_1, \eta_1) = 0$ *and*

$g(\nabla_{\xi_1}\alpha\xi_2 + \mathcal{T}_{\xi_1}\beta\xi_2, BZ) + g(\mathcal{T}_{\xi_1}\alpha\xi_2 + \mathcal{A}_{\beta\xi_2}\xi_1, CZ) = 0$

(c) $h((\nabla\pi_*)(\xi_1, \eta_1), \pi_*\beta\xi_2) = \lambda^2 g(\mathcal{T}_{\xi_1}\alpha\eta_1, \beta\alpha\xi_2)$ *and*

$h((\nabla\pi_*)(\xi_1, \beta\alpha\xi_2), \pi_*Z) - h(\nabla^{\pi}_{\beta\xi_2}\pi_*\beta\xi_1, \pi_*\mathcal{F}CZ)$

$= \lambda^2\{g(\mathcal{A}_{\beta\xi_2}\alpha\xi_1 + g(\beta\xi_1, \beta\xi_2)grad\ln\lambda, \mathcal{F}CZ) + g(\mathcal{T}_{\xi_1}\beta\xi_2, BZ)\}$,

for any $\eta_1, \eta_2 \in \Gamma(\mathcal{D})$, $\xi_1, \xi_2 \in \Gamma(\mathcal{D}^{\theta})$ and $Z \in \Gamma(ker\pi_*)^{\perp}$.

Proof. The distribution $\mathcal{D}^{\theta}$ defines a totally geodesic foliation on M if and only if $g(\nabla_{\xi_1}\xi_2, \mathcal{F}\eta_1) = 0$ and $g(\nabla_{\xi_1}\xi_2, Z) = 0$ for any $\xi_1, \xi_2 \in \Gamma(\mathcal{D}^{\theta})$, $\eta_1 \in \Gamma(\mathcal{D})$ and $Z \in \Gamma(ker\pi_*)^{\perp}$. By using (3.6), (2.2) and (3.3) we get

$$\begin{aligned} g(\nabla_{\xi_1}\xi_2, \mathcal{F}\eta_1) &= g(\nabla_{\xi_1}\alpha\xi_2, \eta_1) + g(\nabla_{\xi_1}\beta\xi_2, \eta_1) \\ &= g(\nabla_{\xi_1}\mathcal{F}\alpha\xi_2, \mathcal{F}\eta_1) - g(\beta\xi_2, \nabla_{\xi_1}\eta_1) \\ &= g(\nabla_{\xi_1}\alpha^2\xi_2, \mathcal{F}\eta_1) + g(\nabla_{\xi_1}\omega\alpha\xi_2, \alpha\eta_1) - g(\beta\xi_2, \nabla_{\xi_1}\eta_1). \end{aligned} \tag{4.5}$$

Moreover, utilizing (2.9), (2.12), (3.3) and Lemma 3.10, we obtain

$$\sin^2\theta g(\nabla_{\xi_1}\xi_2, \mathcal{F}\eta_1) = -g(\mathcal{T}_{\xi_1}\alpha\eta_1, \beta\alpha\xi_2) + \lambda^{-2}g_B((\nabla\pi_*)(\xi_1, \eta_1), \pi_*\beta\xi_2). \tag{4.6}$$

Whereas, using (3.6), (2.2), (3.3) and (3.4), expression (4.6) turns to

$$\begin{aligned} g(\nabla_{\xi_1}\xi_2, Z) &= g(\nabla_{\xi_1}\alpha\xi_2, \mathcal{F}Z) + g(\nabla_{\xi_1}\beta\xi_2, \mathcal{F}Z) \\ &= g(\nabla_{\xi_1}\alpha^2\xi_2, Z) + g(\nabla_{\xi_1}\beta\alpha\xi_2, Z) \\ &\quad + g(\nabla_{\xi_1}\beta\xi_2, BZ) + g(\nabla_{\beta\xi_2}\mathcal{F}\xi_1, \mathcal{F}CZ). \end{aligned} \tag{4.7}$$

By using (2.8), (2.9), (2.10), Proposition 3.7 and Lemma 2.2, we arrive at

$$\begin{aligned} g(\nabla_{\xi_1}\xi_2, Z) &= \cos^2\theta g(\nabla_{\xi_1}\xi_2, Z) + g(\mathcal{T}_{\xi_1}\beta\xi_2, BZ) + g(\mathcal{A}_{\beta\xi_2}\alpha\xi_1, \mathcal{F}CZ) \\ &- \lambda^{-2} g_B((\nabla\pi_*)(\xi_1, \beta\alpha\xi_2), \pi_* Z) \\ &- g(grad\ln\lambda, \beta\xi_2) g(\beta\xi_1, \mathcal{F}CZ) - g(grad\ln\lambda, \beta\xi_1) g(\beta\xi_2, \mathcal{F}CZ) \\ &+ g(\beta\xi_2, \beta\xi_1) g(grad\ln\lambda, \mathcal{F}CZ) + \lambda^{-2} g_B(\nabla^{\pi}_{\beta\xi_2}\pi_*\beta\xi_1, \pi_*\mathcal{F}CZ). \end{aligned}$$

