Palestine Journal of Mathematics

Vol. 12(4)(2023) , 331–339

Skew $(\mu_1 + u\mu_2 + v\mu_3 + w\mu_4 + uv\mu_5 + vw\mu_6 + uw\mu_7 + uvw\mu_8)$ Constacyclic Codes over $F_q[u, v, w]/ < u^2 - 1, v^2 - 1, w^2 - 1, uv - vu, vw - wv, wu - uw >$

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Communicated by Ayman Badawi

MSC 2010 Classifications: Primary 94B05, 94B15; Secondary 16S36, 94B60.

Keywords and phrases: Linear Codes , Cyclic Codes, Skew Polynomial Rings, Skew Constacyclic Codes.

Abstract In this paper, $(\theta - \mu)$**-constacyclic codes (skew constacyclic code)** over $\mathcal{R} = F_q[u, v, w]/ < u^2 - 1, v^2 - 1, w^2 - 1, uv - vu, vw - wv, wu - uw >= F_q + uF_q + vF_q + wF_q + uvF_q + uwF_q + vwF_q + uvwF_q$ are studied. The structure of $(\theta - \mu)$**-constacyclic codes** and their **duals** are provided. The **generator polynomials** and **generating idempotents** of $(\theta - \mu)$**-constacyclic codes** over $\mathcal{R}$ are described. Properties of $(\theta - \mu)$**-constacyclic codes** generated by **monic right divisors** of $x^n - \mu$, where μ is a **unit** element are exhibited.

1 Introduction

Gursoy et al. [5] studied a special class of **linear codes**, called **skew cyclic codes**, over the ring $F_p + vF_p$, and constructed **skew cyclic codes** over the ring $F_p + vF_p$.

AL-Ashker et al. [12] studied **skew constacyclic codes** over finite non-chain rings of the form $F_p + vF_p$, where p is an odd prime and $v^2 = v$.

Dertli et al. [2] studied **skew cyclic codes** over the ring $F_q + uF_q + vF_q + uvF_q$, where $u^2 = 1$, $v^2 = 1$, $uv = vu$, $q = p^m$ and p is an odd prime. Islam et al. [8] studied the structural properties of **skew cyclic** and **skew constacyclic codes** over the ring $F_q + uF_q + vF_q + uvF_q$, where $u^2 = u$, $v^2 = v$, $uv = vu$, $q = p^m$ and p is an odd prime. Further, **generating polynomials** as well as **idempotent generators** for **skew cyclic** and **skew constacyclic codes** codes are determined.

Yue et al. [6] studied **skew cyclic codes** over $\mathcal{R} = F_q + uF_q + vF_q + uvF_q + v^2F_q + uv^2F_q$, where $v^3 = v$, $u^2 = 1, , q = p^m$ and p is an odd prime. **Gray map** from $\mathcal{R}$ to F_q^6 was defined. The **Gray image** of a **linear code** of length n over $\mathcal{R}$ are considered. Ashraf et al. [10] constructed **quantum codes** over F_p from **cyclic codes** over the ring $F_p + uF_p + vF_p + uvF_p + v^2F_p + uv^2F_p$ using **self-orthogonal** property of these classes of codes. Al- Shorbassi et al. [16, 15] studied **Quadratic residue codes** and **skew constacyclic codes** over the ring $F_p + uF_p + vF_p + uvF_p + v^2F_p + uv^2F_p$. The structural properties of $(\theta - \beta)$**-constacyclic codes** over the ring $\mathcal{R}$ are studied. Further, **generating polynomials** and **idempotent generators** for $(\theta - \beta)$**-constacyclic codes** over the ring $\mathcal{R}$ are studied.

Dertli et al. [1] constructed **quantum codes** over F_2 by using the **cyclic codes** over $A_3 = F_2 + uF_2 + vF_2 + wF_2 + uvF_2 + uwF_2 + vwF_2 + uvwF_2$, where $u^2 = u$, $v^2 = v$,$w^2 = w$ and $uv = vu$, $vw = wv$,$uw = wu$. Moreover, the parameters of **quantum codes** over F_2 are determined. Kewat et al. [14] studied **cyclic codes** over the **Frobenius rings** $\mathcal{R} = F_p[u, v, w]/ < u^2 - 1, v^2 - 1, w^2 - 1, uv - vu, vw - wv, uw - wu >= F_p + uF_p + vF_p + wF_p + uvF_p + uwF_p + vwF_p + uvwF_p$, where $u^2 = 0$, $v^2 = 0$,$w^2 = 0$ and $uv = vu$, $vw = wv$,$uw = wu$, and found a unique set of **generators** for these codes and characterize the free **cyclic codes**. Islam et al. [7] constructed **quantum codes** over F_p by using the **cyclic codes** of length n over the ring $F_p[u, v, w]/ < u^2 - 1, v^2 - 1, w^2 - 1, uv - vu, vw - wv, uw - wu >$. Moreover obtained the **self-orthogonal** properties of **cyclic codes** over $\mathcal{R}$. Al-Shorbassi et al. [17] defined **Quadratic residue codes** over the finite commutative ring $\mathcal{R} = F_p + uF_p + vF_p + wF_p + uvF_p + uwF_p + vwF_p + uvwF_p$ by their **generating idempotents**, where p and q are distinct odd prime.

The plan of the paper is organized as follows:

In Section 2, we define the **Gray map** $\Psi : \mathcal{R} \to F_q^8$, and find $\mu_1, \mu_2, \ldots, \mu_8$ which satisfies $\mu_i\mu_j = \begin{cases} \mu_i & \text{if } i = j, \\ 0 & \text{if } i \neq j, \end{cases}$ and $\sum_{i=1}^{8} \mu_i = 1$.where $1 \leq i, j \leq 8$. **In section 3**, we define **cyclic and skew cyclic** codes over $\mathcal{R}$. **In section 4** we define **μ-constacyclic codes** and **$(\theta - \mu)$-constacyclic codes (skew constacyclic code)** over $\mathcal{R}$. Further, we study structural properties, specially, **generating polynomials** and **idempotent generators** for **$(\theta - \mu)$-constacyclic codes**. **Section 5** concludes the paper.

2 Preliminaries: Gray map and linear codes over $\mathcal{R}$

Let $\mathcal{R} = F_q[u, v, w]/ < u^2 - 1, v^2 - 1, w^2 - 1, uv - vu, vw - wv, wu - uw >= F_q + uF_q + vF_q + wF_q + uvF_q + uwF_q + vwF_q + uvwF_q = \{a_1 + ua_2 + va_3 + wa_4 + uva_5 + uwa_6 + vwa_7 + uvwa_8 | a_i \in F_q, 1 \leq i \leq 8\}$, where $q = p^m$. This ring has characteristic q and cardinality q^8 and it is Frobenius rings but not local.

Linear code of length n over $\mathcal{R}$ is an **$\mathcal{R}$-submodule** of $\mathcal{R}^n$ and each member of the code is called **codeword**. Let $c = (c_0, c_1, \ldots, c_{n-1})$, $\acute{c} = (\acute{c}_0, \acute{c}_1, \ldots, \acute{c}_{n-1})$ be any two element of $\mathcal{R}^n$. Then **Euclidean inner product** of c and $\acute{c}$ is defined as $c \cdot \acute{c} = \sum_{i=0}^{n-1} c_i\acute{c}_i$. The **dual of the code** $\mathcal{C}$ is defined by $\mathcal{C}^\perp = \{x \in \mathcal{R}^n | xy = 0, \forall y \in \mathcal{C}\}$. If $\mathcal{C} = \mathcal{C}^\perp$, we say that the code $\mathcal{C}$ is **self dual**, and if $\mathcal{C} \subseteq \mathcal{C}^\perp$ it is called **self-orthogonal**.

