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ON k-SYMPLECTIC AND $k-$COSYMPLECTIC STRUCTURES ON $\mathfrak{sl}(n, \mathbb{R})$

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Abstract. This paper aims to prove that the Lie algebra $\mathfrak{sl}(n, \mathbb{R})$ always admits both a $k-$symplectic and a $k-$cosymplectic structure and the given Reeb vectors associated with these structures.

1 Introduction

The theory of k-symplectic structure on a smooth manifold, was initiated in 1984 by Awane [1] and developed by different authors (see [5], [8] and [11]), who gave a natural generalization of the symplectic structure with Lagrangian foliation. The study of these structures was motivated by some mathematical and physical considerations, like the local study of Pfaffian systems and Nambu's statistical mechanics. Recall that Nambu mechanics is a generalization of Hamiltonian mechanics involving multiple Hamiltonians. For this reason, Awane proposed an approach to formalize it. A k-symplectic structure on an $n(k+1)$-dimensional smooth manifold M will be defined by a n-codimensional foliation and a system of k closed two forms vanishing on the subbundle defined by this foliation and with transversally characteristic spaces. A complete description of these structures can be found in [2] and [3]. The k-cosymplectic formalism initiated in 1998 by Léon in ([12],[13]) is introduced to study symmetries and conservation laws for certain kinds of Hamiltonian classical field theories. The k-cosymplectic structures are the foundations of the k-cosymplectic formalism. A $k-$cosymplectic manifold, as originally defined by León [12] in 1998, is a smooth manifold of dimension $k(n+1)+n$, admitting a family of closed 1-forms $\{\eta^1, \ldots, \eta^k\}$, and a family of closed 2-forms $\{\omega_1, \ldots, \omega_k\}$, and an integrable $nk-$dimensional distribution $\mathcal{F}$ on M such that: $\eta^1 \wedge \ldots \wedge \eta^k \neq 0$; $(\ker \eta^1) \cap \ldots \cap (\ker \eta^k) \cap (\ker \omega_1) \cap \ldots \cap (\ker \omega_k) = \{0\}$; dim $\big((\ker \omega_1) \cap \ldots \cap (\ker \omega_k)\big) = k$, and $\eta^\alpha_{|_\mathcal{F}} = 0, \quad \omega_{\alpha_{|_{\mathcal{F}\times\mathcal{F}}}} = 0$ for all $\alpha \in \{1, \ldots, k\}$. The k-cosymplectic geometry is the generalization to field theories of the standard cosymplectic geometry for non-autonomous mechanics.

The goal of this paper is to equip the Lie algebra $\mathfrak{sl}(n, \mathbb{R})$ with k-symplectic structures and k-cosymplectic structures, and determine the Reeb vectors associated with the latter structures.

This paper is organized as follows. Section 2 is devoted to the review of the main definitions and some useful results. In Section 3, we prove that the Lie algebra of $n \times n$ real traceless matrices always admits a $k-$symplectic structure. Finally, in Section 4, we show some examples of k-cosymplectic structures on $\mathfrak{sl}(n, \mathbb{R})$, and we give the Reeb vectors associated with these structures.

2 Preliminaries

In this section, we review some general concepts on $k-$symplectic geometry and k-cosymplectic geometry. For a complete list of references and more details of the subjects see (Awane (1984); de León and his collaborators (1998); de León, Merino and Salgado (2001))([1],[2],[3],[12],[13]).

k-symplectic structures

A left invariant k-symplectic structure on a connected Lie group G of dimension $n(k+1)$ is equivalent to its associated infinitesimal structure, namely, the Lie algebra $\mathfrak{g}$ of G, a Lie subalgebra $\mathfrak{h}$ of dimension nk and a family $\{\theta^1, ..., \theta^k\}$ of closed forms of degree two, such that:

(i) The exterior system $\{\theta^1, ..., \theta^k\}$ is non-degenerate, i.e. $\bigcap_{1\leq\alpha\leq k} \ker\theta^\alpha = 0$, where $\ker\theta^\alpha = \{x \in \mathfrak{g} \mid \theta^\alpha(x,y) = 0, \forall y \in \mathfrak{g}\}$.

(ii) The Lie subalgebra $\mathfrak{h}$ is a totally isotropic subspace of $\mathfrak{g}$ with respect the system $\{\theta^1, ..., \theta^k\}$, that is,

$$\theta^\alpha(x,y) = 0,$$

for all $x, y \in \mathfrak{h}$, and $\alpha = 1, ..., k$.

The Lie algebra $\mathfrak{g}$ is said to be quadratic if it comes equipped with a non-degenerate, symmetric, bilinear form, $B : \mathfrak{g} \times \mathfrak{g} \to \mathbb{R}$ satisfying,

$$B([x,y],z) = B(x,[y,z]).$$

for any x, y and $z \in \mathfrak{g}$. The bilinear form B is said to be invariant if this property is satisfied.

Definition 2.1. We will say that $(\mathfrak{g}, B, \theta^1, ..., \theta^k, \mathfrak{h})$ is a quadratic k-symplectic Lie algebra if $(\mathfrak{g}, B)$ is quadratic and $(\mathfrak{g}, \theta^1, ..., \theta^k, \mathfrak{h})$ is a $k-$symplectic Lie algebra.