Therefore,

$$\begin{aligned} \sin^2\theta g(\nabla_{\xi_1}\xi_2, Z) &= g(\mathcal{T}_{\xi_1}\beta\xi_2, BZ) + g(\mathcal{A}_{\beta\xi_2}\alpha\xi_1, \mathcal{F}CZ) \\ &+ g(\beta\xi_2, \beta\xi_1) g(\mathcal{H} grad\ln\lambda, \mathcal{F}CZ) \\ &- \lambda^{-2}\{h((\nabla\pi_*)(\xi_1, \beta\alpha\xi_2), \pi_* Z) + h(\nabla^{\pi}_{\beta\xi_2}\pi_*\beta\xi_1, \pi_*\mathcal{F}CZ)\}. \end{aligned} \quad (4.8)$$

Thus, from (4.5) and (4.7) we obtain $(a) \Leftrightarrow (b)$ whereas $(a) \Leftrightarrow (c)$ follows from (4.6) and (4.8). □

We remark that the manifold $M = M_1 \times M_2$ is called a usual product of Riemannian manifold if and only if $\mathcal{D}_{M_1}$ and $\mathcal{D}_{M_2}$ are totally geodesic foliations, where $\mathcal{D}_{M_1}$ and $\mathcal{D}_{M_2}$ are the canonical foliations orthogonal to each other.

Immediate from Theorem 4.1 and Theorem 4.2, we have the following corollaries;

Corollary 4.3. *Let $\pi : (M, g, \mathcal{F}) \to (B, h)$ be a CSS submersion from a l.p.R. manifold $(M, g, \mathcal{F})$ onto a Riemannian manifold (B, h). Then the fibers of π are locally product manifold if and only if*

(a) $g(\hat{\nabla}_{\eta_1}\alpha\eta_2, \alpha\xi_1) + g(\mathcal{T}_{\eta_1}\alpha\eta_2, \beta\xi_1) = 0,$
$g(\mathcal{T}_{U_1}\mathcal{F}\eta_2, CZ) + g(\eta_2, \nabla_{\eta_1}\mathcal{F}BZ) = 0,$

(b) $g(\hat{\nabla}_{\xi_1}\alpha\xi_2, \eta_1) + g(\mathcal{A}_{\beta\xi_2}\xi_1, \eta_1) = 0$
$g(\nabla_{\xi_1}\alpha\xi_2 + \mathcal{T}_{\xi_1}\beta\xi_2, BZ) + g(\mathcal{T}_{\xi_1}\alpha\xi_2 + \mathcal{A}_{\beta\xi_2}\xi_1, CZ) = 0$

for any $\eta_1, \eta_2 \in \Gamma(\mathcal{D})$, $\xi_1, \xi_2 \in \Gamma(\mathcal{D}^\theta)$ and $Z \in \Gamma(ker\pi_*)^\perp$.

Corollary 4.4. *Let π be a CSS submersion from a l.p.R. $(M, g, \mathcal{F})$ onto a Riemannian manifold (B, h). Then the fibers of π are locally product manifold if and only if*

(a) $h((\nabla\pi_*)(\eta_1, \alpha\eta_2), \pi_*\beta\xi_1) = \lambda^2 g(\hat{\nabla}_{\eta_1}\alpha\eta_2, \alpha\xi_1),$
$h((\nabla\pi_*)(\eta_1, \alpha\eta_2), \pi_* CZ) = -\lambda^2 g(\hat{\nabla}_{\eta_1}\alpha BZ + \mathcal{T}_{\eta_1}\beta BZ, \eta_2).$

(b) $h((\nabla\pi_*)(\xi_1, \eta_1), \pi_*\beta\xi_2) = \lambda^2 g(\mathcal{T}_{\xi_1}\alpha\eta_1, \beta\alpha\xi_2),$
$h((\nabla\pi_*)(\xi_1, \beta\alpha\xi_2), \pi_* Z) - h(\nabla^{\pi}_{\beta\xi_2}\pi_*\beta\xi_1, \pi_*\mathcal{F}CZ)$
$= \lambda^2\{g(\mathcal{A}_{\beta\xi_2}\alpha\xi_1 + g(\beta\xi_1, \beta\xi_2) grad\ln\lambda, \mathcal{F}CZ) + g(\mathcal{T}_{\xi_1}\beta\xi_2, BZ)\},$

for any $\eta_1, \eta_2 \in \Gamma(\mathcal{D}), \xi_1, \xi_2 \in \Gamma(\mathcal{D}^\theta)$ and $Z \in \Gamma(ker\pi_*)^\perp$.