Recall the **Gray map** from [7], which defined as follows:
$\Psi : \mathcal{R} \to F_q^8$ $\Psi(a_1 + ua_2 + va_3 + wa_4 + uva_5 + vwa_6 + uwa_7 + uvwa_8) = (\alpha_1, \alpha_2, \ldots, \alpha_8)$. Where
$\alpha_1 = a_1 + a_2 + a_3 + a_4 + a_5 + a_6 + a_7 + a_8,$
$\alpha_2 = a_1 + a_2 + a_3 - a_4 + a_5 - a_6 - a_7 - a_8,$
$\alpha_3 = a_1 + a_2 - a_3 + a_4 - a_5 - a_6 + a_7 - a_8,$
$\alpha_4 = a_1 - a_2 + a_3 + a_4 - a_5 + a_6 - a_7 - a_8,$
$\alpha_5 = a_1 + a_2 - a_3 - a_4 - a_5 + a_6 - a_7 + a_8,$
$\alpha_6 = a_1 - a_2 - a_3 + a_4 + a_5 - a_6 - a_7 + a_8,$
$\alpha_7 = a_1 - a_2 + a_3 - a_4 - a_5 - a_6 + a_7 + a_8,$
$\alpha_8 = a_1 - a_2 - a_3 - a_4 + a_5 + a_6 + a_7 - a_8.$

By Maple one can have the result
$\mu_1 = \gamma[1 + u + v + w + uv + vw + uw + uvw],$
$\mu_2 = \gamma[1 + u + v - w + uv - vw - uw - uvw],$
$\mu_3 = \gamma[1 + u - v + w - uv - vw + uw - uvw],$
$\mu_4 = \gamma[1 - u + v + w - uv + vw - uw - uvw],$
$\mu_5 = \gamma[1 + u - v - w - uv + vw - uw + uvw],$
$\mu_6 = \gamma[1 - u - v + w + uv - vw - uw + uvw],$
$\mu_7 = \gamma[1 - u + v - w - uv - vw + uw + uvw],$
$\mu_8 = \gamma[1 - u - v - w + uv + vw + uw - uvw].$

Note that $\mu_i\mu_j = \begin{cases} \mu_i & \text{if } i = j, \\ 0 & \text{if } i \neq j, \end{cases}$ and $\sum_{i=1}^{8} \mu_i = 1$, where $1 \leq i, j \leq 8$ and $8\gamma \equiv 1(mod\ q)$. This map Ψ can be extended in natural way to F_q^{8n} as $\Psi : \mathcal{R}^n \to F_q^{8n}$ define by $\Psi(r_1, r_2, \ldots, r_n) = (\alpha_1, \ldots, \alpha_n, \alpha_2, , \ldots, \alpha_n, \ldots, \alpha_8, \ldots, \alpha_n)$.

By **Chinese Remainder Theorem**,we have $\mathcal{R} = \oplus_{i=1}^{8} \mu_i\mathcal{R} \cong \oplus_{i=1}^{8} \mu_iF_q$.
Thus, every element of $r \in \mathcal{R}$ can be uniquely expressed $r = \sum_{i=1}^{8} a_i\mu_i$, where $a_i \in F_q$ and

$1 \leq i \leq 8$.

Hamming weight $w_H(a)$ is defined as the number of the non-zero components in $a = (a_1, a_2, \ldots, a_n) \in \mathcal{C}$, and **Hamming distance** between two codewords a and b of $\mathcal{C}$ is defined as $d_H(a, b) = w_H(a - b)$ while the **Hamming distance** for the code $\mathcal{C}$ is denoted by $d_H(\mathcal{C})$ and defined as $d_H(\mathcal{C}) = min\{d_H(a, b)|a \neq b, \forall a, b \in \mathcal{C}\}$. **The Lee weight** of an element $r \in \mathcal{R}$ is defined by $w_L(r) = w_H(\Psi(r))$ and **Lee weight** for the codeword $a = (a_1, a_2, \ldots, a_n) \in \mathcal{R}^n$ is $w_L(a) = \sum_{i=1}^{n} w_L(a_i)$. The **Lee distance** between two codewords $a, b \in \mathcal{R}^n$ is defined as $d_L(a, b) = w_L(a - b) = \sum_{i=1}^{n} w_L(a_i - b_i)$ and the *Lee distance* for the code $\mathcal{C}$ is defined by $d_L(\mathcal{C}) = min\{d_L(a, b)|a \neq b, \forall a, b \in \mathcal{C}\}$. See [3].

Theorem 2.1. *The Gray map* Ψ*:* $\mathcal{R} \to {F_q}^8$ *is linear and* $d_L(x, y) = d_H(\Psi(x), \Psi(x))$.

Proof. Let $x_1 = a_1 + ua_2 + va_3 + wa_4 + uva_5 + vwa_6 + uwa_7 + uvwa_8$ and $x_2 = \acute{a}_1 + u\acute{a}_2 + v\acute{a}_3 + w\acute{a}_4 + uv\acute{a}_5 + vw\acute{a}_6 + uw\acute{a}_7 + uvw\acute{a}_8$, then $\Psi(x_1 + x_2) = (\alpha_1 + \acute{\alpha}_1, \alpha_2 + \acute{\alpha}_2, \ldots, \alpha_8 + \acute{\alpha}_8) = (\alpha_1, \alpha_2, \ldots, \alpha_8) + (\acute{\alpha}_1, \acute{\alpha}_2, \ldots, \acute{\alpha}_8) = \Psi(x) + \Psi(y)$.

For any $\beta \in F_q$, we have $\Psi(\beta x_1) = (\beta\alpha_1, \beta\alpha_2, \beta\alpha_3, \beta\alpha_4, \beta\alpha_5, \beta\alpha_6, \beta\alpha_7, \beta\alpha_8) = \beta(\alpha_1, \alpha_2, \alpha_3, \alpha_4, \alpha_5, \alpha_6, \alpha$ $\beta\Psi(x_1)$. Which implies that Ψ is linear .

Now $d_L(x, y) = w_L(x - y) = w_H(\Psi(x - y)) = w_H(\Psi(x) - \Psi(y)) = d_H(\Psi(x), \Psi(y))$. □

Theorem 2.2. *If* $\mathcal{C}$ *is an* $[n, k, d_L]$ *linear codes over* $\mathcal{R}$*, then* $\Psi(\mathcal{C})$ *ia a* $[8n, k, d_H]$ *linear codes over* F_q.

Proof. By Theorem 2.1, we have that Ψ is linear and bijective, then $|\mathcal{C}| = |\Psi(\mathcal{C})|$, so $\Psi(\mathcal{C})$ ia a $[8n, k, d_H]$ and $d_L = d_H$. □

Theorem 2.3 ([1], Theorem 3). *Let* $\mathcal{C}$ *is a code of length* n *over* $\mathcal{R}$*. If* $\mathcal{C}$ *is self-orthogonal, then* $\Psi(\mathcal{C})$ *is self-orthogonal.*

Theorem 2.4 ([7], Theorem 3). *If* $\mathcal{C}$ *is a code of length* n *over* $\mathcal{R}$*. Then* $\Psi(\mathcal{C}^{\perp}) = (\Psi(\mathcal{C}))^{\perp}$.

3 Cyclic and Skew Cyclic Codes over $\mathcal{R}$

In this section, first we define **cyclic and skew cyclic codes** over $\mathcal{R}$, and then study an important theorems which discuss the algebraic construction of **cyclic and skew cyclic codes** of any length n over $\mathcal{R}$.