Proposition 2.2. *Let $(\mathfrak{g}, B)$ be a quadratic Lie algebra. A k-symplectic structure $(\mathfrak{g}, B, \theta^1, ..., \theta^k, \mathfrak{h})$ may be defined on $\mathfrak{g}$ if and only if there exist skew-symmetric derivations D^α, $\alpha = 1, ..., k$ of $(\mathfrak{g}, B)$ such as*

(i) $\bigcap_{1\leq\alpha\leq k} \ker D^\alpha = 0$

(ii) $D^\alpha\mathfrak{h}\perp\mathfrak{h}, \quad \forall\alpha \in \{1, ..., k\}$ *(the orthogonal $\perp$ is relatively to B).*

Proof. For each θ^α there exists a skew-symmetric derivation D^α such as

$$\theta^\alpha(x,y) = B(D^\alpha x, y), \qquad \forall x, y \in \mathfrak{g} \tag{2.1}$$

B is non-degenerate then $\ker\theta^\alpha = \ker D^\alpha$. □

If $\mathfrak{g}$ is semisimple Lie algebra then, the Killing form B is non-degenerate, and any derivation is inner, so we have:

Corollary 2.3. $(\mathfrak{g}, B, \theta^1, ..., \theta^k, \mathfrak{h})$ *is a quadratic k-symplectic semisimple Lie algebra if and only if there exist $x_1, \ldots, x_k \in \mathfrak{g}$ such as*

(i) If $[x_\alpha, x] = 0$, $\forall\alpha \in \{1, \ldots, k\}$, then $x = 0$.

(ii) $x_\alpha \in [\mathfrak{h}, \mathfrak{h}]^\perp$, $\forall\alpha \in \{1, \ldots, k\}$. Here the orthogonal $\perp$ is relatively to B.

In this case, we denote by $(\mathfrak{g}, x_1, \ldots, x_k, \mathfrak{h})$ the k-symplectic structure on the semisimple Lie algebra $\mathfrak{g}$ relatively to the Killing form.

Remarks and examples

(i) For $k = 1$ we find the case of polarized symplectic Lie algebras. It is well known that every quadratic Lie algebra admitting a symplectic structure must be nilpotent. See [4] for a complete study of symplectic structures on quadratic Lie algebras.

(ii) Let $\mathfrak{so}(4,\mathbb{R}) \cong \mathfrak{sl}(2,\mathbb{R}) \oplus \mathfrak{sl}(2,\mathbb{R})$ be the semisimple Lie algebra. It is of type $A_1 \times A_1$, and has a Chevalley basis $\mathfrak{B} = \{h_1, h_2, x_1, x_2, y_1, y_2\}$, defined as follows:

$$ah_1 + bh_2 + cx_1 + dx_2 + c'y_1 + d'y_2 = \begin{pmatrix} a & c & 0 & 0 \\ c' & -a & 0 & 0 \\ 0 & 0 & b & d \\ 0 & 0 & d' & -b \end{pmatrix},$$

we get the brackets

$$\begin{array}{rclcrcl} [h_1,x_1] &=& 2x_1 & & [h_2,x_2] &=& 2x_2 \\ [h_1,y_1] &=& -2y_1 & & [h_2,y_2] &=& -2y_2 \\ [x_1,y_1] &=& h_1 & & [x_2,y_2] &=& h_2. \end{array} .$$

In this basis the Killing form is given by

$$K = \begin{pmatrix} 8 & 0 & 0 & 0 & 0 & 0 \\ 0 & 8 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 4 & 0 & 0 \\ 0 & 0 & 4 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 4 \\ 0 & 0 & 0 & 0 & 4 & 0 \end{pmatrix}.$$

Let $\mathfrak{h} = span\{x_1,x_2,h_1,h_2\}$ be a 4-dimensional solvable subalgebra of $\mathfrak{so}(4,\mathbb{R})$. Using Corollary 2.3, it is easy to see that $(\mathfrak{so}(4,\mathbb{R}), h_1+h_2, x_1+x_2, \mathfrak{h})$ is a 2-symplectic structure. From (2.1) this 2-symplectic structure is given by $(\theta_1,\theta_2,\mathfrak{h})$ with $2-$ symplectic exterior system:

$$\begin{cases} \theta_1 &= -h_1^* \wedge y_1^* - h_2^* \wedge y_2^* \\ \theta_2 &= x_1^* \wedge y_1^* + x_2^* \wedge y_2^*, \end{cases}$$

Note that $\{h_1^*,h_2^*,x_1^*,x_2^*,y_1^*,y_2^*\}$ is the dual basis associated with $\mathfrak{B}$.

Note that the subalgebra $\mathfrak{h}_2 = span\{x_1,y_1,h_1,h_2\}$ cannot be associated with any k-symplectic structure in $\mathfrak{so}(4,\mathbb{R})$.

(iii) A quadratic Lie algebra may not have k-symplectic structures. For example, the exceptional simple Lie algebra $\mathfrak{g}_2$ does not support a k-symplectic structure. Indeed, the subalgebras of $\mathfrak{g}_2$, which is of dimension 14, is of dimension less than 9 [9].

k-cosymplectic structures

In this subsection, we present some basic facts about $k-$cosymplectic manifolds.

Definition 2.4. (León [12]) Let M be a differentiable manifold of dimension $k(n+1)+n$. A $k-$cosymplectic structure is a family $\left(\eta^\alpha,\omega_\alpha,\mathcal{F} : 1 \le \alpha \le k\right)$, where each η^α is a closed 1-form, each ω_α is a closed 2-form and $\mathcal{F}$ is an integrable $nk-$dimensional distribution on M satisfying:

(i) $\eta^1 \wedge \ldots \wedge \eta^k \neq 0$;

(ii) $(\ker \eta^1) \cap \ldots \cap (\ker \eta^k) \cap (\ker \omega_1) \cap \ldots \cap (\ker \omega_k) = \{0\}$;

(iii) $\dim\left((\ker \omega_1) \cap \ldots \cap (\ker \omega_k)\right) = k$;

(iv) $\eta^\alpha_{|\mathcal{F}} = 0, \quad \omega_{\alpha_{|\mathcal{F}\times\mathcal{F}}} = 0; \quad 1 \le \alpha \le k.$

If $\left(M,\eta^\alpha,\omega_\alpha,\mathcal{F} : 1 \le \alpha \le k\right)$ is an $k-$cosymplectic structure, then there exists a family of k vector fields $\{R_\alpha\}_{1\le\alpha\le k}$, which are called *Reeb vector fields*, characterized by the following conditions:

$$i_\alpha(R_\beta) = \delta_\alpha^\beta, \quad i_\alpha\omega_\beta = 0; \quad 1 \le \alpha,\, \beta \le k$$

In the particular case, if $k=1$, then $(M,\eta,\omega,\mathcal{F})$ is a cosymplectic structure on M of dimension $2n+1$.