We now proceed to the necessary and sufficient conditions for the totally geodesciness of the horizontal distribution $(ker\pi_*)^\perp$.

Theorem 4.5. *Let $\pi : (M, g, \mathcal{F}) \to (B, h)$ be a CSS submersion from a l.p.R. manifold $(M, g, \mathcal{F})$ onto a Riemannian manifold (B, h). Then, the distributions $(ker\pi_*)^\perp$ defines a totally geodesic foliation on M if and only if*

$$\mathcal{A}_Z CW + \mathcal{V}\nabla_Z BW \in \Gamma(\mathcal{D}^\theta)$$

and

$$\begin{aligned}&\lambda^{-2}\{h(\nabla^{\pi}_{Z}\pi_*W,\pi_*\beta\alpha\xi)+h(\nabla^{\pi}_{Z}\pi_*CW,\pi_*\beta\xi)\}\\&=-g(\mathcal{A}_ZBW,\beta\xi)+g(grad\ln\lambda,Z)g(W,\beta\alpha\xi)+g(grad\ln\lambda,W)g(Z,\beta\alpha\xi)\\&-g(Z,W)g(grad\ln\lambda,\beta\alpha\xi)+g(grad\ln\lambda,CW)g(Z,\beta\xi)-g(Z,CW)g(grad\ln\lambda,\beta\xi)\end{aligned}$$

for any $Z,W\in\Gamma((\ker\pi_*)^{\perp})$ *and* $\xi\in\Gamma(\mathcal{D}^{\theta})$.

Proof. For any $Z,W\in\Gamma((ker\pi_*)^{\perp})$ and $\eta\in\Gamma(\mathcal{D})$, use of (3.6), (2.2), (2.10), (2.11) and (3.4) yields

$$g(\nabla_ZW,\eta)=g(\alpha(\mathcal{A}_ZCW+\mathcal{V}\nabla_ZBW),\eta). \tag{4.9}$$

In addition, using (3.6), (2.2), (3.3) and (3.4) for any $\xi\in\Gamma(\mathcal{D}^{\theta})$, we get

$$g(\nabla_ZW,\xi)=g(\nabla_ZW,\alpha^2\xi)+g(\nabla_ZW,\beta\alpha\xi)+g(\nabla_ZBW,\beta\xi)+g(\nabla_ZCW,\beta\xi).$$

Appealing to Proposition 3.7 and Lemma 2.2 and (2.11), preceding equation in turn yields

$$\begin{aligned}g(\nabla_ZW,\xi)=\cos^2\theta g(\nabla_ZW,\xi)&+g(\mathcal{A}_ZBW,\beta\xi)-g(grad\ln\lambda,Z)g(W,\beta\alpha\xi)\\&-g(grad\ln\lambda,W)g(Z,\beta\alpha\xi)+g(Z,W)g(grad\ln\lambda,\beta\alpha\xi)\\&+\lambda^{-2}g_B(\nabla^{\pi}_{Z}\pi_*W,\pi_*\beta\alpha\xi)\\&-g(grad\ln\lambda,Z)g(CW,\beta\xi)-g(grad\ln\lambda,CW)g(Z,\beta\xi)\\&+g(Z,CW)g(grad\ln\lambda,\beta\xi)+\lambda^{-2}g_B(\nabla^{\pi}_{Z}\pi_*CW,\pi_*\beta\xi).\end{aligned}$$

Thus,

$$\begin{aligned}\sin^2\theta g(\nabla_ZW,\xi)=&\,g(\mathcal{A}_ZBW,\beta\xi)-g(grad\ln\lambda,Z)g(W,\beta\alpha\xi)\\&-g(grad\ln\lambda,W)g(Z,\beta\alpha\xi)\\&+g(Z,W)g(grad\ln\lambda,\beta\alpha\xi)-g(grad\ln\lambda,CW)g(Z,\beta\xi)\\&+g(Z,CW)g(grad\ln\lambda,\beta\xi)\\&+\lambda^{-2}\{h(\nabla^{\pi}_{Z}\pi_*W,\pi_*\beta\alpha\xi)-h(\nabla^{\pi}_{Z}\pi_*CW,\pi_*\beta\xi)\}.\end{aligned} \tag{4.10}$$

Hence, proof follows from (4.9) and (4.10). □

On the same line, we discuss the geometry of leaves of the vertical distribution $ker\pi_*$.