Let ρ, σ be maps from $\mathcal{R}^n$ to $\mathcal{R}^n$ given by

$$\rho(c_0, c_1, \ldots, c_{n-1}) = (c_{n-1}, c_0, c_1, \ldots, c_{n-2}),$$

$$\sigma(c_0, c_1, \ldots, c_{n-1}) = (-c_{n-1}, c_0, c_1, \ldots, c_{n-2}).$$

Then $\mathcal{C}$ is said to be **cyclic** if $\rho(\mathcal{C}) = \mathcal{C}$, **negacyclic** if $\sigma(\mathcal{C}) = \mathcal{C}$. A **cyclic code** $\mathcal{C}$ of length n over $\mathcal{R}$ can be regarded as an **ideal** of the ring $\mathcal{R}/ < x^n - 1 >$.

Let θ_t be the **automorphism** $\theta_t : F_q \to F_q$ defined by $\theta_t(a) = a^{p^t}$ and this can be extended to $\mathcal{R}$ defined as : $\theta(r) = a_1^{p^t} + ua_2^{p^t} + va_3^{p^t} + wa_4^{p^t} + uva_5^{p^t} + vwa_6^{p^t} + uwa_7^{p^t} + uvwa_8^{p^t}$. Moreover let $\mathcal{R}[x, \theta_t]$ be the **skew polynomial ring** defined as $\mathcal{R}[x, \theta_t] = \{a_0 + a_1x + \ldots + a_{n-1}x^{n-1}|a_i \in \mathcal{R}, n \in N\}$. This ring is a noncommutative ring. The **addition** in the ring $\mathcal{R}[x, \theta_t]$ is the usual polynomial addition and **multiplication** is defined using the rule as follows $(ax^i)(bx^j) = a\theta_t^i(b)x^{i+j}$. See [19].

Definition 3.1 ([2], Definition 3.1). A subset $\mathcal{C}$ of $\mathcal{R}^n$ is called a skew cyclic code of length n if $\mathcal{C}$ is a submodule of $\mathcal{R}^n$ and if $c = (c_0, c_1, \ldots, c_{n-1}) \in \mathcal{C}$, then $(\theta_t(c_{n-1}), \theta_t(c_0), \ldots, \theta_t(c_{n-2})) \in \mathcal{C}$.

Let $B_1, B_2, \ldots, B_8$ be the linear codes, then $\oplus_{i=1}^{8} B_i = \{a_1+a_2+a_3+a_4+a_5+a_6+a_7+a_8 | a_i \in B_i\}$ and $\otimes_{i=1}^{8} B_i = \{(a_1, a_2, a_3, a_4, a_5, a_6, a_7, a_8) | a_i \in B_i\}$.

Definition 3.2. Let $\mathcal{C}$ be a linear code of length n over $\mathcal{R}$, we define

$$\begin{aligned}
C_1 &= \{a_1 \in {F_q}^n | \textstyle\sum_{i=1}^{8} a_i\mu_i \in \mathcal{C},\ for\ some\ a_j \in {F_q}^n, j \neq 1, 1 \leq j \leq 8\},\\
C_2 &= \{a_2 \in {F_q}^n | \textstyle\sum_{i=1}^{8} a_i\mu_i \in \mathcal{C},\ for\ some\ a_j \in {F_q}^n, j \neq 2, 1 \leq j \leq 8\},\\
C_3 &= \{a_3 \in {F_q}^n | \textstyle\sum_{i=1}^{8} a_i\mu_i \in \mathcal{C},\ for\ some\ a_j \in {F_q}^n, j \neq 3, 1 \leq j \leq 8\},\\
C_4 &= \{a_4 \in {F_q}^n | \textstyle\sum_{i=1}^{8} a_i\mu_i \in \mathcal{C},\ for\ some\ a_j \in {F_q}^n, j \neq 4, 1 \leq j \leq 8\},\\
C_5 &= \{a_5 \in {F_q}^n | \textstyle\sum_{i=1}^{8} a_i\mu_i \in \mathcal{C},\ for\ some\ a_j \in {F_q}^n, j \neq 5, 1 \leq j \leq 8\},\\
C_6 &= \{a_6 \in {F_q}^n | \textstyle\sum_{i=1}^{8} a_i\mu_i \in \mathcal{C},\ for\ some\ a_j \in {F_q}^n, j \neq 6, 1 \leq j \leq 8\},\\
C_7 &= \{a_7 \in {F_q}^n | \textstyle\sum_{i=1}^{8} a_i\mu_i \in \mathcal{C},\ for\ some\ a_j \in {F_q}^n, j \neq 7, 1 \leq j \leq 8\},\\
C_8 &= \{a_8 \in {F_q}^n | \textstyle\sum_{i=1}^{8} a_i\mu_i \in \mathcal{C},\ for\ some\ a_j \in {F_q}^n, j \neq 8, 1 \leq j \leq 8\}.
\end{aligned}$$

Note that $\mathcal{C}_1, \mathcal{C}_2, \ldots, \mathcal{C}_8$ are linear codes over F_q. Moreover, $\mathcal{C} = \oplus_{i=1}^{8} \mu_i \mathcal{C}_i$ and $|\mathcal{C}| = |\mathcal{C}_1||\mathcal{C}_2| \ldots |\mathcal{C}_8|$.

Theorem 3.3 ([1], Theorem 4). *Let $\mathcal{C} = \oplus_{i=1}^{8} \mu_i \mathcal{C}_i$ be a linear code of length n over F_q. Then $\mathcal{C}^{\perp} = \oplus_{i=1}^{8} \mu_i \mathcal{C}_i^{\perp}$.*

Lemma 3.4. *let G_i be the generator matrix of q-ary linear codes $\mathcal{C}_i$ respectively, where $1 \leq i \leq 8$, then the generator matrix of $\mathcal{C}$ and $\Psi(\mathcal{C})$ are*

$$G = \begin{pmatrix} \mu_1\mathcal{C}_1 \\ \mu_2\mathcal{C}_2 \\ \vdots \\ \mu_8\mathcal{C}_8 \end{pmatrix},\ and\ \Psi(G) = \begin{pmatrix} \Psi(\mu_1\mathcal{C}_1) \\ \Psi(\mu_2\mathcal{C}_2) \\ \vdots \\ \Psi(\mu_8\mathcal{C}_8) \end{pmatrix}, respectively.$$

Theorem 3.5 ([7], Theorem 4). *Let $\mathcal{C}$ be a linear code of length n over F_q. Then $\Psi(\mathcal{C}) = \otimes_{i=1}^{8} \mathcal{C}_i$*

Corollary 3.6. *Let $\mathcal{C} = \oplus_{i=1}^{8} \mu_i \mathcal{C}_i$ be a linear code of length n over F_q, where $\mathcal{C}_i$ is the $[n, k_i, d_L(\mathcal{C}_i)]$ linear code over F_q, then $\Psi(\mathcal{C})$ is the $[8n, \sum_{i=1}^{8} k_i, min\{d_L(\mathcal{C}_i)|i = 1, 2, \ldots, 8\}]$.*

Theorem 3.7 ([7], Theorem 6). *Let $\mathcal{C} = \oplus_{i=1}^{8} \mu_i \mathcal{C}_i$ be a linear code of length n over F_q, Then $\mathcal{C}$ is a cyclic code if and only if $\mathcal{C}_i$, for $i = 1, 2, \ldots, 8$ are cyclic codes of length n over F_q.*

