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# BOUNDARY BEHAVIOR OF POWER SERIES IN BICOMPLEX SPACE AND GAP THEOREMS

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**Abstract** The boundary behavior of the power series on the circle of convergence in the bicomplex space is discussed in the current work. The bicomplex Hadamard's gap theorem and other theorems have been proved for the power series in bicomplex space. The unit hypersphere in the bicomplex space is also exemplified. The bicomplex Vivanti-Pringsheim theorem and its extension have been proved for the power series in the bicomplex space. The Ringleb decomposition theorem has been applied to establish the results in the bicomplex space.

## 1 Introduction

A generalization of the complex and the hyperbolic numbers are the bicomplex numbers. A lot of research work has been done in the area of bicomplex integral transform, bicomplex holomorphic functions. Efforts have been made in recent years to extend various theorems [3, 5, 9, 10], Mittag-Leffler function and applications [1, 2, 25, 26] to bicomplex variables from their complex counterparts.

**Definition 1.1** (Bicomplex Number). Segre [24] defined the set of bicomplex numbers as follows:

$$\mathbb{T} = \{\xi = x_0 + i_1x_1 + i_2x_2 + jx_3 \mid x_0,\ x_1,\ x_2,\ x_3\ \in \mathbb{R}\}. \tag{1.1}$$

Alternatively,

$$\mathbb{T} = \{\xi = z_1 + i_2z_2 \mid z_1,\ z_2\ \in \mathbb{C}\}. \tag{1.2}$$

(1.1) and (1.2) represent the bicomplex numbers in the form of real and complex numbers, respectively. Bicomplex numbers are also called 'Tetra numbers.'

Here, we denote the real components of a bicomplex number $\xi$ as $x_0 = \mathrm{Re}(\xi),\ x_1 = \mathrm{Im}_{i_1}(\xi),\ x_2 = \mathrm{Im}_{i_2}(\xi),\ x_3 = \mathrm{Im}_j(\xi)$.

The set of hyperbolic numbers $\mathbb{D}$ is a proper subset of the set of bicomplex numbers $\mathbb{T}$ (see, e.g. [23])

$$\mathbb{D} = \{x_1 + x_3j \mid x_1,\ x_3 \in \mathbb{R},\ j^2 = 1 \text{ and } j \notin \mathbb{R}\} \subseteq \mathbb{T}. \tag{1.3}$$

There are two zero divisors in $\mathbb{T}$ denoted by $e_1$ and $e_2$ such that $e_1 + e_2 = 1,\ e_1.e_2 = 0$ while $e_1 = \dfrac{1+j}{2},\ e_2 = \dfrac{1-j}{2},\ j = i_1i_2$. The idempotent representation [18] for every $\xi = z_1 + i_2z_2 \in \mathbb{T}$ is as follows:

$$\xi = \xi_1e_1 + \xi_2e_2, \tag{1.4}$$

where $e_1,\ e_2 \in \mathbb{T}$ are idempotent elements and $\xi_1 = z_1 - i_1z_2,\ \xi_2 = z_1 + i_1z_2$ are idempotent coefficients.

Using the idempotent form of bicomplex numbers significantly benefits all algebraic operations. Idempotent representation is useful because it allows for term-by-term addition, multiplication, and division, which makes the results easier to evaluate.

There exist two projection mappings $P_{e_1} : \mathbb{T} \to T_{e_1}$, $P_{e_2} : \mathbb{T} \to T_{e_2}$, $T_{e_1}$, $T_{e_2} \subseteq \mathbb{C}$ corresponding to two idempotent coefficients $\xi_1$ and $\xi_2$ for any bicomplex number $\xi = \xi_1 e_1 + \xi_2 e_2 \in \mathbb{T}$, which are given by

$$P_{e_1}(\xi) = \xi_1 \in T_{e_1} \text{ and } P_{e_2}(\xi) = \xi_2 \in T_{e_2}, \tag{1.5}$$

where

$$T_{e_1} = \{\xi_1 : \xi_1 = z_1 - i_1 z_2 \in \mathbb{C}\} \text{ and } T_{e_2} = \{\xi_2 : \xi_2 = z_1 + i_1 z_2 \in \mathbb{C}\}, \tag{1.6}$$

are called the auxiliary complex spaces, and $\mathbb{T} = T_{e_1} \times T_{e_2}$ can be viewed as the product of these auxiliary spaces.