Theorem 4.6. *Let* $\pi:(M,g,\mathcal{F})\to(B,h)$ *be a CSS submersion, where* $(M,g,\mathcal{F})$ *is a l.p.R. manifold and* (B,h) *is a Riemannian manifold. Then, the distributions* $(ker\pi_*)$ *defines a totally geodesic foliation on* M *if and only if*

$$\begin{aligned}h(\nabla^{\pi}_{\beta\xi}\pi_*\beta\mathcal{F}CZ,\pi_*\beta\eta)=&\,\lambda^2g(\beta(\hat{\nabla}_\eta\alpha\xi+\mathcal{T}_\eta\beta\xi),Z)+g(\mathcal{T}_\eta\alpha\xi,CZ)\\&-g(\mathcal{A}_{\beta\xi}\mathcal{F}CZ,\alpha\eta)+g(\beta\xi,\beta\eta)g(\mathcal{H}grad(\ln\lambda),\mathcal{F}CZ)\end{aligned}$$

for any $\eta,\xi\in\Gamma(ker\pi_*)$ *and* $Z\in\Gamma(ker\pi_*)^{\perp}$.

Proof. For any $\eta,\xi\in\Gamma(ker\pi_*)$ and the horizontal vector field Z, using (3.6), (2.2), (3.3) and (3.4), we obtain

$$g(\nabla_\eta\xi,Z)=g(\nabla_\eta\alpha\xi,BZ)+g(\nabla_\eta\alpha\xi,CZ)+g(\nabla_\eta\beta\xi,BZ)+g(\nabla_\eta\beta\xi,CZ).$$

In view of (2.8) and (2.9), we have

$$\begin{aligned}g(\nabla_\eta\xi,Z)=&\,g(\beta(\hat{\nabla}_\eta\alpha\xi+\mathcal{T}_\eta\beta\xi),Z)+g(\mathcal{T}_\eta\alpha\xi,CZ)\\&-g(\mathcal{A}_{\beta\xi}\mathcal{F}CZ,\alpha\eta)-g(\nabla_{\beta\xi}\mathcal{F}CZ,\beta\eta).\end{aligned}$$

Moreover, using (2.12) and Lemma (2.2), above equation yields

$$g(\nabla_\eta \xi, Z) = g(\beta(\hat{\nabla}_\eta \alpha\xi + \mathcal{T}_\eta \beta\xi), Z) + g(\mathcal{T}_\eta \alpha\xi, CZ) - g(\mathcal{A}_{\beta\xi}\mathcal{F}CZ, \alpha\eta)$$
$$+ \lambda^{-2} g(\mathcal{H}grad \ln\lambda, \beta\xi)\lambda^2 g(\beta\eta, \mathcal{F}CZ) + \lambda^{-2} g(\beta\xi, \beta\eta)\lambda^2 g(\mathcal{H}grad \ln\lambda, \mathcal{F}CZ)$$
$$- \lambda^{-2} g(\mathcal{H}grad \ln\lambda, \beta\eta)\lambda^2 g(\beta\xi, \mathcal{F}CZ) - \lambda^{-2} h(\nabla^\pi_{\beta\xi}\pi_* \mathcal{F}CZ, \pi_*\beta\eta).$$

With the fact that π is a CSS submersion, we reach out the expression

$$g_M(\nabla_\eta \xi, Z) = g(\omega(\hat{\nabla}_\eta \alpha\xi + \mathcal{T}_\eta \beta\xi), Z) + g(\mathcal{T}_\eta \alpha\xi, CZ) - g(\mathcal{A}_{\beta\xi}\mathcal{F}CZ, \alpha\eta)$$
$$+ g(\beta\xi, \beta\eta) g(\mathcal{H}grad(\ln\lambda), \mathcal{F}CZ) - \lambda^{-2} h(\nabla^\pi_{\beta\xi}\pi_* \mathcal{F}CZ, \pi_*\beta\eta). \tag{4.11}$$

Thus, proof follows from (4.11). □

Consequently, we deduce the following;

Theorem 4.7. *Let $\pi : (M, g, \mathcal{F}) \to (B, h)$ be a CSS submersion from a l.p.R. manifold $(M, g, \mathcal{F})$ onto a Riemannian manifold (B, h). Then any two of the following conditions imply the third;*

(i) $ker\pi_*$ *defines a totally geodesic foliation on* M_1*;*

(ii) λ *is a constant on* $\Gamma(\nu)$*;*

(iii) $\lambda^{-2} h(\nabla^\pi_{\beta\xi}\pi_* \mathcal{F}CZ \pi_*\beta\eta) = g(\beta(\hat{\nabla}_\eta \alpha\xi + \mathcal{T}_\eta \beta\xi), Z)$
$$+ g(\mathcal{T}_\eta \alpha\xi, CZ) - g(\mathcal{A}_{\beta\xi}\mathcal{F}CZ, \alpha\eta)$$

for any $\eta, \xi \in \Gamma(ker\pi_*)$ and $Z \in \Gamma((ker\pi_*)^\perp)$.