Theorem 3.8 ([1], Proposition 2). *Let $\mathcal{C} = \oplus_{i=1}^{8} \mu_i \mathcal{C}_i$ be a cyclic code of length n over F_q, Then $\mathcal{C} =< \mu_1 f_1, \mu_2 f_2, \ldots, \mu_8 f_8 >$.*

Lemma 3.9. *Let $\mathcal{C} = \oplus_{i=1}^{8} \mu_i \mathcal{C}_i$ be a cyclic code of length n over F_q, Then $\mathcal{C} =< f(x) >$ and $f(x)|(x^n - 1)$, where $f_i(x)$ the generator polynomials of $\mathcal{C}_i$, $i = 1, 2, \ldots, 8$ and $f(x) = \sum_{i=1}^{8} \mu_i f_i(x)$.*

Corollary 3.10. *Let $\mathcal{C} = \oplus_{i=1}^{8} \mu_i \mathcal{C}_i$ be a cyclic code of length n over F_q. Then $|\mathcal{C}| = q^{8n - \sum_{i=1}^{8} deg(f(x))}$, where $f_i(x)$ the generator polynomials of $\mathcal{C}_i$, $i = 1, 2, \ldots, 8$.*

Theorem 3.11. *Let $\mathcal{C} = \oplus_{i=1}^{8} \mu_i \mathcal{C}_i$ be a cyclic code of length n over F_q and $f_i(x)$ be the generator polynomials of $\mathcal{C}_i$ such that $x^n - 1 = h_i(x) f_i(x)$ for $i = 1, 2, \ldots, 8$. Then*
(i) $\mathcal{C}^{\perp} =< \sum_{i=1}^{8} \mu_i h_i^*(x) >$, where $h_i^*(x)$ is the ***reciprocal polynomial*** *of $h_i(x)$ and $h_i^*(x) = x^{deg(h_i(x))} h_i(x^{-1})$, for $i = 1, 2, \ldots, 8$.*
(ii) $|\mathcal{C}^{\perp}| = q^{\sum_{i=1}^{8} deg(f_i(x))}$.
(iii) $\mathcal{C}$ is a self-dual cyclic code if and only if $\mathcal{C}_i$ for $i = 1, 2, \ldots, 8$ are the self-dual cyclic codes over F_q.

Proof. (i) Since $\mathcal{C}=\oplus_{i=1}^{8}\mu_i\mathcal{C}_i$ is a cyclic code of length n over F_q, by Theorem 3.7, $\mathcal{C}_1,\mathcal{C}_2,\ldots,\mathcal{C}_8$ are cyclic code of length n over F_q. So $\mathcal{C}_1^{\perp},\mathcal{C}_2^{\perp},\ldots,\mathcal{C}_8^{\perp}$ are cyclic code of length n over F_q. Hence, by Theorem 3.1 $\mathcal{C}^{\perp}=\oplus_{i=1}^{8}\mu_i\mathcal{C}_i^{\perp}$ is cyclic code of length n over F_q. Take $\mathcal{C}_i^{\perp}=<h_i^*(x)>$, so $\mathcal{C}^{\perp}=<\sum_{i=1}^{8}\mu_i h_i^*(x)>$, where $h_i^*(x)=x^{deg(h_i(x))}h_i(x^{-1})$, for $i=1,2,\ldots,8$.

(ii) $|\mathcal{C}^{\perp}|=|\mathcal{C}_1^{\perp}||\mathcal{C}_2^{\perp}|\ldots|\mathcal{C}_8^{\perp}|=q^{deg(f_1(x))}.q^{deg(f_2(x))}.\ldots.q^{deg(f_8(x))}$
$=q^{\sum_{i=1}^{8}deg(f_i(x))}$.

(iii) It is obvious. □

4 Skew $(\mu_1+u\mu_2+v\mu_3+w\mu_4+uv\mu_5+vw\mu_6+uw\mu_7+uvw\mu_8)$-Constacyclic Codes over $\mathcal{R}$

In this section, we begin definition of μ**-constacyclic code and** $(\theta-\mu)$**-consta cyclic codes (skew constacyclic code)** over $\mathcal{R}=F_q[u,v,w]/<u^2-1,v^2-1,w^2-1,uv-vu,vw-wv,wu-uw>=F_q+uF_q+vF_q+wF_q+uvF_q+uwF_q+vwF_q+uvwF_q$, then we will write all results of μ**-constacyclic code and** $(\theta-\mu)$**-constacyclic codes**.

Let μ be a **unit** in $\mathcal{R}$, $\mathcal{C}$ be a **linear code** of length n over $\mathcal{R}$, and ϱ be a map from $\mathcal{R}^n$ to $\mathcal{R}^n$ given by $\varrho(c_0,c_1,\ldots,c_{n-1})=(\mu c_{n-1},c_0,c_1,\ldots,c_{n-2})$. If $\varrho(\mathcal{C})=\mathcal{C}$, then $\mathcal{C}$ is said to be μ**-constacyclic code**. See [13, 4].

Note that a μ**-constacyclic codes** is **cyclic codes** if $\mu=1$ and **negacyclic** codes if $\mu=-1$.

Definition 4.1. Let $\mu=\mu_1+u\mu_2+v\mu_3+w\mu_4+uv\mu_5+vw\mu_6+uw\mu_7+uvw\mu_8$ be a unit in $\mathcal{R}$, where $\mu_i\in {F_q}^*$ for $i=1,2,\ldots,8$ and θ be the automorphism on $\mathcal{R}$. A linear code $\mathcal{C}$ of length n is said to be skew constacyclic code or specifically $(\theta-\mu)$-constacyclic codes over $\mathcal{R}$ if and only if $\mathcal{C}$ is invariant under the $(\theta-\mu)$-constacyclic shift operator $\vartheta_{\theta,\mu}:\mathcal{R}^n\to\mathcal{R}^n$ defined as $\vartheta_{\theta,\mu}(c)=\vartheta_{\theta,\mu}(c_0,c_1,\ldots,c_{n-1})=(\mu\theta(c_{n-1}),\theta(c_0),\ldots,\theta c_{n-2})$, i.e $\mathcal{C}$ is $(\theta-\mu)$-consta cyclic codes if and only if $\vartheta_{\theta,\mu}(\mathcal{C})=\mathcal{C}$. This code becomes skew cyclic code when $\mu=1$ and skew negacyclic code when $\mu=-1$.

Theorem 4.2. *Let $\mathcal{C}$ be $(\theta-\mu)$-constacyclic code of length n over $\mathcal{R}$, let k be the order of the automorphism and n be the length of the code with $gcd(n,k)=1$. Then $\mathcal{C}$ is a μ-constacyclic code of length n over $\mathcal{R}$.*

Proof. Let $gcd(n,k)=1$, then there exist an integers a,b, such that $ak=1+bn$. Let $c(x)=\sum_{i=0}^{n-1}c_ix^i\in\mathcal{C}$. Then $x^ic(x)\in\mathcal{C}$, $1\le i\le ak$.

Now $x^{ak}c(x)=x^{ak}\sum_{i=0}^{n-1}c_ix^i=\theta_i^{ak}c_0x^{ak}+\theta_i^{ak}c_1x^{ak+1}+\cdots+\theta_i^{ak}c_{n-1}$
$x^{ak+n-1}=c_0x^{1+bn}+c_1x^{2+bn}+\cdots+c_{n-1}x^{n+bn}=\mu^b(c_0x+c_1x^2+\cdots+c_{n-2}$
$x^{n-1}+\mu x^{n-1})$.

