Palestine Journal of Mathematics

Vol. 12(4)(2023) , 306–313

Extended Metric Space of Type (γ, β) and Some Fixed Point Results

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Communicated by Thabet Abdeljawad

MSC 2010 Classifications: Primary 20M99, 13F10; Secondary 13A15, 13M05.

Keywords and phrases: fixed point, extended metric space of type (γ, β), extended $b-metric$ space.

Abstract We initiate a new function, referred to as a controlled metric space of type (γ, β), through the use of the functions $\gamma, \beta : M \times M \to M$ to play a key role in the triangular inequality. Our new concept is to extend the idea of the extended $b-$metric space in the sense of Kamran et al. to a more general type. We base and prove new results on fixed points in such spaces. Also, we give some examples to reinforce our concept and related results..

1 Introduction

The subject of fixed point theorems is one of the most beneficial subjects that help the scientists to demonstrate an existence solution of such models in applied sciences by converting them to suitable forms. The concept of metric space has been stretched in many directions. Baktain [1] and Czerwik [2] expanded the concept of metric spaces to the concept of $b-$metric space and enhanced the Banach contraction theorem [3]. Then, many researchers have benefited from the concept of $b-$metric spaces to demonstrate and enhance many results in fixed point. Abdeljawad et al. [4] extended some results in fixed points to partial $b-$metric spaces. Roshan et al. [5] studied some fixed point theorem in ordered $b-$metric spaces. Also, Shatanawi et al. [6, 7] benefited from comparison functions to obtain new results on $b-$metric spaces. In 2018, Huang et al. [8] applied some fixed point theorems on $b-$metric spaces to differential equations.

In 2017, Kamran et al. [9] enhanced the motif of $b-$metric space to the motif of extended $b-$metric spaces and enhanced the Banach contraction theorem for such spaces. Recently, Mukheimer et al. [10] promoted some results from $b-$metric spaces to extended $b-$metric spaces and they supported their results with non-trivial examples. Huang et al. [11] began studying some results on extended $b-$metric spaces for contractions of rational forms. Recently, Mlaiki et al [12] initiated studying fixed point results in their new space "controlled metric type space".

Henceforth, M stands for a non-empty set.

Definition 1.1. [9] Let $\theta : M \times M \to [1, \infty)$. The function $\rho : M \times M \to [0, \infty)$ is defined as an extended $b-$metric space if $\forall\, \zeta, \varphi, \varrho \in M$, we have

(i) $\rho(\zeta, \varphi) = 0 \iff \zeta = \varphi$,

(ii) $\rho(\zeta, \varphi) = \rho(\varphi, \zeta)$,

(iii) $\rho(\zeta, \varphi) \leq \theta(\zeta, \varphi)[\rho(\zeta, \varrho) + \rho(\varrho, \varphi)]$.

The pair (M, ρ) is referred to as an extended $b-$metric space.

Some examples for (M, ρ) are stated here:

Example 1.2. For $M = [0, \infty)$, set $\theta : M \times M \to [1, \infty)$ and $\rho : M \times M \to [0, \infty)$ via $\theta(\zeta_1, \zeta_2) = 1 + \zeta_1 + \zeta_2$, for all $\zeta_1, \zeta_2 \in M$, and

$$\rho(\zeta_1, \zeta_2) = \begin{cases} \zeta_1 + \zeta_2, & for\ all\ \zeta_1, \zeta_2 \in M\ \zeta_1 \neq \zeta_2, \\ 0, & \zeta_1 = \zeta_2. \end{cases}$$

Example 1.3. For $M = [0, \infty)$, set $\theta : M \times M \to [1, \infty)$ and $\rho : M \times M \to [0, \infty)$ via $\theta(\zeta_1, \zeta_2) = \frac{3+\zeta_1+\zeta_2}{2}$, for all $\zeta_1, \zeta_2 \in M$, and

$\rho(\zeta_1, \zeta_2) = 0$, for all $\zeta_1, \zeta_2 \in M$, $\zeta_1 = \zeta_2$,
$\rho(\zeta_1, \zeta_2) = \rho(\zeta_2, \zeta_1) = 5$, for all $\zeta_, \zeta_2 \in M - \{0\}$, $\zeta_1 \neq \zeta_2$,
$\rho(\zeta_1, 0) = \rho(0, \zeta_1) = 2$, for all $\zeta_1 \in M - \{0\}$.