Proof. Considering (4.11) for any $\eta, \xi \in \Gamma(ker\pi_*)$ and $Z \in \Gamma((ker\pi_*)^\perp)$, , we have

$$g(\nabla_\eta \xi, Z) = g(\beta(\hat{\nabla}_\eta \alpha\xi + \mathcal{T}_\eta \beta\xi), Z) + g(\mathcal{T}_\eta \alpha\xi, CZ) - g(\mathcal{A}_{\beta\xi}\mathcal{F}CZ, \alpha\eta)$$
$$+ g(\beta\xi, \beta\eta) g(\mathcal{H}grad \ln\lambda, \mathcal{F}CZ) - \lambda^{-2} h(\nabla^\pi_{\beta\xi}\pi_* \mathcal{F}CZ, \pi_*\beta\eta).$$

Now, if we take (i) and (iii) into account, then we can easily obtain

$$g(\beta\eta, \beta\xi) g(\mathcal{H}grad \ln\lambda, FCZ) = 0,$$

which shows that λ is a constant on (ν). On the same line, (iii) can be obtained if $(ker\pi_*)$ defines totally geodesic foliation and λ is constant (ν). □

From Theorem 4.1, Theorem 4.2 and Theorem 4.5, we have the following result.

Theorem 4.8. *Let $\pi : (M, g, \mathcal{F}) \to (B, h)$ be a CSS submersion, where $(M, g, \mathcal{F})$ is a l.p.R. manifold and (B, h) is a Riemannian manifold. Then, the total space M is a locally product manifold of the leaves of $\mathcal{D}$, $\mathcal{D}^\theta$ and $(ker\pi_*)^\perp$, i.e., $M = M_\mathcal{D} \times M_{\mathcal{D}^\theta} \times M_{(ker\pi_*)^\perp}$, if and only if*

(a) $h((\nabla\pi_*)(\eta_1, \alpha\eta_2), \pi_*\beta\xi_1) = \lambda^2 g(\hat{\nabla}_{\eta_1}\alpha\eta_2, \alpha\xi_1),$
$h((\nabla\pi_*)(\eta_1, \alpha\eta_2), \pi_* CZ) = -\lambda^2 g(\hat{\nabla}_{\eta_1}\alpha BZ + \mathcal{T}_{\eta_1}\beta BZ, \eta_2).$

(b) $h((\nabla\pi_*)(\xi_1, \eta_1), \pi_*\beta\xi_2) = \lambda^2 g(\mathcal{T}_{\xi_1}\alpha\eta_1, \beta\alpha\xi_2),$
$h((\nabla\pi_*)(\xi_1, \beta\alpha\xi_2), \pi_* Z) - h(\nabla^\pi_{\beta\xi_2}\pi_*\beta\xi_1, \pi_*\mathcal{F}CZ)$
$= \lambda^2\{g(\mathcal{A}_{\beta\xi_2}\alpha\xi_1 + g(\beta\xi_1, \beta\xi_2) grad \ln\lambda, \mathcal{F}CZ) + g(\mathcal{T}_{\xi_1}\beta\xi_2, BZ)\},$

(c) $\mathcal{A}_Z CW + \mathcal{V}\nabla_Z BW \in \Gamma(\mathcal{D}^\theta),$
$\lambda^{-2}\{h(\nabla^\pi_Z \pi_* W, \pi_*\beta\alpha\xi) + h(\nabla^\pi_Z \pi_* CW, \pi_*\beta\xi)$
$= -g(\mathcal{A}_Z BW, \beta\xi) + g(grad \ln\lambda, Z) g(W, \beta\alpha\xi) + g(grad \ln\lambda, W) g(Z, \beta\alpha\xi)$
$-g(Z, W) g(grad \ln\lambda, \beta\alpha\xi) + g(grad \ln\lambda, CW) g(Z, \beta\xi) - g(Z, CW) g(grad \ln\lambda, \beta\xi).$

for any $Z, W \in \Gamma(\ker\pi_*)^{\perp}$, $\eta_1, \eta_2 \in \Gamma(\mathcal{D})$ *and* $\xi, \xi_1, \xi_2 \in \Gamma(\mathcal{D}^{\theta})$, *where* $M_{\mathcal{D}}$, $M_{\mathcal{D}^{\theta}}$ *and* $M_{(ker\pi_*)^{\perp}}$ *are the leaves of the distributions* $\mathcal{D}$, $\mathcal{D}^{\theta}$ *and* $(ker\pi_*)^{\perp}$, *respectively.*

Immediate from Theorem 4.5 and Theorem 4.6, we have

Theorem 4.9. *Let* $\pi : (M, g, \mathcal{F}) \to (B, h)$ *be a CSS submersion from a l.p.R. manifold* $(M, g, \mathcal{F})$ *onto a Riemannian manifold* (B, h). *Then, the total space* M *is a locally product manifold of the leaves of* $ker\pi_*$ *and* $(ker\pi_*)^{\perp}$, *i.e.,* $M = M_{ker\pi_*} \times M_{(ker\pi_*)^{\perp}}$, *if and only if*