**Definition 1.2** (Idempotent Representation). The idempotent representation of a hyperbolic number $w = x_1 + x_2 j \in \mathbb{D}$ can be represented as (see, e.g.[13, p.4])

$$w = w_1 e_1 + w_2 e_2, \quad w_1 = x_1 + x_2,\ w_2 = x_1 - x_2. \tag{1.7}$$

If $w_1, w_2 \geq 0$, then $w$ is said to be a non-negative hyperbolic number, and $\mathbb{D}^+$ represents this set. When $w_1, w_2 > 0$ hyperbolic number $w$, is said to be a positive hyperbolic number. This concept defines the following partial order on hyperbolic numbers [13].

**Definition 1.3** (Partial Order on $\mathbb{D}$). Let $u, v \in \mathbb{D}$, then partial order on $\mathbb{D}$ is defined as [13, p.4]

$$u \preccurlyeq v \text{ if } v - u \in \mathbb{D}^+ \text{ and } u \prec v \text{ if } v - u \in \mathbb{D}^+/\{0\}. \tag{1.8}$$

**Definition 1.4** (Null Cone). Null cone is the set of zero-divisors in $\mathbb{T}$, given by (see, e.g. [13, p.3])

$$\mathbb{NC} = \{\xi = z_1 + i_2 z_2,\ z_1, z_2 \in \mathbb{C} :\ z_1^2 + z_2^2 = 0\}. \tag{1.9}$$

If $\xi \in \mathbb{NC} \ \Rightarrow\ \xi^{-1}$ does not exist.

The product of two non-zero numbers in the bicomplex space is zero iff the first is a complex multiple of $e_1$ and the second is a complex multiple $e_2$ or vice-versa. Let $\xi,\ \eta \in \mathbb{T}$, then

$$\xi\eta = 0 \ \Rightarrow\ \xi = 0 \text{ or } \eta = 0 \text{ or } \xi = ze_1,\ \eta = we_2 \text{ or } \xi = ze_2,\ \eta = we_1, \quad z, w \in \mathbb{C}. \tag{1.10}$$

Non-zero multiples of $e_1$ and $e_2$ are called first and second nil-factors, respectively. For a bicomplex number $\xi = \xi_1 e_1 + \xi_2 e_2 \in \mathbb{T}$, the components $\xi_1 e_1$ and $\xi_2 e_2$ will be located in the first and second nil-planes, respectively (Figure 1).

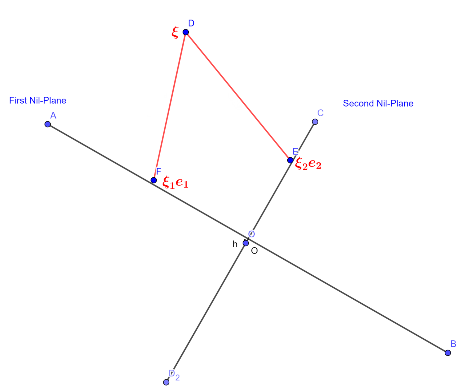


**Figure 1.** Two-dimensional representation of a bicomplex number $\xi$

**Definition 1.5** (j-Modulus). The $j$-modulus of $\xi \in \mathbb{T}$ is given by (see, e.g. [23])

$$|\xi|_j = |z_1 - i_1 z_2|e_1 + |z_1 + i_1 z_2|e_2 \in \mathbb{D}. \tag{1.11}$$

**Definition 1.6** (j-Argument). Let $\xi \in \mathbb{T}$ then the hyperbolic argument of $\xi$ is given by (see, e.g. [12])

$$\arg_j(\xi) = \arg(\xi_1)e_1 + \arg(\xi_1)e_2. \tag{1.12}$$

**Definition 1.7** (Bicomplex Radicals). ([16, p.128]) Consider the bicomplex equation $\xi^n = \omega$, where $\xi = \xi_1 e_1 + \xi_2 e_2$, $\omega = \omega_1 e_1 + \omega_2 e_2 \in \mathbb{T}$. The following two complex equations in variables $\xi_1$, $\xi_2$ are identical to this equation

$$\xi_1^n = \omega_1 \text{ and } \xi_2^n = \omega_2, \tag{1.13}$$

$$\implies \xi_1 = \sqrt[n]{\omega_1} \text{ and } \xi_2 = \sqrt[n]{\omega_2}. \tag{1.14}$$

Riley [21] investigated the convergence of the bicomplex power series in Theorem 1.8.