In our current work, we enhance the concept of extended $b-$metric spaces to a more general form by making use of two specific functions on the set $M \times M$ to play a key role in triangular inequality. Also, we formulate some examples to reinforce our concept and related results.

2 Mains results

We commence this section by introducing this definition:

Definition 2.1. On M, consider $\gamma, \beta : M \times M \to [1, \infty)$. An extended metric of type (γ, β) is a function $\nu : M \times M \to [0, \infty)$ that achieves:

(i) $\nu(\zeta, \varphi) = 0 \iff \zeta = \varphi$,

(ii) $\nu(\zeta, \varphi) = \nu(\varphi, \zeta)$,

(iii) $\nu(\zeta, \varphi) \leq \gamma(\zeta, \varphi)\, \nu(\zeta, \varrho) + \beta(\zeta, \varphi)\, \nu(\varrho, \varphi) \;\; \forall \zeta, \varphi, \varrho \in M$.

Henceforth, (M, ν) is referred to as an extended metric space of type (γ, β).

Remark 2.2. An extended metric space in general is not an extended b-metric.

Example 2.3. Put $M = \{0, 1, 2\}$, set the distance function $\nu : M \times M \to [0, \infty)$ as

$\nu(\zeta, \varphi)$	0	1	2
0	$\nu(0,0) = 0$	$\nu(1,0) = 1$	$\nu(2,0) = \frac{2}{5}$
1	$\nu(0,1) = 1$	$\nu(1,1) = 0$	$\nu(2,1) = \frac{6}{25}$
2	$\nu(0,2) = \frac{2}{5}$	$\nu(1,2) = \frac{6}{25}$	$\nu(2,2) = 0$

and define the function β , γ: $M \times M \to [1, \infty)$ by

$\gamma(\zeta, \varphi)$	0	1	2
0	$\gamma(0,0) = 1$	$\gamma(1,0) = \frac{11}{10}$	$\gamma(2,0) = 1$
1	$\gamma(0,1) = \frac{11}{10}$	$\gamma(1,1) = 1$	$\gamma(2,1) = 1$
2	$\gamma(0,2) = 1$	$\gamma(1,2) = 1$	$\gamma(2,2) = 1$

and

$\beta(\zeta, \varphi)$	0	1	2
0	$\beta(0,0) = 1$	$\beta(1,0) = \frac{6}{5}$	$\beta(2,0) = \frac{151}{100}$
1	$\beta(0,1) = \frac{6}{5}$	$\beta(1,1) = 1$	$\beta(2,1) = \frac{8}{5}$
2	$\beta(0,2) = \frac{151}{100}$	$\beta(1,2) = \frac{8}{5}$	$\beta(2,2) = 1$

It is clearly that (M, ν) is an extended metric space of type (γ, β).

On the other hand, we have

$$1 = \nu(0,1) > \gamma(0,1)\nu(0,2) + \gamma(0,1)\nu(2,1) = 0.704.$$

$$1 = \nu(0,1) > \beta(0,1)\nu(0,2) + \beta(0,1)\nu(2,1) = 0.768.$$

Thus, (M, ν) is not an extended b-metric space through the use of function γ or function β.

Example 2.4. On $M = [0, \infty)$, set $\gamma, \beta : M \times M \to [1, \infty)$ as

$$\gamma(\zeta, \varphi) = \begin{cases} \zeta & if \zeta, \varphi \geq 1, \\ 1 & if\ not \end{cases} \quad \text{and} \quad \beta(\zeta, \varphi) = \begin{cases} 1 & if\ \zeta, \varphi < 1, \\ max\{\zeta, \varphi\}, & ifnot \end{cases}$$

and set $\nu : M \times M \to [0, +\infty)$ as

$$\nu(\zeta, \varphi) = \begin{cases} 0 & \iff \zeta = \eta, \\ \frac{1}{\zeta} & if\ \zeta \geq 1\ and\ \varphi \in [0, 1), \\ \frac{1}{\varphi} & if\ \varphi \geq 1\ and\ \zeta \in [0, 1)], \\ 1 & if\ not. \end{cases}$$

Then, $(M.\nu)$ is an extended metric space of type (γ, β).