(a) $h(\nabla^{\pi}_{\beta\xi}\pi_*\beta\mathcal{F}CZ, \pi_*\beta\eta) = \lambda^2 g(\beta(\hat{\nabla}_\eta\alpha\xi + \mathcal{T}_\eta\beta\xi), Z) + g(\mathcal{T}_\eta\alpha\xi, CZ)$
$-g(\mathcal{A}_{\beta\xi}\mathcal{F}CZ, \alpha\eta) + g(\beta\xi, \beta\eta)g(\mathcal{H}grad(\ln\lambda), \mathcal{F}CZ)$,
for any $\eta, \xi \in \Gamma(ker\pi_*)$ *and* $Z \in \Gamma(ker\pi_*)^{\perp}$.

(b) $\mathcal{A}_Z CW + \mathcal{V}\nabla_Z BW \in \Gamma(\mathcal{D}^{\theta})$
$\lambda^{-2}\{h(\nabla^{\pi}_Z\pi_*W, \pi_*\beta\alpha\xi) + h(\nabla^{\pi}_Z\pi_*CW, \pi_*\beta\xi)\}$
$= -g(\mathcal{A}_Z BW, \beta\xi) + g(grad\ln\lambda, Z)g(W, \beta\alpha\xi) + g(grad\ln\lambda, W)g(Z, \beta\alpha\xi)$
$-g(Z, W)g(grad\ln\lambda, \beta\alpha\xi) + g(grad\ln\lambda, CW)g(Z, \beta\xi) - g(Z, CW)g(grad\ln\lambda, \beta\xi)$,
for any $Z, W \in \Gamma((\ker\pi_*)^{\perp})$ *and* $\xi \in \Gamma(\mathcal{D}^{\theta})$.

5 Totally geodesicness of The CSS submersions

We recall that a differentiable map π between two Riemannian manifolds is called totally geodesic if $\nabla\pi_* = 0$ [7].

Following result gives the characterization for CSS submersions to be totally geodesic map.

Theorem 5.1. *Let* $\pi : (M, g, \mathcal{F}) \to (B, h)$ *be a CSS submersion from a l.p.R. manifold* $(M, g, \mathcal{F})$ *onto a Riemannian manifold* (B, h). *Then* π *defines a totally geodesic map if*

$$\nabla^{\pi}_Z\pi_*W = \pi_*(C(\mathcal{A}_Z\alpha\eta + \mathcal{H}\nabla_Z\beta\eta + \mathcal{A}_Z BW + \mathcal{H}\nabla_Z CY_2) + \beta(\mathcal{V}\nabla_Z\alpha\eta + \mathcal{A}_Z\beta\eta + \mathcal{V}\nabla_Z BW + \mathcal{A}_Z CW)) \tag{5.1}$$

for any $Z \in \Gamma((ker\pi_*)^{\perp})$ *and* $E = \eta + W \in \Gamma(TM)$, *where* $\eta \in \Gamma(ker\pi_*)$ *and* $W \in \Gamma((ker\pi_*)^{\perp})$.

Proof. For any $Z \in \Gamma(ker\pi_*)^{\perp}$ and $E \in \Gamma(TM)$, using (2.2) and (2.12), we get

$$(\nabla\pi_*)(Z, E) = \nabla^{\pi}_Z\pi_*E - \pi_*(\mathcal{F}\nabla_Z\mathcal{F}E)$$

Moreover, applying (2.10), (2.11), (3.3) and (3.4) we obtain

$$\begin{aligned}(\nabla\pi_*)(Z, E) = \nabla^{\pi}_Z\pi_*E - \pi_*(&B\mathcal{A}_Z\alpha\eta + C\mathcal{A}_Z\alpha\eta + \alpha\mathcal{V}\nabla_Z\alpha\eta + \beta\mathcal{V}\nabla_Z\alpha\eta \\ &+ \alpha\mathcal{A}_Z\beta\eta + \beta\mathcal{A}_Z\beta\eta + B\mathcal{H}\nabla_Z\beta\eta + C\mathcal{H}\nabla_Z\beta\eta \\ &+ B\mathcal{A}_Z BW + C\mathcal{A}_Z BW + \alpha\mathcal{V}\nabla_Z BW + \beta\mathcal{V}\nabla_Z BW \\ &+ \alpha\mathcal{A}_Z CW + \beta\mathcal{A}_Z CW + B\mathcal{H}\nabla_Z CW + C\mathcal{H}\nabla_Z CW),\end{aligned}$$