**Theorem 1.8.** *Let $\xi = \xi_1 e_1 + \xi_2 e_2 \in \mathbb{T}$. Define*

$$M(\xi) = \sqrt{\|\xi\|^2 + \sqrt{\|\xi\|^4 - |\xi|^4_{abs}}} = \max(|\xi_1|, |\xi_2|), \tag{1.15}$$

*then $M(\xi)$ is a norm, and if $\sum_{n=0}^{\infty} p_n \xi^n$, $p_n = b_n e_1 + c_n e_2$ is a power series with component series $\sum_{n=0}^{\infty} b_n \xi_1^n$ and $\sum_{n=0}^{\infty} c_n \xi_2^n$, both having the same radius of convergence $R > 0$, then $\sum_{n=0}^{\infty} a_n \xi^n$ converges for $M(\xi) < R$ and diverges for $M(\xi) > R$, where the norm or the real modulus of $\xi$ is given by*

$$\|\xi\| = \sqrt{|z_1|^2 + |z_2|^2} = \frac{1}{\sqrt{2}}\sqrt{|\xi_1|^2 + |\xi_2|^2} = \sqrt{x_0^2 + x_1^2 + x_2^2 + x_3^2}, \tag{1.16}$$

*and*

$$|\xi|_{abs} = \sqrt{|\xi_1||\xi_2|}. \tag{1.17}$$

The decomposition theorem of Ringleb [22] (see, also [21, p.145]) leads to the following result about singularities in bicomplex space:

**Definition 1.9** (Singularity). A bicomplex function

$$f(\xi) = f_1(\xi_1)e_1 + f_2(\xi_2)e_2, \quad \xi = \xi_1 e_1 + \xi_2 e_2 \in \mathbb{T}, \tag{1.18}$$

can have a singularity at $\xi = \xi_0 = \xi_{0_1} e_1 + \xi_{0_2} e_2$ if and only if $f_1(\xi_1)$ has a singularity at $\xi_1 = \xi_{0_1}$ or $f_2(\xi_2)$ has a singularity at $\xi_2 = \xi_{0_2}$.

As a result, $f(\xi)$ has a singularity at every point, where the closure of $f(\xi)$'s area of analyticity intersects one of the nil-planes with respect to $\xi_0$, i.e., the set of points of the form $\xi_0 + a$ for $a$ in a nil-plane.

**Remark 1.10.** A function $f(\xi)$ with a singularity at the origin can have no point of analyticity in one of the nil-planes, then $f(\xi)$ is said to be singular in that nil-plane, even though all points of the nil-plane will, in general, not be a boundary point of the region $\mathbb{T}$ in which $f(\xi)$ is analytic.

**Theorem 1.11** (Taylor's Theorem ). *[21, p.142] In a four-dimensional region $T \subseteq \mathbb{T}$, let $h(\xi)$ be analytic, and $p$ be a point in $T$. Then $h(\xi)$ can be expanded into a generalized Taylor series centered at the point $p$*

$$h(\xi) = h(p) + \frac{(\xi - p)}{1!}h'(p) + \frac{(\xi - p)^2}{2!}h''(p) + \cdots + \frac{(\xi - p)^n}{n!}h^{(n)}(p) + \ldots, \tag{1.19}$$

*whenever $h(\xi)$ is defined, and the series is convergent. If $d$ is the greatest lower bound of $\|\xi - p\|$ for $\xi$ a boundary point of $T$, then the above series converges for $M(\xi) < d\sqrt{2}$. In particular, this implies convergence in the hypersphere $\|\xi - p\| < d$.*

The above theorem's detailed proof may be found in [21, p.142].