Theorem 2.5. *(Darboux coordinates) If M is a $k-$cosymplectic manifold, then for every point of M there exists a local chart of coordinates $(t^\alpha, q^i, p_i^\alpha;\ 1 \le \alpha \le k), \quad 1 \le i \le n$, such that*

$$\eta^\alpha = dt^\alpha, \quad \omega^\alpha = dq^i \wedge dp_i^\alpha; \quad 1 \le \alpha \le k, \quad 1 \le i \le n$$

$$\mathcal{F} = span\left\{\frac{\partial}{\partial p_i^1}, \ldots, \frac{\partial}{\partial p_i^k}\right\}_{1\le i\le n}$$

These are called Darboux or canonical coordinates of the $k-$cosymplectic manifold.

A left invariant k-cosymplectic structure on a connected Lie group G of dimension $k(n+1)+k$ is equivalent to its associated infinitesimal structure, namely, the Lie algebra $\mathfrak{g}$ of G, a Lie subalgebra $\mathfrak{h}$ of dimension nk, a family $\{\eta^1,\dots,\eta^\alpha\}$ of closed forms of degree one and a family $\{\omega^1,...,\omega^k\}$ of closed forms of degree two, such that:

(i) The Lie subalgebra $\mathfrak{h}$ is a totally isotropic subspace of $\mathfrak{g}$ with respect the system $\{\omega_1,...,\omega_k\}$, that is,
$$\omega_\alpha(x,y)=0, \quad \text{for all } x,y\in\mathfrak{h} \text{ and } \alpha=1,...,k$$

(ii) $\eta^1\wedge\dots\wedge\eta^k\neq 0$ and $\eta^\alpha_{|\mathfrak{h}}=0; \quad 1\leq\alpha\leq k;$

(iii) $\big(\cap_{\alpha=1}^k(\ker\eta^\alpha)\big)\cap\big(\cap_{\alpha=1}^k(\ker\omega^k)\big)=\{0\}$ and $\dim\big((\ker\omega^1)\cap\dots\cap(\ker\omega^k)\big)=k.$

Definition 2.6. We will say that $(\mathfrak{g},B,\eta^\alpha,\omega_\alpha,\mathfrak{h})$ is a quadratic k-cosymplectic Lie algebra if $(\mathfrak{g},B)$ is quadratic and $(\mathfrak{g},\eta^\alpha,\omega_\alpha,\mathfrak{h})$ is a k-cosymplectic Lie algebra.

Proposition 2.7. *Let $(\mathfrak{g},B)$ be a quadratic Lie algebra. A k-cosymplectic structure $(\mathfrak{g},B,\eta^1,..,\eta^k,\omega_1,..,\omega_k,\mathfrak{h})$ may be defined on $\mathfrak{g}$ if and only if there exists a family $\{\eta^1,\dots,\eta^\alpha\}$ of forms and a family of skew-symmetric derivations D^α, $\alpha=1,...,k$ of $(\mathfrak{g},B)$ such as*

(i) $D^\alpha\mathfrak{h}\perp\mathfrak{h}$ for all $\alpha=1,...,k$ (the orthogonal $\perp$ is relatively to B).

(ii) $\eta^1\wedge\dots\wedge\eta^k\neq 0$ and $\eta^\alpha_{|\mathfrak{h}}=0, \quad 1\leq\alpha\leq k;$

(iii) $\big(\cap_{\alpha=1}^k(\ker\eta^\alpha)\big)\cap\big(\cap_{\alpha=1}^k(\ker D^\alpha)\big)=\{0\}$ and $\dim\big((\ker D^1)\cap\dots\cap(\ker D^\alpha)\big)=k.$

Corollary 2.8. *$(\mathfrak{g},B,\eta^\alpha,\omega_\alpha,\mathfrak{h})$ is a quadratic k-cosymplectic semisimple Lie algebra if and only if there exist $e_1,\dots,e_k,x_1,\dots,x_k\in\mathfrak{g}$ such as*

(i) $e^1\wedge\dots\wedge e^k\neq 0$ and $e^\alpha_{|\mathfrak{h}}=0 \quad \alpha\in\{1,\dots,k\}$;

(ii) $x_\alpha\in[\mathfrak{h},\mathfrak{h}]^\perp$, $\forall\alpha\in\{1,\dots,k\}$ (the orthogonal $\perp$ is relatively to B);

*(iii) $\big(\cap_{\alpha=1}^k(\ker e^\alpha)\big)\cap\big(\cap_{\alpha=1}^k(\ker D^\alpha)\big)=\{0\}$ and $\dim\big((\ker D^1)\cap\dots\cap(\ker D^\alpha)\big)=k.$
Where $D^\alpha=ad_{x_\alpha}$; $\eta^\alpha=e^\alpha$, $\forall\alpha\in\{1,\dots,k\}$ and $\{e^1,\dots,e^k\}$ is the dual of $\{e_1,\dots,e_k\}$.*

In this case, we denote by $(\mathfrak{g},e^1,\dots,e^k,x_1,\dots,x_k,\mathfrak{h})$ the k-cosymplectic structure on the semisimple Lie algebra $\mathfrak{g}$ relatively to the Killing form B.