for any $\eta \in \Gamma(ker\pi_*)$ and $W \in \Gamma(ker\pi_*)^{\perp}$ such that $E = \eta + W \in \Gamma(TM)$. Since the vertical vector fields are π-related to zero vector field in B, therefore we get the expression

$$\begin{aligned}(\nabla\pi_*)(Z, E) = \nabla^{\pi}_Z\pi_*W - \pi_*(&C(\mathcal{A}_Z\alpha\eta + \mathcal{H}\nabla_Z\beta\eta + \mathcal{A}_Z BW + \mathcal{H}\nabla_Z CW) \\ &+ \beta(\mathcal{V}\nabla_Z\alpha\eta + \mathcal{A}_Z\beta\eta + \mathcal{V}\nabla_Z BW + \mathcal{A}_Z CW)),\end{aligned}$$

from which the result follows.

□

We remark that a CSS submersion is called a $(\beta\mathcal{D}^{\theta}, \nu)$-totally geodesic map if $(\nabla\pi_*)(\beta\xi, Z) = 0$, for any $\xi \in \Gamma(\mathcal{D}^{\theta})$ *and* $Z \in \Gamma(\nu)$.

Thus, we have the following characterization;

Theorem 5.2. *Let $\pi : (M, g, \mathcal{F}) \to (B, h)$ be a CSS submersion, where $(M, g, \mathcal{F})$ be a l.p.R and (B, h), a Riemannian manifold. Then, π defines a $(\beta\mathcal{D}^\theta, \nu)$-totally geodesic map if and only if π is horizontally homothetic map.*

Proof. By using Lemma 2.2,

$$(\nabla\pi_*)(\beta\xi, Z) = \beta\xi(\ln\lambda)\pi_* Z + Z(\ln\lambda)\pi_*\beta\xi - g(\beta\xi, Z)\pi_*(grad\ln\lambda),$$

for any vector field $\xi \in \Gamma(\mathcal{D}^\theta)$ and $Z \in \Gamma(\nu)$. Simply, we obtain $(\nabla\pi_*)(\beta\xi, Z) = 0$, if π is a horizontally homothetic map.

Nevertheless, if $(\nabla\pi_*)(\beta\xi, Z) = 0$, we have

$$\beta\xi(\ln\lambda)\pi_* Z + Z(\ln\lambda)\pi_*\beta\xi = 0. \tag{5.2}$$

By taking inner product (5.2) with $\pi_*\beta\xi$, we get

$$g(grad\ln\lambda, \beta\xi)h(\pi_* Z, \pi_*\beta\xi) + g(grad\ln\lambda, Z)h(\pi_*\beta\xi, \pi_*\beta\xi) = 0.$$

which in turn yields that λ is a constant on $\Gamma(\nu)$. Likewise, taking inner product of (5.2) with $\pi_* Z$, we obtain

$$g(grad\ln\lambda, \beta\xi)h(\pi_* Z, \pi_* Z) + g(grad\ln\lambda, Z)h(\pi_*\beta\xi, \pi_* Z) = 0.$$

Consequently, λ is a constant on $\Gamma(\beta\mathcal{D}^\theta)$. Thus, λ is a constant on $\Gamma(ker\pi_*)^\perp$, which completes the proof. □

We establish the another characterization.

Theorem 5.3. *A CSS submersion $\pi : (M, g, \mathcal{F}) \to (B, h)$ from a l.p.R. manifold $(M, g, \mathcal{F})$ onto a Riemannian manifold (B, h) is a totally geodesic map if and only if*

(a) $C\mathcal{T}_{\eta_1}\alpha\eta_2 + \beta\hat{\nabla}_{\eta_1}\alpha\eta_2 = 0$

(b) $C(\mathcal{T}_{\eta_1}\alpha\xi_1 + \mathcal{A}_{\beta\xi_1}\eta_1) + \beta(\hat{\nabla}_{\eta_1}\alpha\xi_1 + \mathcal{T}_{\eta_1}\beta\xi_1) = 0$ *and*

(c) $C(\mathcal{T}_\eta BZ + \mathcal{H}\nabla_\eta CZ) + \beta(\hat{\nabla}_\eta BZ + \mathcal{T}_\eta CZ) = 0,$

for any $\eta \in \Gamma(ker\pi_*)$, $\eta_1, \eta_2 \in \Gamma(\mathcal{D})$, $\xi_1 \in \Gamma(\mathcal{D}^\theta)$, and $Z \in \Gamma(ker\pi_*)^\perp$.