Motivated by the work of Luna-Elizarrarás et al. [13, 14] on the bicomplex holomorphic functions, residue theorem, and power series in bicomplex variables, in this paper, we have discussed the boundary behavior of bicomplex power series and established the bicomplex Vivanti-Pringsheim theorem. For the study of power series in the bicomplex space, Abel's theorem and Cauchy-Hadamard's theorem are already introduced [13]. Continuing this way, we have discussed power series in bicomplex space for further properties.

The criterion for singular points, one of the most crucial results in complex analysis, is the Vivanti-Pringsheim theorem. It was given by Vivanti [28] in 1893 and proved by Pringsheim [19] in 1894.

The straightforward approach for identifying the region of convergence of a power series to a circle of convergence extends up to the nearest singular point in the disposition of the function it represents. There will be at least one singular point on the circle of convergence for every power series with a positive radius of convergence.

On the circle of convergence, the power series may diverge or converge everywhere or converge at some points and diverge at others. The power series has a natural boundary at the circle of convergence if every point on it is a singular point [11].

The geometric series $\sum_{n=0}^{\infty} z^n$ has a radius of convergence one and $z = 1$ as its only singular point. Vivanti-Pringsheim theorem is a generalization of this example.

**Theorem 1.12** (Vivanti-Pringsheim Theorem [11, 20]). *Suppose $a_k \geq 0$ for $k \geq 0$. If the power series $\sum_{k=0}^{\infty} a_k z^k$ has its radius $l$ of convergence satisfying $0 < l < \infty$, then $z = l$ is a singular point.*

Another result is given by Dienes (see, e.g., [6, p.227]), which examined the singular point of the series in connection with the argument of the coefficients $a_n$.

**Theorem 1.13** (Dienes Theorem). *[11, p.45] Suppose $f(z) = \sum_{n=0}^{\infty} a_n z^n$ has the radius of convergence equal to 1 and $|\arg a_n| \leq \alpha < \frac{\pi}{2}$, $n \geq 1$ for some constant $\alpha > 0$. Then 1 is a singular point of the series.*

In the Vivanti-Pringsheim theorem, we have just seen that the sum of a power series with positive coefficients can not be continued directly past the real positive point on the circle of convergence. Now we shall be concerned with conditions that make it impossible to continue the sum of a power series beyond the disk of convergence in any direction so that the circle of convergence is a natural boundary ( [4, p.146]).

The convergence of a Taylor series $\sum_{k=0}^{\infty} c_k (z-c)^k$ is stopped only by the nearest singular point to $z = c$ [6]. Hadamard clarified the presence of power series with natural boundaries in 1892. Mordell's 1927 [17] demonstration of Hadamard's gap theorem is the simplest (see, also [20]).

**Theorem 1.14** (Hadamard's Gap Theorem). *( [4, p.146])$f(z) = \sum_{n=0}^{\infty} a_n z^n$ with a finite radius of convergence can not be continued beyond its circle of convergence if $a_n = 0$ except for $n = n_k$, where*

$$\frac{n_{k+1}}{n_k} \geq \lambda > 1, \tag{1.20}$$

*i.e., the circle of convergence is the natural boundary of the function.*

After Hadamard's theorem, a much deeper result is Fabry's gap theorem (see, e.g., [4]) as follows:

**Theorem 1.15** (Fabry's Theorem ). *( [6, p.376]) If $\frac{\lambda_n}{n} \to \infty$ and the radius of convergence of*

$$f(z) = \sum_{n=0}^{\infty} a_n z^{\lambda_n}, \tag{1.21}$$

*is 1, $f(z)$ is not continued beyond the $|z| = 1$.*

Consider the following Hadamard's theorem in which the term $\lambda_{\nu+1} - \lambda_\nu > \theta\lambda_\nu,\ \theta > 0$, is known as the Hadamard gaps.

**Theorem 1.16** (Hadamard). *( [6, p.231]) If* $\lim |\sqrt[\lambda_\nu]{a_{\lambda_\nu}}| = 1$ *and* $\lambda_{\nu+1} - \lambda_\nu > \theta\lambda_\nu,\ \theta > 0$, *the circle* $|z| = 1$ *is a singular line for* $f(z) = \sum a_{\lambda_\nu} z^{\lambda_\nu}$.