3 The n-symplectic structure on $\mathfrak{sl}(n,\mathbb{R})$

Let $\mathfrak{sl}(n,\mathbb{R})$ be the Lie algebra of $n\times n$ real traceless matrices, we use the basis given by the following elements $\mathfrak{B}=\{(E_{i,j})_{1\leq i\neq j\leq n},(F_i)_{1\leq i\leq n-1}\}$, here $E_{i,j}$ is the matrix such that the (i,j)-component is 1 and the other components are all zero and $F_i=E_{i,i}-E_{n,n}$, we get the brackets

$$\begin{array}{lcll|lcll}
[E_{i,j},E_{j,i}] & = & F_i-F_j & 1\leq i\neq j<n & [F_i,E_{n,i}] & = & -2E_{n,i} & 1\leq i<n \\
[E_{i,n},E_{n,i}] & = & F_i & 1\leq i<n & [F_i,E_{i,j}] & = & E_{i,j} & 1\leq i<n,\ 1\leq j\leq n \\
[E_{i,j},E_{j,k}] & = & E_{i,k} & 1\leq i\neq k\leq n & [F_j,E_{i,j}] & = & -E_{i,j} & 1\leq i\leq n,\ 1\leq j<n \\
[F_i,E_{n,j}] & = & -E_{n,j} & 1\leq i\neq j<n & [F_i,E_{i,n}] & = & 2E_{i,n} & 1\leq i<n \\
[F_j,E_{i,n}] & = & E_{i,n} & 1\leq i\neq j<n & & & &
\end{array}$$

the unspecified brackets are either zero or given by antisymmetry. Recall that the Killing form in $\mathfrak{sl}(n,\mathbb{R})$ is given by $B(x,y)=2ntr(xy)$, for $x,y\in\mathfrak{sl}(n,\mathbb{R})$. In the basis $\mathfrak{B}$, the Killing form is given by

$$B(F_i,F_i)=4n,\ 1\leq i<n,\ B(F_i,F_j)=2n \ \text{ and } \ B(E_{i,j},E_{j,i})=2n\ ;\ 1\leq i\neq j\leq n.$$

It is easy to see that
$$\mathfrak{sl}(n,\mathbb{R})=\mathfrak{aff}(n-1,\mathbb{R})\rtimes\mathbb{R}^{n-1},$$

with $\mathfrak{aff}(n-1,\mathbb{R}) \simeq span\{(E_{i,j})_{1\leq j\leq n}^{1\leq i\leq n-1},(F_i)_{1\leq i\leq n-1}\}$ and $\mathbb{R}^{n-1} \simeq span\{(E_{n,i})_{1\leq i\leq n-1}\}$. On the other hand, the derived ideal of $\mathfrak{aff}(n-1,\mathbb{R})$ is given in the basis $\mathfrak{B}$ by

$$[\mathfrak{aff}(n-1,\mathbb{R}),\mathfrak{aff}(n-1,\mathbb{R})] = span\{(E_{i,j})_{1\leq j\leq n}^{1\leq i\leq n-1},(F_i-F_j)_{1\leq i\neq j\leq n-1}\}.$$

Recall that the Lie algebra of affine transformations $\mathfrak{aff}(n,\mathbb{R})$ is a 1-symplectic Lie algebra of dimension $n(n+1)$ (see [7][6] for more details).
For $a_\alpha \in \mathfrak{sl}(n,\mathbb{R})$, $\alpha\in\{1,\dots,n\}$ let

$$a_\alpha = \sum_{1\leq i\neq j\leq n} a_{i,j}^{(\alpha)}E_{i,j} + \sum_{1\leq i\leq n-1}\lambda_i^{(\alpha)}F_i$$

the second condition of Corollary 2.3

$$B(a_\alpha,E_{i,j})=0 \quad and \quad B(a_\alpha,F_i-F_j)=0$$

implies that

$$\begin{cases} a_{j,i}^{(\alpha)}=0, \quad 1\leq i\neq j\leq n, \quad i\neq n\\ \text{and}\\ \lambda_i^{(\alpha)}=\lambda_j^{(\alpha)}, \quad i,j\in\{1,\cdots,n-1\}\end{cases}$$

Hence

$$a_\alpha = \sum_{i=1}^{n-1} a_{i,n}^{(\alpha)}E_{i,n} + \sum_{i=1}^{n-1}\lambda_i^{(\alpha)}F_i$$

with $\lambda_1^{(\alpha)}=\ldots=\lambda_{n-1}^{(\alpha)}$.
Note the following

$$a_\alpha = \sum_{i=1}^{n-1} a_{i,n}^{(\alpha)}E_{i,n} + \lambda_1^{(\alpha)}\sum_{i=1}^{n-1}F_i.$$

Lemma 3.1. *Let $x\in\mathfrak{g}$, with $x=\sum_{1\leq i\neq j\leq n}x_{i,j}E_{i,j}+\sum_{1\leq i\leq n-1}\mu_iF_i$. Then for all $\alpha\in\{1,\dots,n\}$ we have:*

$$\begin{aligned}[a_\alpha,x] = &\sum_{1\leq i\neq j\leq n-1}\Big(a_{i,n}^{(\alpha)}x_{n,j}E_{i,j} - a_{i,n}^{(\alpha)}x_{j,i}E_{j,n} - \mu_jx_{i,n}E_{i,n}\Big)\\ &+\sum_{1\leq i\leq n-1}\Big(a_{i,n}^{(\alpha)}x_{n,i}F_i + n\lambda_1^{(\alpha)}(x_{i,n}E_{i,n}-x_{n,i}E_{n,i}) - 2\mu_ia_{i,n}^{(\alpha)}E_{i,n}\Big)\end{aligned}$$