On the opposite side, we have

$$\nu(0, \frac{1}{2}) = 1 > \gamma(0, \frac{1}{2})\nu(0, 3) + \gamma(0, \frac{1}{2})\nu(3, \frac{1}{2}) = \frac{1}{2} + \frac{1}{2} = \frac{2}{3}.$$

$$\nu(0, \frac{1}{2}) = 1 > \beta(0, \frac{1}{2})\nu(0, 3) + \beta(0, \frac{1}{2})\nu(3, \frac{1}{2}) = \frac{1}{2} + \frac{1}{2} = \frac{2}{3}.$$

So, (M, ν) is an extended metric space of type (γ, β) which is not an extended $b-$metric space through the use of the function γ or β.

Now, we are going to present the topological definitions for the extended metric space of type (γ, β), which will be used in our current works.

Definition 2.5. On (M, ν), let $\{\xi_n\}_{n \in N}$ be a sequence on M. Then:

(i) $\{\xi_n\}_{n \in N}$ converges to some $\xi \in M$ if for each $\varepsilon > 0$, $\exists$ an integer n_ϵ such that $\nu(\xi_n, \xi) < \varepsilon$ for each $n > n_\epsilon$. It can be written mathematically as $\lim_{n \to \infty} \nu(\xi_n, \xi) = 0$.

(ii) $\{\xi_n\}_{n \in N}$ is a Cauchy sequence $\iff$ for each positive number ε, $\nu(\xi_m, \xi_n) < \varepsilon$. $\forall m > n > n_\epsilon$, where n_ϵ is a positive integer.

(iii) The space (M, ν) is said to be a complete extended metric space of type (γ, β) if and only if each Cauchy sequence converges to M.

Definition 2.6. On (M, ν), take $\xi \in M$ and $c > 0$.

(i) The open set $\omega(\xi, c)$ is defined as

$$\omega(\xi, c) = \{\eta \in M, \nu(\xi, \eta) < c\},$$

(ii) The map $T : M \to M$ is called continuous at $\xi \in M$ if $\forall\ \varepsilon > 0$, $\exists\ m > 0$ such that $T(\omega(\xi, m)) \subseteq \omega(T\xi, \varepsilon)$.

Remark: T is continuous at $\xi \in M$, then $\xi_n \to \xi$ implies that $T\xi_n \to T\xi$ when n approaches to ∞.

Now, we state and prove our first result.

Theorem 2.7. *Impose (M, ν) is complete. Assume there exists $k \in (0, 1)$ such that $T : M \to M$ satisfies*

$$\nu(T\zeta, T\varphi) \leq k\nu(\zeta, \varphi), \tag{2.1}$$

for all $\zeta, \varphi \in M$. For $\xi_0 \in M$, Put $\xi_n = T^n \xi_0$. Also, assume that

$$\sup_{m \geq 1} \lim_{i \to \infty} \frac{\gamma(\xi_{i+1}, \xi_m)}{\gamma(\xi_i, \xi_m)} \beta(\xi_i, \xi_m) < \frac{1}{k}. \tag{2.2}$$

Then T holds only one fixed point in M.

Proof. Assume $\{\xi_n = T^n \xi_0\}$ in M holds the assumptions of theorem. Condition (2.1) implies that