## 2 Polar Form of Bicomplex Numbers and Vivanti-Pringsheim Theorem

In this section, we discuss the polar form of the bicomplex number and some of its properties. The polar form of a bicomplex number plays a crucial role when generalizing complex analysis results into bicomplex outcomes where the polar form of a complex number is used. The bicomplex version of the Vivanti-Pringsheim theorem is presented in this study. We can find Taylor's series about a point in the polar form with the help of this polar form.

Futagawa [7], in 1928, discussed the polar representation of bicomplex numbers. Let $\xi \in \mathbb{T}$ then $\xi = l_0 + l_1 i_1 + l_2 i_2 + l_3 j$. In the exponential form, $\xi$ can be written as

$$\xi = \exp(\delta + i_1\alpha - i_2\beta + j\gamma), \tag{2.1}$$

where $e^\delta = r,\ -\infty < \delta < \infty,\ -\pi \le \alpha < \pi,\ -\frac{\pi}{2} \le \beta < \frac{\pi}{2},\ -\infty < \gamma < \infty$, and $e^\delta,\ e^\gamma,\ \alpha,\ \beta$ are called the first modulus, the second modulus, the first vectorial angle, the second vectorial angle, respectively, of the bicomplex number $\xi$.
Since from [7, p.186],

$$\|\xi\| = \sqrt{l_0^2 + l_1^2 + l_2^2 + l_3^2}, \tag{2.2}$$

where

$$\begin{aligned} l_0 &= r(\cos\alpha\cos\beta\cosh\gamma - \sin\alpha\sin\beta\sinh\gamma), && (2.3)\\ l_1 &= r(\sin\alpha\cos\beta\cosh\gamma + \cos\alpha\sin\beta\sinh\gamma), && (2.4)\\ l_2 &= -r(\cos\alpha\sin\beta\cosh\gamma + \sin\alpha\cos\beta\sinh\gamma), && (2.5)\\ l_3 &= r(\cos\alpha\cos\beta\sinh\gamma - \sin\alpha\sin\beta\cosh\gamma). && (2.6) \end{aligned}$$

On simplifications, we get

$$\|\xi\|^2 = r^2\cosh 2\gamma, \quad r = e^\delta. \tag{2.7}$$

**Remark 2.1.** Futagawa [7] called the term $\sqrt{l_0^2 + l_1^2 + l_2^2 + l_3^2}$ as the absolute value of $\xi$ in his work.

**Remark 2.2.** $\|\xi\|^2 = 1$ for $r = 1$ i.e. $(\delta = 0)$, $\gamma = 0$ in the exponential form. The unit hypersphere in bicomplex space is thus the set of points

$$U = \left\{\xi = e^{i_1\alpha - i_2\beta} : -\pi \le \alpha < \pi,\ -\frac{\pi}{2} \le \beta < \frac{\pi}{2}\right\}. \tag{2.8}$$

To understand the position of singularity for the bicomplex power series at the boundary of the circle of convergence, consider the following example:

Consider the bicomplex series $\sum_{n=0}^{\infty} \xi^n = \frac{1}{1-\xi},\ \|\xi\| < 1$, has a radius of convergence is one and $\xi = 1$ as its only singular point, the function is analytic in $\mathbb{T}/\{1\}$. Given bicomplex geometric series can be written as for $\xi \in \mathbb{T}$

$$\begin{aligned} \sum_{n=0}^{\infty} \xi^n &= \left(\sum_{n=0}^{\infty} \xi_1^n\right) e_1 + \left(\sum_{n=0}^{\infty} \xi_2^n\right) e_2 \\ &= \left(\frac{1}{1-\xi_1}\right) e_1 + \left(\frac{1}{1-\xi_1}\right) e_2, \quad |\xi_1| < 1,\ |\xi_2| < 1. \end{aligned} \tag{2.9}$$

Here

$$|\xi_1| < 1 \text{ and } |\xi_2| < 1 \Rightarrow \|\xi\| = \sqrt{\frac{|\xi_1|^2 + |\xi_2|^2}{2}} < 1. \tag{2.10}$$

Since $\sum_{n=0}^{\infty} \xi_1^n, \ \sum_{n=0}^{\infty} \xi_2^n$ have radii of convergence one and $\xi_1 = 1, \ \xi_2 = 1$ are singular points respectively, so by the application of the Ringleb decomposition theorem, $\sum_{n=0}^{\infty} \xi^n$ has a radius of convergence 1 and $\xi = 1$ is a singular point.