The system $[a_\alpha,x]=0$, $\alpha\in\{1,\dots,n\}$ is equivalent to

$$\lambda_1^{(\alpha)}x_{n,i}=a_{i,n}^{(\alpha)}x_{n,j}=0\ ,\ 1\leq i,j\leq n-1 \tag{3.1}$$

and

$$n\lambda_1^{(\alpha)}x_{i,n}-2\mu_ia_{i,n}^{(\alpha)}-\sum_{1\leq j\neq i\leq n-1}(\mu_ja_{i,n}^{(\alpha)}+a_{j,n}^{(\alpha)}x_{i,j})=0\ ,\ i\in\{1,\dots,n-1\} \tag{3.2}$$

Let $a_\alpha=\sum_{i=1}^{n-1}a_{i,n}^{(\alpha)}E_{i,n}+\lambda_1^{(\alpha)}\sum_{i=1}^{n-1}F_i$, for all $\alpha\in\{1,\cdots,n\}$. Consider M the matrix per blocks defined by

$$M = \left(A_{ij}\right)_{1\leq i,j\leq n} \text{ with } A_{ij} = \begin{array}{cl} \begin{pmatrix} 0 & 0 & \cdots & 0 & \cdots & 0 & 0 \\ \vdots & & & & & & \vdots \\ 0 & 0 & \cdots & 0 & \cdots & 0 & 0 \\ a_{1n}^{(i)} & a_{2n}^{(i)} & \cdots & \hat{a}_{in}^{(i)} & \cdots & a_{n-1,n}^{(i)} & -n\lambda_1^{(i)} \\ 0 & 0 & \cdots & 0 & \cdots & 0 & 0 \\ \vdots & & & & & & \vdots \\ 0 & 0 & \cdots & 0 & \cdots & 0 & 0 \end{pmatrix} & \begin{array}{l} \\ \\ \\ \leftarrow \text{row} j \ \ (\text{for } j \neq n) \\ \\ \\ \\ \end{array} \\ \begin{array}{c} \uparrow \\ \text{column } i \end{array} & \end{array}$$

where the circumflex ˆ indicates that the underlying term has been deleted. We also set

$$A_{i,n} = \begin{pmatrix} 2a_{1,n}^{(i)} & a_{1,n}^{(i)} & \cdots & a_{1,n}^{(i)} & a_{1,n}^{(i)} \\ a_{2,n}^{(i)} & 2a_{2,n}^{(i)} & a_{2,n}^{(i)} & \cdots & a_{2,n}^{(i)} \\ \vdots & & & & \vdots \\ a_{n-2,n}^{(i)} & \cdots & a_{n-2,n}^{(i)} & 2a_{n-2,n}^{(i)} & a_{n-2,n}^{(i)} \\ a_{n-1,n}^{(i)} & a_{n-1,n}^{(i)} & \cdots & a_{n-1,n}^{(i)} & 2a_{n-1,n}^{(i)} \end{pmatrix}; \quad \text{for all } i \text{ in } \{1, \cdots, n\}$$

Next, taking into account the systems (3.1) and (3.2), we get

$$\left([a_\alpha, x] = 0, \ \ \alpha = 1, \ldots, k\right) \qquad \text{if and only if} \qquad \begin{cases} x_{ni} = 0, \quad i = 1, \ldots, n-1, \\ \text{and} \\ MX = 0. \end{cases} \tag{3.3}$$

where

$$X^T = \left(\hat{x}_{1,1}, x_{1,2}, ..., x_{1,n}, x_{2,1}, \hat{x}_{2,2}, x_{2,3}, ..., x_{2,n}, .., x_{n-1,1}, x_{n-1,2}, .., x_{n-1,n-2}, \hat{x}_{n-1,n-1}, x_{n-1,n}, \mu_1, .., \mu_{n-1}\right)$$

such that the circumflex ˆ indicates that the underlying term has been deleted.
Keeping the previous notations, we have

Theorem 3.2. *Let* $x_\alpha = \sum\limits_{i=1}^{n-1} a_{i,n}^{(\alpha)} E_{i,n} + \lambda_1^{(\alpha)} \sum\limits_{i=1}^{n-1} F_i$. *For all* $n \geq 2$, $\left(\text{sl}(n, \mathbb{R}), x_1, \ldots, x_n, \mathfrak{aff}(n-1, \mathbb{R})\right)$ *is a n-symplectic strucure if and only if* $\det(M) \neq 0$.

Example 3.3. For all $n \geq 2$,

$$\left(\mathfrak{sl}(n, \mathbb{R}), E_{1,n}, \ldots, E_{n-1,n}, F_1 + \ldots + F_{n-1}, \mathfrak{aff}(n-1, \mathbb{R})\right)$$

is a n-symplectic structure.

Proof. We start with a calculation related to the bracket $[a_\alpha, x] = 0$ for all $a_\alpha \in \{E_{1,n}, \ldots, E_{n-1,n}\}$. By the systems (3.1) and (3.2), we find

$$\begin{cases} x_{n,i} = 0 & i \in \{1, \cdots, n-1\}, \\ \mu_i + \sum_{j=1}^{n-1} \mu_j = 0 & i \in \{1, \cdots, n-1\}, \\ x_{i,j} = 0 & 1 \leq i \neq j \leq n-1. \end{cases}$$

On the other hand, the linear system

$$\mu_i + \sum_{j=1}^{n-1} \mu_j = 0 \qquad i \in \{1, \cdots, n-1\}$$

is equivalent to the system $AX = 0$ where

$$A = \begin{pmatrix} 2 & 1 & 1 & \dots & 1 & 1 \\ 1 & 2 & 1 & \dots & 1 & 1 \\ \vdots & \dots & \dots & \dots & \dots & \vdots \\ 1 & 1 & \cdots & 1 & 2 & 1 \\ 1 & 1 & 1 & \cdots & 1 & 2 \end{pmatrix} \quad and \quad X = \begin{pmatrix} \mu_1 \\ \mu_2 \\ \vdots \\ \mu_{n-1} \end{pmatrix}.$$