Hence $\mu^bx^{ak}c(x)=c_0x+c_1x^2+\cdots+c_{n-2}x^{n-1}+c_{n-1}\mu\in\mathcal{C}$. So $\mathcal{C}$ is a μ-constacyclic code of length n over $\mathcal{R}$. □

Corollary 4.3. *Let $gcd(n,k)=1$. If $f(x)$ is a right divisor of $x^n-\mu$ in the skew polynomial ring $\mathcal{R}[x,\theta]$, then $f(x)$ is a factor of $x^n-\mu$ in the polynomial ring $\mathcal{R}[x]$.*

Definition 4.4 ([11], Definition 4.1). Let $\mathcal{C}$ be a linear code of length n over $\mathcal{C}$ and $n=ml$. Then $\mathcal{R}$ is said to be an $\mu-quasi-twisted$ code if for any

$$(c_{0\ 0},c_{0\ 1},\ldots,c_{0\ l-1},\ldots,c_{m-1\ 0},c_{m-1\ 1},\ldots,c_{m-1\ l-1})\in\mathcal{C}\ \ implies$$

$$(\mu c_{m-1\ 0},\mu c_{m-1\ 1},\ldots,\mu c_{m-1\ l-1},,\ldots,c_{m-2\ 0},c_{m-2\ 1},\ldots,c_{m-2\ l-1})\in\mathcal{C}.$$

Theorem 4.5 ([8], Theorem 4). *A linear code $\mathcal{C}$ of length n over $\mathcal{R}$ is skew cyclic if and only if $\Psi(\mathcal{C})$ is skew quasi-cyclic code of length $8n$ over F_q of index 8.*

Theorem 4.6. *Let $\mathcal{C}$ be a $(\theta-\mu)$-constacyclic code of length n over $\mathcal{R}$ and $gcd(n,k)=l$. Then $\mathcal{C}$ is a μ-quasi-twisted code of index l over $\mathcal{R}$.*

Proof. Since $gcd(n,k) = l$, there exits two integers a,b, such that $ak = 1 + bn$. Let $\mathcal{C}$ be a $(\theta - \mu)$-constacyclic code and $c = (c_{0\ 0}, c_{0\ 1}, \ldots, c_{0\ l-1}, \ldots, c_{m-1\ 0}, c_{m-1\ 1}, \ldots, c_{m-1\ l-1}) \in \mathcal{C}$, so we have $\vartheta_\mu(c), \vartheta_\mu^2(c), \ldots, \vartheta_\mu^l(c)$ are belong to $\mathcal{C}$.

$\vartheta_\mu^l(c) = (\theta^l(\mu c_{m-1\ 0}), \ldots, \theta^l(\mu c_{m-1\ l-1}), \theta^l(c_{0\ 0}), \ldots, \theta^l(c_{0\ l-1}), \ldots, \theta^l(c_{m-2\ 0}), \ldots, \theta^l(c_{m-2\ l-1}))$.

Which implies that $\vartheta_\mu^{1+bn}(c) = (\theta^{1+bn}(c_{m-1\ 0}), \ldots, \theta^{1+bn}(c_{m-1\ l-1}), \ldots, \theta^{1+bn}(\mu c_{m-2\ 0}), \ldots, \theta^{1+bn}(\mu c_{m-2\ l-1})) = (\theta^{ak}(c_{m-1\ 0}), \ldots, \theta^{ak}(c_{m-1\ l-1}), \ldots, \theta^{ak}(\mu c_{m-2\ 0}), \ldots, \theta^{ak}(\mu c_{m-2\ l-1})) = (c_{m-1\ 0}, , \ldots, c_{m-1\ l-1}, \ldots, \mu c_{m-2\ 0}, \ldots, \mu c_{m-2\ l-1})$.

So $\mu\vartheta_\mu^{1+bn}(c) = (\mu c_{m-1\ 0}, , \ldots, \mu c_{m-1\ l-1}, \ldots, c_{m-2\ 0}, \ldots, c_{m-2\ l-1}) \in \mathcal{C}$. Therefore $\mathcal{C}$ is a μ-quasi-twisted code of index l over $\mathcal{R}$. □

Theorem 4.7. *Let $\mu = \mu_1 + u\mu_2 + v\mu_3 + w\mu_4 + uv\mu_5 + vw\mu_6 + uw\mu_7 + uvw\mu_8$ be a unit in $\mathcal{R}$, $\mathcal{C} = \oplus_{i=1}^8 \mu_i \mathcal{C}_i$ be a linear code of length n over $\mathcal{R}$. Then $\mathcal{C}$ is μ-constacyclic codes over $\mathcal{R}$ if and only if $\mathcal{C}_1$, $\mathcal{C}_2$, $\mathcal{C}_3$, $\mathcal{C}_4$, $\mathcal{C}_5$, $\mathcal{C}_6$, $\mathcal{C}_7$ and $\mathcal{C}_8$ are skew μ_1^*-constacyclic code, skew μ_2^*-constacyclic code, skew μ_3^*-constacyclic code, skew μ_4^*-constacyclic code, skew μ_5^*-constacyclic code, skew μ_6^*-constacyclic code, skew μ_7^*-constacyclic code and skew μ_8^*-constacyclic code of length n over F_q respectively.*

Proof. By definition of gray map, take $\mu_1^* = \mu_1 + \mu_2 + \mu_3 + \mu_4 + \mu_5 + \mu_6 + \mu_7 + \mu_8$, $\mu_2^* = \mu_1 + \mu_2 + \mu_3 - \mu_4 + \mu_5 - \mu_6 - \mu_7 - \mu_8, \ldots, \mu_8^* = \mu_1 - \mu_2 - \mu_3 - \mu_4 + \mu_5 + \mu_6 + \mu_7 - \mu_8$.

Let $r = (r_0, r_1, \ldots, r_{n-1}) \in \mathcal{C}$, where $r_i = \mu_1 a_i + \mu_2 b_i + \mu_3 c_i + \mu_4 d_i + \mu_5 e_i + \mu_6 f_i + \mu_7 g_i + \mu_8 h_i$, $0 \le i \le n-1$.

Let $a = (a_0, a_1, \ldots, a_{n-1})$, $b = (b_0, b_1, \ldots, b_{n-1})$, $c = (c_0, c_1, \ldots, c_{n-1})$, $d = (d_0, d_1, \ldots, d_{n-1})$, $e = (e_0, e_1, \ldots, e_{n-1})$, $f = (f_0, f_1, \ldots, f_{n-1})$, $g = (g_0, g_1, \ldots, g_{n-1})$ and $h = (h_0, f_1, \ldots, h_{n-1})$, so $a \in \mathcal{C}_1$, $b \in \mathcal{C}_2$, $c \in \mathcal{C}_3$, $d \in \mathcal{C}_4$, $e \in \mathcal{C}_5$, $f \in \mathcal{C}_6$,$g \in \mathcal{C}_7$ and $h \in \mathcal{C}_8$.

Suppose that $\mathcal{C}_1$, $\mathcal{C}_2$, $\mathcal{C}_3$, $\mathcal{C}_4$, $\mathcal{C}_5$, $\mathcal{C}_6$, $\mathcal{C}_7$ and $\mathcal{C}_8$ are skew μ_1^*-constacyclic code, skew μ_2^*-constacyclic code, skew μ_3^*-constacyclic code, skew μ_4^*-consta cyclic code, skew μ_5^*-constacyclic code, skew μ_6^*-constacyclic code, skew μ_7^*-constacyclic code and skew μ_8^*-constacyclic code of length n over F_q respectively. So $\vartheta_{\mu_1^*}(a) \in \mathcal{C}_1$, $\vartheta_{\mu_2^*}(b) \in \mathcal{C}_2$, $\vartheta_{\mu_3^*}(c) \in \mathcal{C}_3$, $\vartheta_{\mu_4^*}(d) \in \mathcal{C}_4$, $\vartheta_{\mu_5^*}(e) \in \mathcal{C}_5$, $\vartheta_{\mu_6^*}(f) \in \mathcal{C}_6$, $\vartheta_{\mu_7^*}(g) \in \mathcal{C}_7$ and $\vartheta_{\mu_8^*}(h) \in \mathcal{C}_8$.