In this section, bicomplex versions of the Vivanti-Pringsheim theorem have been established. In Theorem 2.3, we have considered real coefficients in the bicomplex power series and analyzed the cases for different radii values of convergence of the component power series. In Theorem 2.5, bicomplex power series with hyperbolic coefficients, is studied.

The following Vivanti-Pringsheim theorem in bicomplex space is an extension of the above example and is a bicomplex extension of complex Vivanti-Pringsheim Theorem 1.12.

**Theorem 2.3.** *Suppose $a_n \geq 0$ for $n \geq 0$. If the power series $f(\xi) = \sum_{n=0}^{\infty} a_n \xi^n, \ \xi \in \mathbb{T}$ has its radius $\mathcal{R}$ of convergence satisfying $0 < \mathcal{R} < \infty$, then $\xi = \mathcal{R}$ is a singular point of $f$.*

*Proof.* Consider the bicomplex power series

$$f(\xi) = \sum_{n=0}^{\infty} a_n \xi^n, \quad a_n \geq 0, \tag{2.11}$$

where $\xi = \xi_1 e_1 + \xi_2 e_2 \in \mathbb{T}$.
In terms of idempotent components,

$$\begin{aligned} f(\xi) &= \sum_{n=0}^{\infty} a_n \xi^n \\ &= \sum_{n=0}^{\infty} a_n \xi_1^n e_1 + \sum_{n=0}^{\infty} a_n \xi_2^n e_2 \\ &= f_1(\xi_1) e_1 + f_2(\xi_2) e_2, \quad a_n \geq 0, \end{aligned} \tag{2.12}$$

where $f_i(\xi_i) = \sum_{n=0}^{\infty} a_n \xi_i^n, \ (i = 1, 2)$.

Let $\mathcal{R}_i, \ (i = 1, 2)$ be the radius of convergence for the corresponding complex power series $f_i(\xi_i)$. The radius of convergence $\mathcal{R}$ of $f(\xi)$ can not be zero since otherwise $\mathcal{R}_1 = 0 = \mathcal{R}_2$.

For $0 < \mathcal{R} < \infty$, we consider the following cases:
One of $\mathcal{R}_1$ or $\mathcal{R}_2$ is zero:
If $\mathcal{R}_1 = 0, \ \mathcal{R}_2 \neq 0$, then the set of points of convergence of $\sum_{n=0}^{\infty} a_n \xi^n$ is confined to the second nil-plane. And the set of points of convergence of $\sum_{n=0}^{\infty} a_n \xi^n$ is confined to the first nil-plane if $\mathcal{R}_1 \neq 0, \ \mathcal{R}_2 = 0$. The convergence region for $f(\xi)$ will be empty in these two circumstances.

For $\mathcal{R} > 0$, the series $f_i(\xi_i), \ (i = 1, 2)$ converges and have radius of convergence $\mathcal{R}_1 = \mathcal{R}_2 = \mathcal{R}$. The other case $(\mathcal{R}_1 \neq \mathcal{R}_2)$ is not possible [21, p.141].
Also $a_n \geq 0$, then by the Vivanti-Pringsheim Theorem 1.12, $\xi_1 = \mathcal{R}, \ \xi_2 = \mathcal{R}$ are singular points for $f_1(\xi_1), \ f_2(\xi_2)$ respectively.

By the application of the Ringleb decomposition theorem, $f(\xi)$ has the singularity $\mathcal{R} = \mathcal{R}e_1 + \mathcal{R}e_2$. Hence $\xi = \mathcal{R}$, is the singular point. □

A simple form of the above theorem in bicomplex space is given below.

