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On Degree Product Eigenvalues and Degree Product Energy of Graphs

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Abstract.
Let G be a simple connected graph with n-vertices and m-edges. In this paper we introduce the concept of degree product matrix $DP(G)$ and degree product energy $E_{DP}(G)$ of a graph G and obtain the bounds for the degree product eigenvalues and degree product energy of any connected graph G.

1 Introduction

In this paper we consider simple, undirected and unweighted graphs. Let $G = (V, E)$ be such a graph with a vertex set $V(G) = \{v_1, v_2, \dots, v_n\}$ and edge set $E(G)$, where $|V(G)| = n$ and $|E(G)| = m$. For $v_i \in V(G)$, let d_i be the degree of vertex v_i. Let $A(G)$ be the adjacency matrix of the graph G is a square matrix of order n whose (i, j)- entry is equal to unity if the vertices v_i and v_j are adjacent and is equal to zero otherwise. The eigenvalues of adjacency matrix $A(G)$ are denoted by $\lambda_1, \lambda_2, \dots, \lambda_n$ and since they are real it can be ordered as $\lambda_1 \geq \lambda_2 \geq \dots \geq \lambda_n$. The energy of graph G was first defined by Gutman in 1978 as [6]

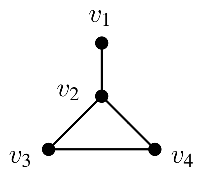
$$E(G) = \sum_{i=1}^{n} |\lambda_i| .$$

Its mathematical properties were extensively investigated in the literature. For details see the book [10], the recent articles [4, 5, 7, 8, 11, 12, 13, 14, 18] and references cited theirin. Motivated by work on maximum degree energy [1], bounds for the degree sum eigenvalues and degree sum energy [17], we introduce in this paper a new matrix called *degree product matrix* defined as $DP(G) = [dp_{ij}]$, in which

$$dp_{ij} = \begin{cases} d_i d_j & \text{if } i \neq j \\ 0 & otherwise. \end{cases}$$

Denote the eigenvalues of the degree product matrix $DP(G)$ by $\gamma_1, \gamma_2, \dots, \gamma_n$ and label them in non-increasing order as $\gamma_1 \geq \gamma_2 \geq \dots \geq \gamma_n$. In addition $\phi_{DP}(G, \gamma) = det(\gamma I_n - DP(G))$ will be referred as the DP-characteristic polynomial of G, where I_n is the identity matrix of order n. The energy of degree product matrix $DP(G)$ is defined as

$$E_{DP}(G) = \sum_{i=1}^{n} |\gamma_i|.$$



$$DP(G) = \begin{array}{c} \\ v_1 \\ v_2 \\ v_3 \\ v_4 \end{array} \begin{array}{c} \begin{array}{cccc} v_1 & v_2 & v_3 & v_4 \end{array} \\ \begin{bmatrix} 0 & 3 & 2 & 2 \\ 3 & 0 & 6 & 6 \\ 2 & 6 & 0 & 4 \\ 2 & 6 & 4 & 0 \end{bmatrix} \end{array}$$

Fig. 1: Graph G and its degree product matrix.

$\gamma_1 = -6.7788, \gamma_2 = -4, \gamma_3 = -1.3171 \ and \ \gamma_4 = 12.0960.$
Therefore, $E_{DP}(G) = 24.1919$.

In this paper, we obtain bounds for the degree product eigenvalues and degree product energy of graph G.

2 Bounds for the eigenvalues and energy

Lemma 2.1. *Let G be any graph with n-vertices and let $\gamma_1 \geq \gamma_2 \geq \ldots \geq \gamma_n$ be its degree product eigenvalues.*

(i) $\sum\limits_{i=1}^{n} \gamma_i = 0.$

(ii) $\sum\limits_{i=1}^{n} {\gamma_i}^2 = 2M, \ \ where\ M = \sum\limits_{1 \leq i < j \leq n} (d_i d_j)^2.$

Proof.

$$\sum_{i=1}^{n} \gamma_i = trace[DP(G)] = 0.$$

For $i = 1, 2, \ldots, n$ the (i, i) entry of $(DP(G))^2$ is equal to

$$\begin{aligned} \sum_{j=1}^{n} (d_i\, d_j)(d_j\, d_i) &= \sum_{j=1}^{n} (d_i\, d_j)^2 \\ \sum_{i=1}^{n} \gamma_i^2 &= trace[DP(G)]^2 \\ &= \sum_{i=1}^{n} \sum_{j=1}^{n} (d_i\, d_j)^2 \\ &= 2 \sum_{1 \leq i < j \leq n} (d_i d_j)^2 \\ &= 2M. \end{aligned}$$

□

Theorem 2.2. *Let G be any graph with n-vertices. Then $\gamma_1 \leq \sqrt{\frac{2M(n-1)}{n}}$.*