$$\nu(\xi_n, \xi_{n+1}) \leq k^n \nu(\xi_0, \xi_1) \tag{2.3}$$

For integers n and m with $m \geq n$, we acquire

$$\begin{aligned}
\nu(\xi_n, \xi_m) &\leq \gamma(\xi_n, \xi_m)\nu(\xi_n, \xi_{n+1}) + \beta(\xi_n, \xi_m)\nu(\xi_{n+1}, \xi_m) \\
&\leq \gamma(\xi_n, \xi_m)\nu(\xi_n, \xi_{n+1}) + \beta(\xi_n, \xi_m)\gamma(\xi_{n+1}, \xi_m)\nu(\xi_{n+1}, \xi_{n+2}) \\
&+ \beta(\xi_n, \xi_m)\beta(\xi_{n+1}, \xi_m)\nu(\xi_{n+2}, \xi_m) \\
&\leq \gamma(\xi_n, \xi_m)\nu(\xi_n, \xi_{n+1}) + \beta(\xi_n, \xi_m)\gamma(\xi_{n+1}, \xi_m)\nu(\xi_{n+1}, \xi_{n+2}) \\
&+ \beta(\xi_n, \xi_m)\beta(\xi_{n+1}, \xi_m)\gamma(\xi_{n+2}, \xi_m)\nu(\xi_{n+2}, \xi_{n+3}) \\
&+ \beta(\xi_n, \xi_m)\beta(\xi_{n+1}, \xi_m)\beta(\xi_{n+2}, \xi_m)\nu(\xi_{n+3}, \xi_m) \\
&\leq ... \\
&\leq \gamma(\xi_n, \xi_m)\nu(\xi_n, \xi_{n+1}) + \beta(\xi_n, \xi_m)\gamma(\xi_{n+1}, \xi_m)\nu(\xi_{n+1}, \xi_{n+2}) \\
&+ \beta(\xi_n, \xi_m)\beta(\xi_{n+1}, \xi_m)\gamma(\xi_{n+2}, \xi_m)\nu(\xi_{n+2}, \xi_{n+3}) \\
&+ \\
&\vdots \\
&+ \beta(\xi_n, \xi_m)\beta(\xi_{n+1}, \xi_m)\dots\beta(\xi_{m-2}, \xi_m)\nu(\xi_{m-1}, \xi_m).
\end{aligned}$$

By taking advantage of $\gamma(a, b) \geq 1$ and $\beta(a, b) \geq 1$, the following inequality can be deduced:

$$\begin{aligned}
\nu(\xi_n, \xi_m) \leq\ & \beta(\xi_{n-1}, \xi_m)\gamma(\xi_n, \xi_m)\nu(\xi_n, \xi_{n+1}) + \beta(\xi_{n-1}, \xi_m)\beta(\xi_n, \xi_m)\gamma(\xi_{n+1}, \xi_m)\nu(\xi_{n+1}, \xi_{n+2}) \\
&+ \beta(\xi_{n-1}, \xi_m)\beta(\xi_n, \xi_m)\beta(\xi_{n+1}, \xi_m)\gamma(\xi_{n+2}, \xi_m)\nu(\xi_{n+2}, \xi_{n+3}) \\
&+ ... + \beta(\xi_{n-1}, \xi_m)\beta(\xi_n, \xi_m)\beta(\xi_{n+1}, \xi_m)\dots\beta(\xi_{m-2}, \xi_m)\gamma(\xi_{m-1}, \xi_m)\nu(\xi_{m-1}, \xi_m) \\
&= \sum_{i=n}^{m-1} \alpha(\xi_i, \xi_m) \prod_{j=n}^{i} \beta(\xi_{j-1}, \xi_m)\nu(\xi_i, \xi_{i+1}).
\end{aligned} \tag{2.4}$$

By making use of Inequality (2.3), Inequality (2.4) will be transferred to

$$\nu(\xi_n, \xi_m) \leq \sum_{i=n}^{m-1} \alpha(\xi_i, \xi_m) \prod_{j=n}^{i} \beta(\xi_{j-1}, \xi_m)k^i \nu(\xi_0, \xi_1). \tag{2.5}$$

Set

$$\alpha(\xi_i, \xi_m) \prod_{j=n}^{i} \beta(\xi_{j-1}, \xi_m)k^i \nu(\xi_0, \xi_1) := I_i. \tag{2.6}$$

Then

$$\sup_{m>1} \lim_{i\to+\infty} \frac{I_{i+1}}{I_i} = \sup_{m>1} \lim_{i\to\infty} \frac{\gamma(\xi_{i+1}, \xi_m)}{\gamma(\xi_i, \xi_m)}\beta(\xi_i, \xi_m)k < 1.$$

The ratio test will ensure that the sequence

$$\Big(\sum_{i=n}^{m-1} \alpha(\xi_i, \xi_m) \prod_{j=n}^{i} \beta(\xi_{j-1}, \xi_m)k^i \nu(\xi_0, \xi_1)\Big)$$

is Cauchy in real numbers. As a consequence, the sequence (ξ_n) is Cauchy in (M, ν). As a result of the completeness of (M, ν), we reach to

$$\lim_{n\to\infty} \nu(\xi_n, \zeta) = 0. \tag{2.7}$$

Claim: $T\zeta = \zeta$. The triangular inequality and (2.1) imply

$$\begin{aligned}\nu(\zeta, T\zeta) &\leq \gamma(\zeta, T\zeta)\nu(\zeta, \xi_{n+1}) + \beta(\zeta, T\zeta)\nu(\xi_{n+1}, T\zeta) \\ &\leq \gamma(\zeta, T\zeta)\nu(\zeta, \xi_{n+1}) + k\beta(\zeta, T\zeta)\nu(\xi_n, \zeta).\end{aligned}$$