**Corollary 2.4.** *Let the power series $\sum_{m=0}^{\infty} a_m \xi^m, \ \xi \in \mathbb{T}$ have a radius of convergence unity and $a_m \geq 0$ for all values of $m$. Then $\xi = 1$ is a singular point.*

*Proof.* We prove this result by contradiction. Suppose that $\xi = 1$ is a regular point. Then for a point $\rho$ between 0 and 1 on the real axis, there exists a hypersphere centered at $\rho$, which contains the point 1, and function $f(\xi)$ is regular within the hypersphere. The Taylor series about the point $\rho$ is given by

$$\sum_{k=0}^{\infty} \frac{f^{(k)}(\rho)}{k!}(\xi-\rho)^k. \tag{2.13}$$

It is convergent at a point $\xi = 1+\rho,\ \rho > 0$. Also,

$$f^{(m)}(\rho) = \sum_{k=m}^{\infty} k(k-1)\dots(k-m+1)a_m\rho^{k-m}, \tag{2.14}$$

then series (2.13) becomes

$$\sum_{k=0}^{\infty} \frac{(\xi-\rho)^k}{k!} \sum_{k=m}^{\infty} k(k-1)\dots(k-m+1)a_m\rho^{k-m}, \tag{2.15}$$

which is a double series of positive terms, and it is convergent for $\xi = 1+\rho$. Breaking the series into idempotent components and reversing the order of summation in complex domain as done in [27, p.215]. Combining the idempotent components to obtain the value of the series in the bicomplex domain, we get

$$\sum_{k=0}^{\infty} \frac{(\xi-\rho)^k}{k!} \sum_{k=m}^{\infty} k(k-1)\dots(k-m+1)a_m\rho^{k-m} = \sum_{m=0}^{\infty} a_m\xi^m, \tag{2.16}$$

which is the original series, convergent for $\xi = 1+\rho$, which contradicts the fact that the radius of convergence is unity.

Hence 1 is the singular point of the series. □

Another proof is based on the method applied by Pringsheim for the complex space (see, e.g., [27]).

*Proof.* Since there is at least one singularity $e^{i_1\theta - i_2\phi}$, say, on the unit hypersphere as defined in (2.8), the bicomplex Taylor's series about $\rho e^{i_1\theta-i_2\phi+j\psi}$, $0<\rho<1$ is given by

$$\sum_{k=0}^{\infty} \frac{f^{(k)}(\rho e^{i_1\theta-i_2\phi+j\psi})}{k!}(\xi-\rho e^{i_1\theta-i_2\phi+j\psi})^k, \tag{2.17}$$

and, since $e^{i_1\theta-i_2\phi}$, is a singularity, it has the radius of convergence $1-\rho$.
Also,

$$f^{(m)}(\rho) = \sum_{k=m}^{\infty} k(k-1)\dots(k-m+1)a_m\rho^{k-m}, \tag{2.18}$$

and $a_m \geq 0,\ \ m \geq 0$, we get

$$|f^{(m)}(\rho e^{i_1\theta-i_2\phi+j\psi})| \leq f^{(m)}(\rho), \quad m \geq 0,\ \psi \to 0. \tag{2.19}$$

Hence, the radius of convergence does not exceed $1-\rho$, accordingly, $\xi = 1$ is a singularity. □

The following theorem is the hyperbolic version of Theorem 2.3.

**Theorem 2.5.** *If the bicomplex power series* $\sum_{n=0}^{\infty} a_n\xi^n,\ \xi \in \mathbb{T}$, $a_n = a_{n_1} + ja_{n_4}$ *is hyperbolic number with* $a_{n_1} \geq |a_{n_4}|$, $\forall n \in \mathbb{N}$, *has its radius* $\mathcal{R}$ *of convergence satisfying* $0 < \mathcal{R} < \infty$ *then* $\xi = \mathcal{R}$ *is a singular point.*

*Proof.* Consider the bicomplex power series

$$f(\xi) = \sum_{n=0}^{\infty} a_n\xi^n, \quad \xi,\ a_n \in \mathbb{T}. \tag{2.20}$$

Writing

$$\begin{aligned} a_n &= a_{n_1} + i_1 a_{n_2} + i_2 a_{n_3} + j a_{n_4} \\ &= b_n e_1 + c_n e_2. \end{aligned} \tag{2.21}$$

Here, $b_n = (a_{n_1} + a_{n_4}) + i_1(a_{n_2} - a_{n_3})$ and $c_n = (a_{n_1} - a_{n_4}) + i_1(a_{n_2} + a_{n_3})$.