For all $n \geq 2$, A is a $(n-1) \times (n-1)$-matrix of maximal rank, so we get $X = 0$.
On the other hand, if $a_\alpha = F_1 + \ldots + F_{n-1}$, we obtain

$$[F_1 + \ldots + F_{n-1}, x_{i,n} E_{i,n}] = 2x_{i,n}, \quad \text{for } 1 \leq i \leq n-1,$$

which completes the proof. □

The $2-$symplectic structures on $\mathfrak{sl}(2,\mathbb{R})$

(i) Let $\mathfrak{sl}(2,\mathbb{R})$ be the special linear algebra. Let $\{e_1, e_2, e_3\}$ the classical basis of $\mathfrak{sl}(2,\mathbb{R})$, that is satisfying

$$[e_1, e_2] = e_2, \quad [e_1, e_3] = -e_3, \quad [e_2, e_3] = e_1.$$

In this basis the Killing form writes

$$B = \begin{pmatrix} 2 & 0 & 0 \\ 0 & 0 & 2 \\ 0 & 2 & 0 \end{pmatrix}.$$

Let $y = y_1 e_1 + y_2 e_2$ and $z = z_1 e_1 + z_2 e_2 \in \mathrm{sl}(2,\mathbb{R})$, for $\mathfrak{h} = span\{e_1, e_2\}$. A direct calculation gives that $(\mathfrak{sl}(2,\mathbb{R}), y, z, \mathfrak{h})$ is a 2-symplectic strucure if is only if

$$y_1 z_2 - y_2 z_1 \neq 0$$

$$\begin{cases} \theta_1 & = y_2 e_1^* \wedge e_3^* - y_1 e_2^* \wedge e_3^* \\ \theta_2 & = z_2 e_1^* \wedge e_3^* - z_1 e_2^* \wedge e_3^*. \end{cases}$$

(ii) Let $\mathfrak{B} = \{E_{12}, E_{21}, F_1 = E_{11} - E_{22}\}$ be the classical basis of $\mathfrak{sl}(2,\mathbb{R})$ with the Lie brackets given by

$$[E_{12}, E_{21}] = F_1, \quad [E_{12}, F_1] = -2E_{12}, \quad \text{and} \quad [E_{21}, F_1] = 2E_{21}.$$

We denote $\mathfrak{h} = span\{E_{12}, F_1\} = \mathfrak{aff}(1,\mathbb{R})$, we have $[\mathfrak{h}, \mathfrak{h}] = span\{E_{12}\}$.
Let a_1 and a_2 be in $\mathfrak{sl}(2,\mathbb{R})$. Corollary 2.1, $a_\alpha \in [\mathfrak{h}, \mathfrak{h}]^\perp$, $\alpha \in \{1, 2\}$, implies that

$$a_1 = a_{12}^{(1)} E_{12} + \lambda_1^{(1)} F_1 \quad \text{and} \quad a_2 = a_{12}^{(2)} E_{12} + \lambda_1^{(2)} F_1$$

Let $x \in \mathfrak{sl}(2,\mathbb{R})$ such that $x = x_{12} E_{12} + x_{21} E_{21} + \mu_1 F_1$, we obtain

$$\begin{cases} [a_1, x] & = 0 \\ [a_2, x] & = 0 \end{cases} \quad \text{if and only if} \quad \begin{cases} (-2\lambda_1^{(1)} x_{12} + 2a_{12}^{(1)} \mu_1) E_{12} + 2\lambda_1^{(1)} x_{21} E_{21} - a_{12}^{(1)} x_{21} F_1 & = 0 \\ (-2\lambda_1^{(2)} x_{12} + 2a_{12}^{(2)} \mu_1) E_{12} + 2\lambda_1^{(2)} x_{21} E_{21} + a_{12}^{(2)} x_{21} F_1 & = 0 \end{cases}$$

Then

$$\big([a_\alpha, x] = 0\,,\ \alpha \in \{1, 2\}\big) \qquad \text{if and only if} \qquad \begin{cases} x_{21} & = 0 \\ MX & = 0 \end{cases}$$

where $M = \begin{pmatrix} -2\lambda_1^{(1)} & 2a_{12}^{(1)} \\ -2\lambda_1^{(2)} & 2a_{12}^{(2)} \end{pmatrix}$ and $X = \begin{pmatrix} x_{12} \\ \mu_1 \end{pmatrix}$.

Therefore

$(\mathfrak{sl}(2,\mathbb{R}), a_1, a_2, \mathfrak{h})$ is a $2-$symplectic structure if and only if $a_{12}^{(1)}\lambda_1^{(2)} - a_{12}^{(2)}\lambda_1^{(1)} \neq 0$.

Setting $a_1 = E_{12}$ and $a_2 = F_1$, allows to obtain the $2-$symplectic structure $(\mathfrak{sl}(2,\mathbb{R}), E_{12}, F_1, \mathfrak{h})$.