Proof. Consider the Cauchy-Schwarz inequality,

$$\left(\sum_{i=1}^{n} a_i b_i \right)^2 \leq \left(\sum_{i=1}^{n} a_i^2 \right) \left(\sum_{i=1}^{n} b_i^2 \right).$$

Choosing $a_i = 1$, $b_i = \gamma_i$ for $i = 2, 3, \ldots, n$, then

$$\left(\sum_{i=2}^{n}\gamma_i\right)^2 \le (n-1)\sum_{i=2}^{n}\gamma_i^2 \tag{2.1}$$

From Lemma (2.1) ,

$\sum_{i=2}^{n}\gamma_i = -\gamma_1$ and $\sum_{i=2}^{n}{\gamma_i}^2 = -\gamma_1^2 + 2M,$

Then Eq. (2.1) becomes

$$\begin{aligned}(-\gamma_1)^2 &\le (n-1)(2M-\gamma_1^2)\\ \gamma_1 &\le \sqrt{\frac{2M(n-1)}{n}}.\end{aligned}$$

□

Theorem 2.3. *Let G be any graph with n-vertices. Then*

$$\sqrt{2M} \le E_{DP}(G) \le \sqrt{2Mn}.$$

Proof. Consider the Cauchy-Schwarz inequality,

$$\left(\sum_{i=1}^{n}a_ib_i\right)^2 \le \left(\sum_{i=1}^{n}a_i^2\right)\left(\sum_{i=1}^{n}b_i^2\right).$$

Put $a_i = 1$ and $b_i = |\gamma_i|$ in Cauchy-Schwarz inequality, we get

$$\begin{aligned}\left(\sum_{i=1}^{n}|\gamma_i|\right)^2 &\le n\sum_{i=1}^{n}\gamma_i^2\\ (E_{DP}(G))^2 &\le n(2M)\\ E_{DP}(G) &\le \sqrt{2nM}\end{aligned}$$

which is an upperbound.

Now, $(E_{DP}(G))^2 = \left(\sum_{i=1}^{n}|\gamma_i|\right)^2 \ge \sum_{i=1}^{n}|\gamma_i|^2 = 2M,$

$\Rightarrow E_{DP}(G) \ge \sqrt{2M}$ which is a lower bound. □

Theorem 2.4. *Let G be any graph with n-vertices and $\triangle$ be the absolute value of the determinant of the degree product matrix $DP(G)$. Then*

$$\sqrt{2M + n(n-1)\triangle^{2/n}} \le E_{DP}(G) \le \sqrt{2Mn}.$$

Proof. By the definition of degree product energy,

$$\begin{aligned}(E_{DP}(G))^2 = (\sum_{i=1}^{n}\gamma_i)^2 &= \sum_{i=1}^{n}\gamma_i^2 + 2\sum_{i<j}|\gamma_i|\,|\gamma_j|\\ &= 2M + \sum_{i\neq j}|\gamma_i|\,|\gamma_j|. \end{aligned} \tag{2.2}$$

Since for nonnegative number the Arithmetic mean is greater than Geometric mean ,

$$\begin{aligned}\frac{1}{n(n-1)}\sum_{i\neq j}|\gamma_i|\,|\gamma_j| &\geq \left(\prod_{i\neq j}|\gamma_i|\,|\gamma_j|\right)^{\frac{1}{n(n-1)}}\\ &= \left(\prod_{i=1}^{n}|\gamma_i|^{2(n-1)}\right)^{\frac{1}{n(n-1)}}\\ &= \prod_{i=1}^{n}|\gamma_i|^{2/n}\\ &= \triangle^{2/n}. \qquad (2.3)\end{aligned}$$

Combining Eq. (2.2) and Eq. (2.3) we get a lower bound.

Consider,

$$P = \sum_{i=1}^{n}\sum_{j=1}^{n}(|\gamma_i|-|\gamma_j|)^2 \qquad (2.4)$$

On simplifying,

$$= 4Mn - 2(E_{DP}(G))^2$$

Since $P \geq 0$,

$$4Mn - 2(E_{DP}(G))^2 \geq 0$$

$$\Rightarrow E_{DP}(G) \leq \sqrt{2Mn} \quad \text{which is an upper bound.}$$

□

Lemma 2.5. *[9] Let $a_1, a_2, \ldots, a_n$ be non-negative numbers. Then*

$$n\left[\frac{1}{n}\sum_{i=1}^{n}a_i - \left(\prod_{i=1}^{n}a_i\right)^{1/n}\right] \leq n\sum_{i=1}^{n}a_i - \left(\sum_{i=1}^{n}\sqrt{a_i}\right)^2 \leq n(n-1)\left[\frac{1}{n}\sum_{i=1}^{n}a_i - \left(\prod_{i=1}^{n}a_i\right)^{1/n}\right].$$