Proof. (a) For any $\eta_1, \eta_2 \in \Gamma(\mathcal{D})$, using (2.12) (2.2) and (2.8), we reach to

$$(\nabla\pi_*)(\eta_1, \eta_2) = -\pi_*(F(\mathcal{T}_{\eta_1}\alpha\eta_2 + \hat{\nabla}_{\eta_1}\alpha\eta_2)).$$

With the help of (3.3) and (3.4), further we obtain

$$(\nabla\pi_*)(\eta_1, \eta_2) = -\pi_*(B\mathcal{T}_{\eta_1}\alpha\eta_2 + C\mathcal{T}_{\eta_1}\alpha\eta_2 + \alpha\hat{\nabla}_{\eta_1}\alpha\eta_2 + \beta\hat{\nabla}_{\eta_1}\alpha\eta_2).$$

As tangential vector fields on $(\ker\pi_*)$ are $\pi-$related to zero on B, thus above equation takes the form

$$(\nabla\pi_*)(\eta_1, \eta_2) = -\pi_*(C\mathcal{T}_{\eta_1}\alpha\eta_2 + \beta\hat{\nabla}_{\eta_1}\alpha\eta_2).$$

Hence, $\nabla\pi_*)(\eta_1, \eta_2) = 0$ if and only if $(C\mathcal{T}_{\eta_1}\alpha\eta_2 + \beta\hat{\nabla}_{\eta_1}\alpha\eta_2 = 0$.

(b) For any $\eta_1 \in \Gamma(\mathcal{D}), \xi_1 \in \Gamma(\mathcal{D}^\theta)$, making use of (2.2) and (2.12) gives

$$(\nabla\pi_*)(\eta_1, \xi_1) = -\pi_*(\mathcal{F}\nabla_{\eta_1}\mathcal{F}\xi_1).$$

Moreover, on using (2.8), (2.9), (3.3) and (3.4), we obtain

$$(\nabla\pi_*)(\eta_1,\xi_1) = -\pi_*(B\mathcal{T}_{\eta_1}\alpha\xi_1 + C\mathcal{T}_{\eta_1}\alpha\xi_1 + \alpha\hat{\nabla}_{\eta_1}\alpha\xi_1 + \beta\hat{\nabla}_{\eta_1}\alpha\xi_1 + \alpha\mathcal{T}_{\eta_1}\beta\xi_1 + \beta\mathcal{T}_{\eta_1}\beta\xi_1 + B\mathcal{A}_{\beta\xi_1}\eta_1 + C\mathcal{A}_{\beta\xi_1}\eta_1).$$

Since π_* kills the vertical vector field on TM, therefore arrive at

$$(\nabla\pi_*)(\eta_1,\xi_1) = \pi_*(C(\mathcal{T}_{\eta_1}\alpha\xi_1 + \mathcal{A}_{\beta\xi_1}\eta_1) + \beta(\hat{\nabla}_{\eta_1}\alpha\xi_1 + \mathcal{T}_{\eta_1}\beta\xi_1)).$$

Thus, $(\nabla\pi_*)(\eta_1,\xi_1) = 0$ if and only if $C(\mathcal{T}_{\eta_1}\alpha\xi_1 + \mathcal{A}_{\beta\xi_1}\eta_1) + \beta(\hat{\nabla}_{\eta_1}\alpha\xi_1 + \mathcal{T}_{\eta_1}\beta\xi_1) = 0$.

(c) Now for any $\eta \in \Gamma(ker\pi_*),\ Z \in \Gamma((ker\pi_*)^\perp)$, in view of (2.2) and (2.12) we have

$$(\nabla\pi_*)(\eta, Z) = -\pi_*(\mathcal{F}\nabla_\eta\mathcal{F}Z).$$

Moreover, use of (2.8), (2.9), (3.3) and (3.4) gives

$$(\nabla\pi_*)(\eta, Z) = -\pi_*(B\mathcal{T}_\eta BZ + C\mathcal{T}_\eta BZ + \alpha\hat{\nabla}_\eta BZ + \beta\hat{\nabla}_\eta BZ + \alpha\mathcal{T}_\eta CZ + \beta\mathcal{T}_\eta CZ + B\mathcal{A}_{CZ}\eta + C\mathcal{A}_{CZ}\eta).$$

Since $B\mathcal{T}_\eta BZ + \alpha\hat{\nabla}_\eta BZ + \alpha\mathcal{T}_\eta CZ + B\mathcal{A}_{CZ}\eta \in \Gamma(ker\pi_*)$, we derive

$$(\nabla\pi_*)(\eta, Z) = -\pi_*(C(\mathcal{T}_\eta BZ + \mathcal{A}_{CZ}\eta) + \beta(\hat{\nabla}_\eta BZ + \mathcal{T}_\eta CZ)).$$

Therefore, we obtain $(\nabla\pi_*)(\eta, Z) = 0$ if and only if $C(\mathcal{T}_\eta BZ + \mathcal{A}_{CZ}\eta) + \beta(\hat{\nabla}_\eta BZ + \mathcal{T}_\eta CZ) = 0$. □

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