The $3-$symplectic structures on $\mathfrak{sl}(3,\mathbb{R})$

Let $\mathfrak{sl}(3,\mathbb{R})$, be the Lie algebra of 3×3 real traceless matrices. We will use $\mathfrak{B} = \{E_{12}, E_{13}, E_{21}, E_{23}, E_{31}, E_{32}, F_1, F_2\}$ to be the basis of $\mathfrak{sl}(2,\mathbb{R})$, where $F_1 = E_{11} - E_{33}$ and $F_2 = E_{22} - E_{33}$. We have the Lie brackets

$$\begin{array}{llll}
[E_{12}, E_{21}] = F_1 - F_2, & [E_{21}, F_1] = E_{21}, & [F_1, E_{13}] = 2E_{13}, & [F_2, E_{13}] = E_{13} \\
[E_{12}, F_1] = -E_{12}, & [E_{21}, F_2] = -E_{21}, & [F_1, E_{23}] = E_{23}, & [F_2, E_{23}] = 2E_{23} \\
[E_{12}, F_2] = E_{12}, & [E_{21}, E_{13}] = E_{23}, & [F_1, E_{31}] = -2E_{31}, & [F_2, E_{31}] = -E_{31} \\
[E_{12}, E_{23}] = E_{13}, & [E_{21}, E_{32}] = -E_{31}, & [F_1, E_{32}] = -E_{32}, & [F_2, E_{32}] = -2E_{32} \\
[E_{12}, E_{31}] = -E_{32}, & [E_{13}, E_{31}] = F_1, & [E_{13}, E_{32}] = E_{12}, & [E_{23}, E_{31}] = E_{21} \\
[E_{23}, E_{32}] = F_2 & & & .
\end{array}$$

In this basis the Killing form is given by

$$B = \begin{pmatrix}
12 & 0 & 0 & 0 & 6 & 0 & 0 & 0 \\
0 & 0 & 0 & 6 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 6 & 0 \\
0 & 6 & 0 & 0 & 0 & 0 & 0 & 0 \\
6 & 0 & 0 & 0 & 12 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 6 \\
0 & 0 & 6 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 6 & 0 & 0
\end{pmatrix}$$

Let $\mathfrak{h} = \mathfrak{aff}(2,\mathbb{R}) \simeq span\{E_{12}, E_{13}, E_{21}, E_{23}, F_1, F_2\}$, its derived Lie algebra is

$$[\mathfrak{h},\mathfrak{h}] = span\{E_{12}, E_{13}, E_{21}, E_{23}, F_1 - F_2\}.$$

Take $a_1, a_2, a_2 \in [\mathfrak{h},\mathfrak{h}]^{\perp}$ (the orthogonal $\perp$ is relatively to B). Using Lemma 3.1, it follows that, for all $\alpha \in \{1,2,3\}$, the vectors a_α are of the form:

$$\begin{aligned}
a_1 &= a_{1,3}^{(1)}E_{1,3} + a_{2,3}^{(1)}E_{2,3} + \lambda_1^{(1)}(F_1 + F_2) \\
a_2 &= a_{1,3}^{(2)}E_{1,3} + a_{2,3}^{(2)}E_{2,3} + \lambda_1^{(2)}(F_1 + F_2) \\
a_3 &= a_{1,3}^{(3)}E_{1,3} + a_{2,3}^{(3)}E_{2,3} + \lambda_1^{(3)}(F_1 + F_2)
\end{aligned}$$

Choose $x \in \mathfrak{sl}(3,\mathbb{R})$ such that:

$$x = x_{1,2}E_{1,2} + x_{2,1}E_{2,1} + x_{1,3}E_{1,3} + x_{2,3}E_{2,3} + x_{3,1}E_{3,1}E_{3,1} + x_{3,2}E_{3,2} + \mu_1 F_1 + \mu_2 F_2,$$

$$\big([a_\alpha, x] = 0, \quad \alpha \in \{1,2,3\}\big) \qquad \text{if and only if} \qquad \begin{cases} x_{31} = x_{3,2} = 0 \\ MX = 0 \end{cases}$$

where

$$M = \begin{pmatrix}
a_{2,3}^{(1)} & -3\lambda_1^{(1)} & 0 & 0 & 2a_{1,3}^{(1)} & a_{1,3}^{(1)} \\
0 & 0 & a_{1,3}^{(1)} & -3\lambda_1^{(1)} & a_{2,3}^{(1)} & 2a_{2,3}^{(1)} \\
a_{2,3}^{(2)} & -3\lambda_1^{(2)} & 0 & 0 & 2a_{1,3}^{(2)} & a_{1,3}^{(2)} \\
0 & 0 & a_{1,3}^{(2)} & -3\lambda_2^{(1)} & a_{2,3}^{(2)} & 2a_{2,3}^{(2)} \\
a_{2,3}^{(3)} & -3\lambda_1^{(3)} & 0 & 0 & 2a_{1,3}^{(3)} & a_{1,3}^{(3)} \\
0 & 0 & a_{1,3}^{(3)} & -3\lambda_1^{(3)} & a_{2,3}^{(3)} & 2a_{2,3}^{(3)}
\end{pmatrix} \quad and \quad X = \begin{pmatrix} x_{1,2} \\ x_{2,1} \\ x_{1,3} \\ x_{2,3} \\ \mu_1 \\ \mu_2 \end{pmatrix}.$$

We conclude that $(\mathfrak{sl}(3,\mathbb{R}), a_1, a_2, a_3, \mathfrak{h})$ is a 3−symplectic structure if, and only if $\det M \neq 0$. A direct calculation gives that $(\mathfrak{sl}(3,\mathbb{R}), E_{13}, E_{23}, F_1+F_2, \mathfrak{h})$ is a 3-symplectic structure. Using (2.1), we find that

$$\begin{cases} \theta_1 & = -12F_1^* \wedge E_{31}^* - 6E_{21}^* \wedge E_{32}^* - 6F_2^* \wedge E_{31}^* \\ \theta_2 & = -6F_1^* \wedge E_{32}^* - 6E_{12}^* \wedge E_{31}^* - 12F_2^* \wedge E_{32}^* \\ \theta_3 & = 18E_{13}^* \wedge E_{31}^* + 18E_{23}^* \wedge E_{32}^*. \end{cases}$$