Theorem 2.6. *Let G be a connected graph with n vertices. Then*

$$\sqrt{2M + n(n-1)\triangle^{2/n}} \leq E_{DP}(G) \leq \sqrt{2M(n-1) + n\triangle^{2/n}}.$$

Proof. Let $a_i = |\gamma_i|^2$, $i = 1, 2, \ldots, n$ and

$$\begin{aligned}K &= n\left[\frac{1}{n}\sum_{i=1}^{n}|\gamma_i|^2 - \left(\prod_{i=1}^{n}|\gamma_i|^2\right)^{1/n}\right]\\ &= n\left[\frac{2M}{n} - \left(\prod_{i=1}^{n}|\gamma_i|\right)^{2/n}\right]\\ &= n\left[\frac{2M}{n} - \triangle^{2/n}\right]\\ &= 2M - n\,\triangle^{2/n}.\end{aligned}$$

By Lemma (2.5)

$$K \leq n\sum_{i=1}^{n}|\gamma_i|^2 - \left(\sum_{i=1}^{n}|\gamma_i|\right)^2 \leq (n-1)K$$

that is

$$2M - n\triangle^{2/n} \leq 2nM - (E_{DP}(G))^2 \leq (n-1)(2M - n\triangle^{2/n}).$$

Simplification of above equation leads to the desired result.

□

Theorem 2.7. *[16] Let a_i and b_i, $1 \leq i \leq n$ are nonnegative real numbers. Then*

$$\sum_{i=1}^{n} a_i^2 \sum_{i=1}^{n} b_i^2 \leq \frac{1}{4}\left(\sqrt{\frac{M_1M_2}{m_1m_2}} + \sqrt{\frac{m_1m_2}{M_1M_2}}\right)^2 \left(\sum_{i=1}^{n} a_ib_i\right)^2.$$

where $M_1 = max_{1\leq i\leq n}(a_i)$; $M_2 = max_{1\leq i\leq n}(b_i)$; $m_1 = min_{1\leq i\leq n}(a_i)$ and $m_2 = min_{1\leq i\leq n}(b_i)$.

Theorem 2.8. *[15] Let a_i and b_i, $1 \leq i \leq n$ are positive real numbers . Then*

$$\sum_{i=1}^{n} a_i^2 \sum_{i=1}^{n} b_i^2 - \left(\sum_{i=1}^{n} a_ib_i\right)^2 \leq \frac{n^2}{4}(M_1M_2 - m_1m_2)^2.$$

where M_i and m_i are defined similar to the Theorem (2.7).

Theorem 2.9. *[2] Let a_i and b_i, $1 \leq i \leq n$ are nonnegative real numbers. Then*

$$\left| n\sum_{i=1}^{n} a_ib_i - \sum_{i=1}^{n} a_i \sum_{i=1}^{n} b_i \right| \leq \alpha(n)\,(A-a)\,(B-b).$$

where a, b, A and B are real constants, that for each i, $1 \leq i \leq n$, $a \leq a_i \leq A$ and $b \leq b_i \leq B$. Further, $\alpha(n) = n\left\lceil \frac{n}{2} \right\rceil \left(1 - \frac{1}{n}\left\lceil \frac{n}{2} \right\rceil\right)$.

Theorem 2.10. *[3] Let a_i and b_i, $1 \leq i \leq n$ are nonnegative real numbers. Then*

$$\sum_{i=1}^{n} b_i^2 + rR\sum_{i=1}^{n} a_i^2 \leq (r+R)(\sum_{i=1}^{n} a_ib_i),$$

where r and R are real constants, so that for each i, $1 \leq i \leq n$, holds $ra_i \leq b_i \leq Ra_i$.

Theorem 2.11. *Let G be a graph of order n. Then*

$$E_{DP}(G) \geq \sqrt{2Mn - \frac{n^2}{4}(\gamma_1 - \gamma_{min})^2}$$

where $\gamma_1 = \gamma_{max} = max_{1\leq i\leq n}|\gamma_i|$ and $\gamma_{min} = min_{1\leq i\leq n}|\gamma_i|$.