With the help of (2.7), we reach to $\nu(\zeta, T\zeta) = 0$, that means $T\zeta = \zeta$. To verify the uniqueness, put $\eta \in M$ in such a way that $T\eta = \eta$ and $\zeta \neq \eta$. Hence,

$$0 < \nu(\zeta, \eta) = \nu(T\zeta, T\eta) \leq k\nu(\zeta, \eta)$$

is a contradiction. Thus $\zeta = \eta$, and we conclude that ζ is unique. □

Example 2.8. Let $M = \{0, 1, 2\}$. Consider the extended metric space ν type (γ, β) and the functions γ, β given as

$\nu(\zeta, \varphi)$	0	1	2
0	$\nu(0,0) = 0$	$\nu(1,0) = 1$	$\nu(2,0) = \frac{2}{5}$
1	$\nu(0,1) = 1$	$\nu(1,1) = 0$	$\nu(2,1) = \frac{6}{25}$
2	$\nu(0,2) = \frac{2}{5}$	$\nu(1,2) = \frac{6}{25}$	$\nu(2,2) = 0$

$\gamma(\zeta, \varphi)$	0	1	2
0	$\gamma(0,0) = 1$	$\gamma(1,0) = \frac{6}{5}$	$\gamma(2,0) = \frac{151}{100}$
1	$\gamma(0,1) = \frac{6}{5}$	$\gamma(1,1) = 1$	$\gamma(2,1) = \frac{8}{5}$
2	$\gamma(0,2) = \frac{151}{100}$	$\gamma(1,2) = \frac{8}{5}$	$\gamma(2,2) = 1$

and

$\beta(\zeta, \varphi)$	0	1	2
0	$\beta(0,0) = 1$	$\beta(1,0) = \frac{6}{5}$	$\beta(2,0) = \frac{8}{5}$
1	$\beta(0,1) = \frac{6}{5}$	$\beta(1,1) = 1$	$\beta(2,1) = \frac{33}{20}$
2	$\beta(0,2) = \frac{8}{5}$	$\beta(1,2) = \frac{33}{20}$	$\beta(2,2) = 1$

Choose $T0 = 2, T1 = T2 = 1$, and $k = \frac{3}{5}$. It is clear that condition (1) in theorem 2.7 is satisfied. In addition, we have

$$\sup_{m \geq 1} \lim_{i \to \infty} \frac{\gamma(\xi_{i+1}, \xi_m)}{\gamma(\xi_i, \xi_m)} \beta(\xi_i, \xi_m) = 1 < 2 = \frac{1}{k}.$$

Thus, condition (2) of theorem 2.7 has been accomplished.

Remark: Theorem 2 of [9] is a special case of our Theorem 2.7.

Corollary 2.9. *Suppose (M, ρ) is a complete extended b−metric space. Assume there exists $k \in (0, 1)$ such that $T : M \to M$ satisfies*

$$\rho(T\zeta, T\varphi) \leq k\rho(\zeta, \varphi),$$

for all $\zeta, \varphi \in M$. For $\xi_0 \in M$, Put $\xi_n = T^n\xi_0$. Also, assume that

$$\sup_{m \geq 1} \lim_{i \to \infty} \theta(\xi_{i+1}, \xi_m) < \frac{1}{k}. \tag{2.8}$$

Then, T holds only one fixed point in M.

Proof. The proof comes from Theorem 2.7 by noting that (M, ρ) is an extended metric space of type (θ, θ) once we take $\theta = \gamma = \beta$. □

Definition 2.10. Let $\Gamma(\zeta_0) = \{\zeta_0, T\zeta_0, T^2\zeta_0, ...\}$ stands to the orbit of $\zeta_0 \in M$ for the map $T : M \to M$. We say that the function $H : M \to R$ is T-orbitally lower semi-continuous at $\zeta \in M$ if $\zeta_n \to \zeta$ with $\{\zeta_n\} \subset \Gamma(\zeta_0)$, then $H(\zeta) \leq \lim_{n\to\infty} \inf H(\zeta_n)$.