4 The $(n-1)$-cosymplectic structure on $\mathfrak{sl}(n,\mathbb{R})$

Let $\mathfrak{sl}(n,\mathbb{R})$ be the Lie algebra of $n\times n$ real traceless matrices, we will use the same previous basis $\mathfrak{B} = \{(E_{i,j})_{1\leq i\neq j\leq n}, (F_i)_{1\leq i\leq n-1}\}$, where $E_{i,j}$ is the matrix such that the (i,j)-component is 1 and the other components are all zero and $F_i = E_{i,i} - E_{n,n}$. It is straightforward to check that in basis $\mathfrak{B}$, the Killing form of $\mathfrak{sl}(n,\mathbb{R})$ is given by

$$B(F_i,F_i) = 4n,\ 1\leq i<n,\ B(F_i,F_j) = 2n \quad \text{and} \quad B(E_{i,j},E_{j,i}) = 2n\ ;\ 1\leq i\neq j\leq n.$$

Furthermore we have:

$$\mathfrak{h} = span\{(E_{i,j})_{1\leq i\neq j\leq n-1}, (F_p)_{1\leq p\leq n-1}\} \simeq \mathfrak{gl}(n-1,\mathbb{R})$$

is a Lie subalgebra of $\mathfrak{sl}(n,\mathbb{R})$ whose derived Lie subalgebra is

$$[\mathfrak{h},\mathfrak{h}] = span\{(E_{i,j})_{1\leq i\neq j\leq n-1}, (F_i-F_j)_{1\leq i\neq j\leq n-1}\}.$$

For $a_\alpha \in \mathfrak{sl}(n,\mathbb{R})$, $\alpha\in\{1,\ldots,n-1\}$ let

$$a_\alpha = \sum_{1\leq i\neq j\leq n} a_{i,j}^{(\alpha)} E_{i,j} + \sum_{1\leq i\leq n-1} \lambda_i^{(\alpha)} F_i.$$

The second condition of Corollary 2.8

$$B(a_\alpha, E_{i,j}) = 0 \quad \textit{and} \quad B(a_\alpha, F_i - F_j) = 0$$

implies that

$$\begin{cases} a_{j,i}^{(\alpha)} = 0; \quad 1\leq i\neq j\leq n-1 \\ \text{and} \\ \lambda_i^{(\alpha)} = \lambda_j^{(\alpha)}; \quad i,j\in\{1,\cdots,n-1\} \end{cases}$$

Hence

$$a_\alpha = \sum_{i=1}^{n-1} a_{i,n}^{(\alpha)} E_{i,n} + \sum_{j=1}^{n-1} a_{n,j}^{(\alpha)} E_{n,j} + \sum_{i=1}^{n-1} \lambda_i^{(\alpha)} F_i,$$

with $\lambda_1^{(\alpha)} = \ldots = \lambda_{n-1}^{(\alpha)}$.
Therefore, we deduce

$$a_\alpha = \sum_{i=1}^{n-1} a_{i,n}^{(\alpha)} E_{i,n} + \sum_{j=1}^{n-1} a_{n,j}^{(\alpha)} E_{n,j} + \lambda^{(\alpha)} \sum_{i=1}^{n-1} F_i; \quad \alpha\in\{1,\ldots,n-1\}.$$

Proposition 4.1. $\big(\mathfrak{sl}(n,\mathbb{R}),\ E_{\alpha,n}^*,\ \sum_{i=1}^{n-1}(F_i^*\wedge E_{n,\alpha}^*) + \sum_{1\leq j\neq\alpha\leq n-1}(E_{j,\alpha}^*\wedge E_{n,\alpha}^*),\ \mathfrak{h}\big)$ *is a $(n-1)-$cosymplectic Lie algebra and $R_\alpha = E_{\alpha,n}$ are the Reeb vectors associated with this structure.*

Proof. Set $\eta^\alpha = E_{\alpha,n}^*$ and $\omega_\alpha = \sum_{i=1}^{n-1} F_i^*\wedge E_{n,\alpha}^* + \sum_{1\leq j\neq\alpha\leq n-1} E_{j,\alpha}^*\wedge E_{n,\alpha}^*$, for all $\alpha\in\{1,\ldots,n-1\}$. We get, respectively, $(\ker\omega_1)\cap\ldots\cap(\ker\omega_k) = span\{E_{\alpha,n}\}_{1\leq\alpha\leq n-1}$ and $(\ker\eta^1)\cap\ldots\cap(\ker\eta^k) = span\{E_{i,j}\ ;\ 1\leq i\leq n; 1\leq j\leq n-1; i\neq j\}$. □

Example 4.2. (i) For $n=2$, it is easy to verify that $\big(\mathfrak{sl}(2,\mathbb{R}), E_{12}^*, F_1^*\wedge E_{21}^*, span\{F_1\}\big)$ is a cosymplectic structure, with the caracteristic or Reeb vector is $R = E_{12}$.

(ii) A direct computation yields $(\mathfrak{sl}(3,\mathbb{R}),\ \eta^1,\ \eta^2,\ \omega_1,\omega_2,\mathfrak{h})$ where

$$\begin{cases} \mathfrak{h} & = span\{E_{12}, E_{21}, F_1, F_2\} \simeq \mathfrak{gl}(2,\mathbb{R}) \\ \eta^1 & = E_{13}^* \quad \text{and} \quad \eta^2 = E_{23}^* \\ \omega_1 & = F_1^* \wedge E_{31}^* + F_2^* \wedge E_{31}^* + E_{21}^* \wedge E_{32}^* \\ \omega_2 & = F_1^* \wedge E_{32}^* + F_2^* \wedge E_{32}^* + E_{12}^* \wedge E_{31}^* \end{cases}$$

is a 2-cosymplectic structure, with the Reeb vectors $R_1 = E_{13}$ and $R_2 = E_{23}$.

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