Proof. Suppose $\gamma_1, \gamma_2, \ldots, \gamma_n$ are the eigenvalues of $DP(G)$. We assume that $a_i = 1$ and $b_i = |\gamma_i|$, which by Theorem (2.8) implies

$$\begin{aligned}
\sum_{i=1}^{n} 1^2 \sum_{i=1}^{n} |\gamma_i|^2 - \left(\sum_{i=1}^{n} |\gamma_i|\right)^2 &\leq \frac{n^2}{4}(\gamma_1 - \gamma_{min})^2 \\
2Mn - (E_{DP}(G))^2 &\leq \frac{n^2}{4}(\gamma_1 - \gamma_{min})^2 \\
E_{DP}(G) &\geq \sqrt{2Mn - \frac{n^2}{4}(\gamma_1 - \gamma_{min})^2}.
\end{aligned}$$

□

Theorem 2.12. *Suppose zero is not an eigenvalue of $DP(G)$, then*

$$E_{DP}(G) \geq \frac{2\sqrt{\gamma_1\,\gamma_{min}}\sqrt{2Mn}}{\gamma_1 + \gamma_{min}}.$$

where $\gamma_1 = \gamma_{max} = max_{1\leq i\leq n}|\gamma_i|$ and $\gamma_{min} = min_{1\leq i\leq n}|\gamma_i|$.

Proof. Suppose $\gamma_1, \gamma_2, \ldots, \gamma_n$ are the eigenvalues of $DP(G)$. We assume that $a_i = |\gamma_i|$ and $b_i = 1$, which by Theorem (2.7) implies

$$\begin{aligned}
\sum_{i=1}^{n} |\gamma_i|^2 \sum_{i=1}^{n} 1^2 &\leq \frac{1}{4}\left(\sqrt{\frac{\gamma_1}{\gamma_{min}}} + \sqrt{\frac{\gamma_{min}}{\gamma_1}}\right)^2 \left(\sum_{i=1}^{n} |\gamma_i|\right)^2 \\
2Mn &\leq \frac{1}{4}\left(\frac{(\gamma_1 + \gamma_{min})^2}{\gamma_1\, \gamma_{min}}\right) (E_{DP}(G))^2 \\
E_{DP}(G) &\geq \frac{2\sqrt{\gamma_1\, \gamma_{min}}\sqrt{2Mn}}{\gamma_1 + \gamma_{min}}.
\end{aligned}$$

□

Theorem 2.13. *Let G be a graph of order n. Let $\gamma_1 \geq \gamma_2 \geq \ldots \geq \gamma_n$ be the eigenvalues of $DP(G)$. Then*

$$E_{DP}(G) \geq \frac{2M + n\,\gamma_1\gamma_{min}}{\gamma_1 + \gamma_{min}}.$$

where $\gamma_1 = \gamma_{max} = max_{1\leq i\leq n}|\gamma_i|$ and $\gamma_{min} = min_{1\leq i\leq n}|\gamma_i|$.

Proof. We assume that $b_i = |\gamma_i|$, $a_i = 1$, $R = |\gamma_1|$ and $r = |\gamma_{min}|$. Then by Theorem (2.10), we get

$$\begin{aligned}
\sum_{i=1}^{n} |\gamma_i|^2 + \gamma_1\gamma_{min} \sum_{i=1}^{n} 1^2 &\leq (\gamma_1 + \gamma_{min}) \sum_{i=1}^{n} |\gamma_i| \\
2M + n\,\gamma_1\,\gamma_{min} &\leq (\gamma_1 + \gamma_{min})\, E_{DP(G)}.
\end{aligned}$$

On simplification we get the desired result. □

Theorem 2.14. *Let G be a graph of order n. Let $\gamma_1 \geq \gamma_2 \geq \ldots \geq \gamma_n$ be the eigenvalues of $DP(G)$. Then*

$$E_{DP}(G) \geq \sqrt{2Mn - \alpha(n)(\gamma_1 - \gamma_{min})^2}.$$

where $\gamma_1 = \gamma_{max} = max_{1\leq i\leq n}|\gamma_i|$ and $\gamma_{min} = min_{1\leq i\leq n}|\gamma_i|$ and $\alpha(n) = n\left\lceil \frac{n}{2}\right\rceil \left(1 - \frac{1}{n}\left\lceil \frac{n}{2}\right\rceil\right)$.

Proof. We assume that $a_i = |\gamma_i| = b_i$, $A \leq |\gamma_i| \leq B$ and $a \leq |\gamma_n| \leq b$, then by Theorem (2.9) we get

$$\begin{aligned}
\left| n\sum_{i=1}^{n} |\gamma_i|^2 - (\sum_{i=1}^{n} |\gamma_i|)^2 \right| &\leq \alpha(n)(\gamma_1 - \gamma_{min})^2 \\
\left|2Mn - (E_{DP(G)})^2\right| &\leq \alpha(n)(\gamma_1 - \gamma_{min})^2 \\
E_{DP(G)} &\geq \sqrt{2Mn - \alpha(n)(\gamma_1 - \gamma_{min})^2}.
\end{aligned}$$

□

Corollary 2.15. *If $\alpha(n) = \frac{n^2}{4}$, then*

$$E_{DP(G)} \geq \sqrt{2Mn - \frac{n^2}{4}(\gamma_1 - \gamma_{min})^2}.$$

This shows that inequality in the Theorem (2.14) is stronger than the inequality in the Theorem (2.12).

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