Now $\vartheta_\mu(r) = \vartheta_{\mu_1+u\mu_2+v\mu_3+w\mu_4+uv\mu_5+vw\mu_6+uw\mu_7+uvw\mu_8}(r) = (\mu\theta(r_{n-1}), \theta(r_0), \ldots, \theta(c_{r-2})) = ((\mu_1 + u\mu_2 + v\mu_3 + w\mu_4 + uv\mu_5 + vw\mu_6 + uw\mu_7 + uvw\mu_8)\theta(r_{n-1}), \theta(r_0), \ldots, \theta(c_{r-2})) = \mu_1\vartheta_{\mu_1^*}(a) + \mu_2\vartheta_{\mu_2^*}(b) + \mu_3\vartheta_{\mu_3^*}(c) + \mu_4\vartheta_{\mu_4^*}(d) + \mu_5\vartheta_{\mu_5^*}(e) + \mu_6\vartheta_{\mu_6^*}(f) + \mu_7\vartheta_{\mu_7^*}(g) + \mu_8\vartheta_{\mu_8^*}(h) \in \oplus_{i=1}^8 \mu_i\mathcal{C}_i = \mathcal{C}$, which implies that $\mathcal{C}$ is μ-constacyclic code over $\mathcal{R}$.

Conversely, let $a = (a_0, a_1, \ldots, a_{n-1})$, $b = (b_0, b_1, \ldots, b_{n-1})$, $c = (c_0, c_1, \ldots, c_{n-1})$, $d = (d_0, d_1, \ldots, d_{n-1})$, $e = (e_0, e_1, \ldots, e_{n-1})$, $f = (f_0, f_1, \ldots, f_{n-1})$, $g = (g_0, g_1, \ldots, g_{n-1})$ and $h = (h_0, f_1, \ldots, h_{n-1})$, where $a \in \mathcal{C}_1$, $b \in \mathcal{C}_2$, $c \in \mathcal{C}_3$, $d \in \mathcal{C}_4$, $e \in \mathcal{C}_5$, $f \in \mathcal{C}_6$,$g \in \mathcal{C}_7$ and $h \in \mathcal{C}_8$.

Suppose that $\mathcal{C}$ is μ-constacyclic code over $\mathcal{R}$, so $\vartheta_\mu(r) = \vartheta_{\mu_1+u\mu_2+v\mu_3+w\mu_4+uv\mu_5+vw\mu_6+uw\mu_7+uvw\mu_8}(r) \in \mathcal{C}$.

Also $\vartheta_\mu(r) = \mu_1\vartheta_{\mu_1^*}(a) + \mu_2\vartheta_{\mu_2^*}(b) + \mu_3\vartheta_{\mu_3^*}(c) + \mu_4\vartheta_{\mu_4^*}(d) + \mu_5\vartheta_{\mu_5^*}(e) + \mu_6 \vartheta_{\mu_6^*}(f) + \mu_7\vartheta_{\mu_7^*}(g) + \mu_8\vartheta_{\mu_8^*}(h)$. So we have directly $\vartheta_{\mu_1^*}(a) \in \mathcal{C}_1$, $\vartheta_{\mu_2^*}(b) \in \mathcal{C}_2$, $\vartheta_{\mu_3^*}(c) \in \mathcal{C}_3$, $\vartheta_{\mu_4^*}(d) \in \mathcal{C}_4$, $\vartheta_{\mu_5^*}(e) \in \mathcal{C}_5$, $\vartheta_{\mu_6^*}(f) \in \mathcal{C}_6$, $\vartheta_{\mu_7^*}(g) \in \mathcal{C}_7$ and $\vartheta_{\mu_8^*}(h) \in \mathcal{C}_8$.

Hence $\mathcal{C}_1$, $\mathcal{C}_2$, $\mathcal{C}_3$, $\mathcal{C}_4$, $\mathcal{C}_5$, $\mathcal{C}_6$, $\mathcal{C}_7$ and $\mathcal{C}_8$ are skew μ_1^*-constacyclic code, skew μ_2^*-constacyclic code, skew μ_3^*-constacyclic code, skew μ_4^*-constacyclic code, skew μ_5^*-constacyclic code, skew μ_6^*-constacyclic code, skew μ_7^*-constacyclic code and skew μ_8^*-constacyclic code of length n over F_q respectively.

□

Lemma 4.8 ([18], Lemma 3.1). *Let $\mathcal{C}$ be a code of length n over $\mathcal{R}$. Then $\mathcal{C}$ is $(\theta-\mu)$-constacyclic if and only if $\mathcal{C}^\perp$ is $(\theta - \mu^{-1})$-constacyclic. In particular, if $\mu^2 = 1$, then $\mathcal{C}$ is $(\theta - \mu)$-constacyclic if and only if $\mathcal{C}^\perp$ is $(\theta - \mu)$-constacyclic.*

Corollary 4.9. *Let $\mu = \mu_1 + u\mu_2 + v\mu_3 + w\mu_4 + uv\mu_5 + vw\mu_6 + uw\mu_7 + uvw\mu_8$ be a unit in $\mathcal{R}$ and let $\mathcal{C} = \oplus_{i=1}^8 \mu_i\mathcal{C}_i$ be a $(\theta - \mu)$-constacyclic code of length n over $\mathcal{R}$. Then the dual code*

$\mathcal{C}^{\perp}=\oplus_{i=1}^{8}\mu_i\mathcal{C}_i{}^{\perp}$ is $(\theta-\mu^{-1})$-constacyclic code over $\mathcal{R}$, where $\mathcal{C}_1{}^{\perp}$, $\mathcal{C}_2{}^{\perp}$, $\mathcal{C}_3{}^{\perp}$, $\mathcal{C}_4{}^{\perp}$, $\mathcal{C}_5{}^{\perp}$, $\mathcal{C}_6{}^{\perp}$, $\mathcal{C}_7{}^{\perp}$ and $\mathcal{C}_8{}^{\perp}$are $(\theta-(\mu_1^*)^{-1})$-constacyclic code, $(\theta-(\mu_2^*)^{-1})$-constacyclic code, $(\theta-(\mu_3^*)^{-1})$-constacyclic code, $(\theta-(\mu_4^*)^{-1})$-constacyclic code, $(\theta-(\mu_5^*)^{-1})$-constacyclic code, $(\theta-(\mu_6^*)^{-1})$-constacyclic code, $(\theta-(\mu_7^*)^{-1})$-constacyclic code and $(\theta-(\mu_8^*)^{-1})$-constacyclic code of length n over F_q respectively, where $n=mk$ and $k=|<\theta>|$ the order of the ring automorphism.