Theorem 2.11. *Assume (M, ν) is complete. Choose $\zeta_0 \in M$ and consider the map $T : M \to M$. Suppose $\exists\ k \in (0, 1)$ with*

$$\nu(T\varrho, T^2\varrho) \leq k\nu(\varrho, T\varrho), for\ each\ \varrho \in \Gamma(\zeta_0). \tag{2.9}$$

Take $\zeta_n = T^n\zeta_0$. Suppose that

$$\sup_{m\geq 1} \lim_{i\to\infty} \frac{\gamma(\xi_{i+1}, \xi_m)}{\gamma(\xi_i, \xi_m)} \beta(\xi_i, \xi_m) < \frac{1}{k}. \tag{2.10}$$

Then, $\zeta_n \to \xi \in M$. Moreover, $\xi \in M$ is a fixed point of $T \iff$ the functional $\zeta \to \nu(\zeta, T\zeta)$ is T-orbitally lower semi-continuous at ξ.

Proof. Selecting $\xi_0 \in M$. Then, consider the following iterative sequence $(\xi_n = T^n\xi_0)$. By benefiting from (2.9), we reach to

$$\nu(\xi_n, \xi_{n+1}) = \nu(T^n\xi_0, T^{n+1}\xi_0) \leq k^n\nu(\xi_0, \xi_1).$$

One can imitate our proof in Theorem 2.7 to conclude that (ξ_n) is Cauchy in (M, ν) and as a result, (ξ_n) approaches to some $\xi \in M$. Utilizing the property of G at ξ to find

$$\nu(\xi, \xi) \leq \lim_{n\to+\infty} \inf \nu(T^n\xi_0, T^{n+1}\xi_0) \leq \lim_{n\to+\infty} \inf k^n\nu(\xi_0, \xi_1) = 0.$$

Thus, we have $\xi = T\xi$ and hence, ξ is a fixed point. Note that the reverse direction is obvious. □

Now, we use Definition 2.10 and Theorem 2.11 to formulate a generalization for Theorem 3 in [9].

Corollary 2.12. *Suppose (M, ρ) is a complete extended $b-$metric space. Choose $\xi \in M$ and consider the map $T : M \to M$. Assume there exists $k \in (0, 1)$ such that $T : M \to M$ satisfies*

$$\rho(T\xi, T^2\xi) \leq k\rho(\xi, T\xi),$$

for all $\xi \in T^n\xi_0$. Also, assume that

$$\sup_{m\geq 1} \lim_{i\to\infty} \rho(\xi_{i+1}, \xi_m) < \frac{1}{k}. \tag{2.11}$$

Then, $\zeta_n \to \xi \in M$. Moreover, $\xi \in M$ is a fixed point of $T \iff$ the functional $\zeta \to \nu(\zeta, T\zeta)$ is T-orbitally lower semi-continuous at ξ.

Proof. The proof comes from Theorem 2.11 by noting that (M, ρ) is an extended metric space of type (θ, θ) once we take $\theta = \gamma = \beta$. □

Theorem 2.13. *On a set M, consider the $T : M \times M$. Impose (M, ν) is complete, and there exists $a \in (0, \frac{1}{2})$ such that*

$$\nu(T\zeta, T\varphi) \leq a[\nu(\zeta, T\zeta) + \nu(\varphi, T\varphi)] \tag{2.12}$$

for all $\zeta, \varphi\ \in M$. Take $\xi_n = T^n\xi_0$. Moreover, assume

$$\sup_{m\geq 1} \lim_{i\to\infty} \frac{\gamma(\xi_{i+1}, \xi_m)}{\gamma(\xi_i, \xi_m)} \beta(\xi_i, \xi_m) < \frac{1}{a} - 1. \tag{2.13}$$

Also, for each $\xi \in M$, impose

$$\beta(\xi, T\xi) < \frac{1}{a}. \tag{2.14}$$

Then, T possesses a unique fixed point.