Proof. Since $\mu=\mu_1+u\mu_2+v\mu_3+w\mu_4+uv\mu_5+vw\mu_6+uw\mu_7+uvw\mu_8$ is fixed by the automorphism θ and n is a multiple of k, so by Lemma 4.6, we have $\mathcal{C}^{\perp}$ is $(\theta-\mu^{-1})$-constacyclic code over $\mathcal{R}$. Now, $\mu=\mu_1+u\mu_2+v\mu_3+w\mu_4+uv\mu_5+vw\mu_6+uw\mu_7+uvw\mu_8=\sum_{i=1}^{8}\mu_i\mu_i^*$, we have directly $\mathcal{C}_1{}^{\perp}$, $\mathcal{C}_2{}^{\perp}$, $\mathcal{C}_3{}^{\perp}$, $\mathcal{C}_4{}^{\perp}$, $\mathcal{C}_5{}^{\perp}$, $\mathcal{C}_6{}^{\perp}$, $\mathcal{C}_7{}^{\perp}$ and $\mathcal{C}_8{}^{\perp}$are $(\theta-(\mu_1^*)^{-1})$-constacyclic code, $(\theta-(\mu_2^*)^{-1})$-constacyclic code, $(\theta-(\mu_3^*)^{-1})$-constacyclic code, $(\theta-(\mu_4^*)^{-1})$-constacyclic code, $(\theta-(\mu_5^*)^{-1})$-constacyclic code, $(\theta-(\mu_6^*)^{-1})$-constacyclic code, $(\theta-(\mu_7^*)^{-1})$-constacyclic code and $(\theta-(\mu_8^*)^{-1})$-constacyclic code of length n over F_q respectively. □

Theorem 4.10. *Let $\mathcal{C}=\oplus_{i=1}^{8}\mu_i\mathcal{C}_i$ be a $(\theta-\mu)$-constacyclic code of length n over $\mathcal{R}$. Then $\mathcal{C}$ has a generating polynomial $f(x)$ in $\mathcal{R}[x,\theta]$ such that $f(x)$ is a right divisor of $x^n-\mu=x^n-(\mu_1+u\mu_2+v\mu_3+w\mu_4+uv\mu_5+vw\mu_6+uw\mu_7+uvw\mu_8)$.*

Proof. Let $f_i(x)$ be generator of C_i for $1\le i\le 8$ in $\mathcal{R}[x,\theta]$. Then $\mu_1f_1(x)$, $\mu_2f_2(x)$,…,$\mu_8f_8(x)$ are generators of $\mathcal{C}$. Let$f(x)=\sum_{i=1}^{8}\mu_if_i(x)$ and $\Gamma=<f(x)>$. Clearly, $\Gamma=<f(x)>\subseteq\mathcal{C}$. On the other hand, $\mu_1f(x)=\mu_1f_1(x)\in\Gamma$, $\mu_2f(x)=\mu_2f_2(x)\in\Gamma$,…, $\mu_8f(x)=\mu_8f_8(x)\in\Gamma$. Therefore, $\mathcal{C}\subseteq\Gamma=<f(x)>$ and hence $\mathcal{C}=\Gamma=<f(x)>$.

Since $f_1(x)$, $f_2(x)$,…,$f_8(x)$ are right divisors of $x^n-\alpha_1$,$x^n-\alpha_1$,…,$x^n-\alpha_8$ respectively, so there exist $h_1(x)$, $h_2(x)$,…,$h_8(x)$ such that $x^n-\mu_1=h_1(x)*f_1(x)$, $x^n-\mu_2=h_2(x)*f_2(x)$,…, $x^n-\mu_8=h_8(x)*f_8(x)$. Now $[\sum_{i=1}^{8}\mu_ih_i(x)]*f(x)=\mu_1h_1(x)*f(x)+\mu_2h_2(x)*f(x)+\cdots+\mu_8h_8(x)*f(x)=x^n-\mu$. Therefore, $f(x)$ is a right divisor $x^n-\mu=x^n-(\mu_1+u\mu_2+v\mu_3+w\mu_4+uv\mu_5+vw\mu_6+uw\mu_7+uvw\mu_8)$. □

Corollary 4.11. *Each left submodule of $\mathcal{R}[x,\theta]/<x^n-\mu>$ is generated by one element where $\mu=\mu_1+u\mu_2+v\mu_3+w\mu_4+uv\mu_5+vw\mu_6+uw\mu_7+uvw\mu_8$ is a unit in $\mathcal{R}$.*

Theorem 4.12. *Let $\mathcal{C}=\oplus_{i=1}^{8}\mu_i\mathcal{C}_i$ be a $(\theta-\mu)$-constacyclic code of length n over $\mathcal{R}$ and $gcd(n,k)=1$, $gcd(n,q)=1$. Then there exist an idempotent generator $e(x)=\sum_{i=1}^{8}\mu_ie_i(x)$ in $\mathcal{R}[x,\theta]/<x^n-\mu>$, where $e_1(x)$, $e_2(x)$,…, $e_8(x)$ are idempotent generators of $\mathcal{C}_1$, $\mathcal{C}_2$,…, $\mathcal{C}_8$ respectively.*

Proof. Let $e_1(x)$, $e_2(x)$,…, $e_8(x)$ are idempotent generators of $\mathcal{C}_1$, $\mathcal{C}_2$,…, $\mathcal{C}_8$ respectively. Then by [[9], Theorem 16], we conclude that $e(x)=\sum_{i=1}^{8}\mu_ie_i(x)$ is an idempotent generator of $\mathcal{C}$ in $\mathcal{R}[x,\theta]/<x^n-\mu>$. □

Note that it cab easily proved that $\mathcal{C}$ is self-dual if and only if $\mu_1=1$, $\mu_1+\mu_2=\pm1$, $\mu_1+\mu_2+\mu_3=\pm1$, $\mu_1+\mu_2+\mu_3+\mu_4=\pm1$, $\mu_1+\mu_2+\mu_3+\mu_4+\mu_5=\pm1$, $\mu_1+\mu_2+\mu_3+\mu_4+\mu_5+\mu_6=\pm1$, $\mu_1+\mu_2+\mu_3+\mu_4+\mu_5+\mu_6+\mu_7=\pm1$ and $\mu_1+\mu_2+\mu_3+\mu_4+\mu_5+\mu_6+\mu_7+\mu_8=\pm1$.

Example 4.13. Let θ be a Frobenius automorphism of $F_{25}=F_{5^2}$, where $\theta:F_{25}\to F_{25}$ defined by $\theta(\alpha)=\alpha^5$.

Now, $q=5$ and take $n=6$, then
$x^6-1=(x-1)(x+1)(x^2+x+1)(x^2-x+1)$ *and* $x^6+1=(x+2)(x+3)(x^2+2x-1)(x^2+3x-1)$ *modulo* 5.

Take $f_1(x)=f_2(x)=f_3(x)=f_4(x)=(x-1)$ and $f_5(x)=f_6(x)=f_7(x)=f_8(x)=(x^2+2x-1)$. Then $\mathcal{C}_1=<f_1(x)>$, $\mathcal{C}_2=<f_2(x)>$, $\mathcal{C}_3=<f_3(x)>$, $\mathcal{C}_4=<f_4(x)>$, $\mathcal{C}_5=<f_5(x)>$, $\mathcal{C}_6=<f_6(x)>$, $\mathcal{C}_7=<f_7(x)>$ and $\mathcal{C}_8=<f_8(x)>$ are skew cyclic codes of length 8 over F_{25}.

Let $f(x)=\sum_{i=1}^{8}\mu_if_i(x)=(\mu_1+\mu_2+\mu_3+\mu_4)f_1(x)+(\mu_5+\mu_6+\mu_7+\mu_8)$
$f_5(x)=(1-u-v-w+uvw)(x-1)+(1+u+v+w-uvw)(x^2+2x-1)$.

Then $\mathcal{C}=<f(x)>$ is a skew cyclic code of length 8 over $F_{25}+uF_{25}+vF_{25}+wF_{25}+uvF_{25}+uwF_{25}+vwF_{25}+uvwF_{25}$. By Theorem 4.5 we have $\Psi(\mathcal{C})$ is a skew quasi-cyclic code of length 48 and index 8 over F_{25}.

Example 4.14. Let θ be a Frobenius automorphism of $F_9 = F_{3^2}$, where $\theta : F_9 \to F_9$ defined by $\theta(\alpha) = \alpha^3$. Now, $q = 3$ and take $n = 4$, then
$x^4 - 1 = (x+1)(x+2)(x^2+1)$ *and* $x^4 + 1 = (x^2 + x + 2)(x^2 + 2x + 2)$ *modulo* 3.

Take $f_1(x) = f_2(x) = f_3(x) = f_4(x) = (1 + x)$ and $f_5(x) = f_6(x) = f_7(x) = f_8(x) = (1+x+x^2+x^3+x^4)$, and let let $\mu_1 = 1$, $\mu_1+\mu_2 = -1$, $\mu_1+\mu_2+\mu_3 = 1$, $\mu_1+\mu_2+\mu_3+\mu_4 = -1$, $\mu_1+\mu_2+\mu_3+\mu_4+\mu_5 = 1$, $\mu_1+\mu_2+\mu_3+\mu_4+\mu_5+\mu_6 = -1$, $\mu_1+\mu_2+\mu_3+\mu_4+\mu_5+\mu_6+\mu_7 = 1$ and $\mu_1 + \mu_2 + \mu_3 + \mu_4 + \mu_5 + \mu_6 + \mu_7 + \mu_8 = -1$.

Now this system can be solved by **Maple** as follows:
solve($\mu_1 = 1, \mu_1+\mu_2 = -1, \mu_1+\mu_2+\mu_3 = 1, \mu_1+\mu_2+\mu_3+\mu_4 = -1, \mu_1+\mu_2+\mu_3+\mu_4+\mu_5 = 1, \mu_1 + \mu_2 + \mu_3 + \mu_4 + \mu_5 + \mu_6 = -1, \mu_1 + \mu_2 + \mu_3 + \mu_4 + \mu_5 + \mu_6 + \mu_7 = 1, \mu_1 + \mu_2 + \mu_3 + \mu_4 + \mu_5 + \mu_6 + \mu_7 + \mu_8 = -1), [\mu_1, \mu_2, \mu_3, \mu_4, \mu_5$
$, \mu_6, \mu_7, \mu_8]$ *mod* 3. To have $\mu_1 = 1, \mu_2 = 1, \mu_3 = 2, \mu_4 = 1, \mu_5 = 2, \mu_6 = 1, \mu_7 = 2$ and $\mu_8 = 1$. Since $\mu = \mu_1 + u\mu_2 + v\mu_3 + w\mu_4 + uv\mu_5 + vw\mu_6 + uw\mu_7 + uvw\mu_8$, so we have that $\mu = 1 + u - 2v + w - 2uv + vw - 2uw + uvw$.

Compute $\mu_1+\mu_2+\mu_3+\mu_4$ and $\mu_5+\mu_6+\mu_7+\mu_8$ by Maple to have that $\mu_1+\mu_2+\mu_3+\mu_4 = 1 - u - v - w + uvw$ and $\mu_5 + \mu_6 + \mu_7 + \mu_8 = 1 + u + v + w - uvw$.

Then we have $f(x) = \sum_{i=1}^{8} \mu_i f_i(x) = (\mu_1 + \mu_2 + \mu_3 + \mu_4) f_1(x) + (\mu_5 + \mu_6 + \mu_7 + \mu_8) f_5(x)$ $= (1-u-v-w+uvw)(1+x)+(1+u+v+w-uvw)(1+x+x^2+x^3+x^4)$ is *aright divisor* of $x^4 - \mu = x^4 - (1+u-2v+w-2uv+vw-2uw+uvw)$ in $\mathcal{R}[x, \theta]$ and by Theorem 4.2 since $gcd(n, k) = gcd(4, 3) = 1$, then $C =< f(x) >$ is a $(1 + u - 2v + w - 2uv + vw - 2uw + uvw)$ -constacyclic code of length 4 over $R = F_9 + uF_9 + vF_9 + uvF_9 + v^2F_9 + uv^2F_9$.

Example 4.15. Let θ be a Frobenius automorphism of $F_{27} = F_{3^3}$, where $\theta : F_{27} \to F_{27}$ defined by $\theta(\alpha) = \alpha^3$. Now, $q = 3$ and take $n = 4$, then $x^4 - 1 = (x-1)(x+1)(x^2+1)$ *and* $x^4 + 1 = (x^2 + x + 2)(x^2 + 2x + 2)$ *modulo* 3.

Take $f_1(x) = f_2(x) = f_3(x) = f_4(x) = f_5(x) = f_6(x) = f_7(x) = (x + 1)$ and $f_8(x) = (x^2 + x + 2)$, let $\mu_1 = 1$, $\mu_1 + \mu_2 = 1$, $\mu_1 + \mu_2 + \mu_3 = -1$, $\mu_1 + \mu_2 + \mu_3 + \mu_4 = -1$, $\mu_1+\mu_2+\mu_3+\mu_4+\mu_5 = -1$, $\mu_1+\mu_2+\mu_3+\mu_4+\mu_5+\mu_6 = 1$, $\mu_1+\mu_2+\mu_3+\mu_4+\mu_5+\mu_6+\mu_7 = 1$ and $\mu_1 + \mu_2 + \mu_3 + \mu_4 + \mu_5 + \mu_6 + \mu_7 + \mu_8 = 1$.

Now this system can be solved by **Maple** as follows:
solve($\mu_1 = 1, \mu_1+\mu_2 = 1, \mu_1+\mu_2+\mu_3 = -1, \mu_1+\mu_2+\mu_3+\mu_4 = -1, \mu_1+\mu_2+\mu_3+\mu_4+\mu_5 = -1, \mu_1 + \mu_2 + \mu_3 + \mu_4 + \mu_5 + \mu_6 = 1, \mu_1 + \mu_2 + \mu_3 + \mu_4 + \mu_5 + \mu_6 + \mu_7 = 1, \mu_1 + \mu_2 + \mu_3 + \mu_4 + \mu_5 + \mu_6 + \mu_7 + \mu_8 = 1), [\mu_1, \mu_2, \mu_3, \mu_4, \mu_5,$
$\mu_6, \mu_7, \mu_8]$ *mod* 3. To have $\mu_1 = 1, \mu_2 = 0, \mu_3 = 1, \mu_4 = 0, \mu_5 = 0, \mu_6 = 2, \mu_7 = 0$ and $\mu_8 = 0$. Since $\mu = \mu_1 + u\mu_2 + v\mu_3 + w\mu_4 + uv\mu_5 + vw\mu_6 + uw\mu_7 + uvw\mu_8$, so we have that $\mu = 1 + v + 2vw$.

Also $f(x) = \sum_{i=1}^{8} \mu_i f_i(x) = (\mu_1 + \mu_2 + \mu_3 + \mu_4 + \mu_5 + \mu_6 + \mu_7) f_1(x) + \mu_8$
$f_8(x) = (1+u+v+w+2uv+2vw+2uw+uvw)(x+1) + (1-u-v-w+uv+vw+uw-uvw)(x^2+x+2)$. Then $\mathcal{C} =< f(x) >= (1+u+v+w+2uv+2vw+2uw+uvw)(x+1)+(1-u-v-w+uv+vw+uw-uvw)(x^2+x+2)$ is a self-dual skew $(1+v+2vw)$-constacyclic code of length 4 over $\mathcal{R} = F_{27} + uF_{27} + vF_{27} + wF_{27} + uvF_{27} + uwF_{27} + vwF_{27} + uvwF_{27}$.

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Received: 2022-12-04
Accepted: 2023-07-18