Proof. Set $\xi_n = T\xi_{n-1}$. Then,

$$\begin{aligned}\nu(\xi_n, \xi_{n+1}) &= \nu(T\xi_{n-1}, T\xi_n)\\ &\leq a[\nu(\xi_{n-1}, T\xi_{n-1}) + \nu(\xi_n, T\xi_n)]\\ &= a[\nu(\xi_{n-1}, \xi_n) + \nu(\xi_n, \xi_{n+1})].\end{aligned}$$

Thus, $\nu(\xi_n, \xi_{n+1}) \leq \left(\frac{a}{1-a}\right) \nu(\xi_{n-1}, \xi_n)$. By induction, we get

$$\nu(\xi_n, \xi_{n+1}) \leq \Big(\frac{a}{1-a}\Big)^n \nu(\xi_1, \xi_0). \tag{2.15}$$

Claim: $\{\xi_n\}$ is Cauchy. For $n, m \in N$ with $m \geq n$, we gain

$$\nu(\xi_n, \xi_m) \leq \gamma(\xi_n, \xi_m)\nu(\xi_n, \xi_{n+1}) + \beta(\xi_n, \xi_m)\nu(\xi_{n+1}, \xi_m).$$

As in the proof of Theorem 2.7, we acquire

$$\nu(\xi_n, \xi_m) \leq \sum_{i=n}^{m-1} \alpha(\xi_i, \xi_m) \prod_{j=n}^{i} \beta(\xi_{j-1}, \xi_m)\nu(\xi_i, \xi_{i+1}). \tag{2.16}$$

By employing Inequalities (2.15) and (2.16), we acquire

$$\nu(\xi_n, \xi_m) \leq \sum_{i=n}^{m-1} \alpha(\xi_i, \xi_m) \prod_{j=n}^{i} \beta(\xi_{j-1}, \xi_m)\Big(\frac{a}{1-a}\Big)^n \nu(\xi_1, \xi_0).$$

Since $0 \leq a < \frac{1}{2}$, we have $\frac{a}{1-a} < 1$. As in Theorem 2.7, we reach to $\{\xi_n\}$ is Cauchy. The completeness of (M, ν) ensures $u \in M$ such that $\xi_n \to \xi$ in M. If $\xi \neq T\xi$, then

$$\begin{aligned}0 < \nu(\xi, T\xi) &\leq \gamma(\xi, T\xi)\nu(\xi, \xi_{n+1}) + \beta(\xi, T\xi)\nu(\xi_{n+1}, T\xi)\\ &\leq \gamma(\xi, T\xi)\nu(\xi, \xi_{n+1}) + \beta(\xi, T\xi)[a\nu(\xi_n, \xi_{n+1}) + a\nu(\xi, T\xi)].\end{aligned} \tag{2.17}$$

With some calculations and help with assumptions of theorem, one can reach to the following impossible inequality: $0 < \nu(\xi, T\xi) < \nu(\xi, T\xi)$. Thus, $T\xi = \xi$. If $v = Tv$, then

$$\begin{aligned}\nu(\xi, v) = \nu(T\xi, Tv) &\leq a[\nu(\xi, T\xi) + \nu(v, Tv)]\\ &= a[\nu(\xi, \xi) + \nu(v, v)] = 0.\end{aligned}$$

Therefore, $\xi = v$ and hence, ξ is a unique fixed point. □

Corollary 2.14. *Suppose (M, ρ) is a complete extended $b-$metric space. Consider the map $T : M \times M$. Assume there exists $a \in (0, \frac{1}{2})$ such that*

$$\rho(T\zeta, T\varphi) \leq a[\rho(\zeta, T\zeta) + \rho(\varphi, T\varphi)] \tag{2.18}$$

for all $\zeta, \varphi \in M$. For $\xi \in M$, put $\xi_n = T^n\xi_0$. Moreover, assume

$$\sup_{m\geq 1} \lim_{i\to\infty} \alpha(\xi_{i+1}, \xi_m) < \frac{1}{a} - 1. \tag{2.19}$$

Also, for each $\xi \in M$, impose

$$\alpha(\xi, T\xi) < \frac{1}{a}. \tag{2.20}$$

Then, T possesses a unique fixed point.

Proof. The proof comes from Theorem 2.13 by noting that (M, ρ) is a complete extended $b-$metric space of type (α, α) once we take $\gamma = \beta = \alpha$. □

Future work: Our future work is to generalize the results found in [14, 15, 16, 17] to extended $b-$metric space.

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Received: 2022-10-29
Accepted: 2023-02-02