By using idempotent representation, bicomplex power series can be represented as

$$\begin{aligned} f(\xi) &= \sum_{n=0}^{\infty} a_n \xi^n = \sum_{n=0}^{\infty} (b_n e_1 + c_n e_2)(\xi_1 e_1 + \xi_2 e_2)^n = \sum_{n=0}^{\infty} (b_n e_1 + c_n e_2)(\xi_1^n e_1 + \xi_2^n e_2) \\ &= \sum_{n=0}^{\infty} b_n \xi_1^n e_1 + \sum_{n=0}^{\infty} c_n \xi_2^n e_2 \\ &= f_1(\xi_1) e_1 + f_2(\xi_2) e_2. \end{aligned} \tag{2.22}$$

Since,

$$\begin{aligned} & b_n \geq 0 \text{ and } c_n \geq 0, \\ \Rightarrow \quad & a_{n_1} + a_{n_4} \geq 0,\ a_{n_2} - a_{n_3} = 0 \text{ and } a_{n_1} - a_{n_4} \geq 0,\ a_{n_2} + a_{n_3} = 0, \\ \Rightarrow \quad & a_{n_1} \geq |a_{n_4}| \text{ and } a_{n_2} = a_{n_3} = 0, \qquad (2.23) \\ \Rightarrow \quad & a_n = (a_{n_1} + a_{n_4}) e_1 + (a_{n_1} - a_{n_4}) e_2 = a_{n_1} + j a_{n_4}. \qquad (2.24) \end{aligned}$$

Hence, $a_n$ is a hyperbolic number such that $a_{n_1} \geq |a_{n_4}|$.
Let $\mathcal{R}_i,\ (i = 1, 2)$ be the radius of convergence for the complex power series. The radius of convergence $\mathcal{R}$ of $f(\xi)$ can not be zero since otherwise $\mathcal{R}_1 = 0 = \mathcal{R}_2$.

For $0 < \mathcal{R} < \infty$, we consider the following cases:
One of $\mathcal{R}_1$ or $\mathcal{R}_2$ is zero:
If $\mathcal{R}_1 = 0,\ \mathcal{R}_2 \neq 0$, then the set of points of convergence of $\sum_{n=0}^{\infty} a_n \xi^n$ is confined to the second nil-plane. And the set of points of convergence of $\sum_{n=0}^{\infty} a_n \xi^n$ is confined to the first nil-plane if $\mathcal{R}_1 \neq 0,\ \mathcal{R}_2 = 0$. The convergence region for $f(\xi)$ will be empty in these two circumstances.

For $\mathcal{R} > 0$, the series $f_i(\xi_i),\ (i = 1, 2)$ converges and have radius of convergence $\mathcal{R}_1 = \mathcal{R}_2 = \mathcal{R}$, another case $(\mathcal{R}_1 \neq \mathcal{R}_2)$ is not possible [21, p.141].
Also $a_n \geq 0,\ c_n \geq 0$ then by the Vivanti-Pringsheim Theorem 1.12, $\xi_1 = \mathcal{R},\ \xi_2 = \mathcal{R}$ are singular points for $f_1(\xi_1),\ f_2(\xi_2)$ respectively.

By the application of the Ringleb decomposition theorem, $f(\xi)$ has the singularity $\mathcal{R} = \mathcal{R}e_1 + \mathcal{R}e_2$. Hence $\xi = \mathcal{R}$, is the singular point. □

**Remark 2.6.** Hyperbolic numbers enter the picture when analyzing series with real coefficients in bicomplex space for the proof of Theorem 2.5. As a reason, it can be regarded as the hyperbolic Vivanti-Pringsheim theorem.

Now, we derive the bicomplex version of Theorem 1.13.

**Theorem 2.7** (Bicomplex Dienes Theorem). *Suppose $f(\xi) = \sum_{n=0}^{\infty} a_n \xi^n,\ \xi \in \mathbb{T}$ has the radius of convergence equal to 1 and $|\arg a_n|_j \preccurlyeq \alpha < \frac{\pi}{2},\ n \geq 1$ for some constant $\alpha > 0$. Then 1 is a singular point of the series.*

*Proof.* From equation (2.22), we have bicomplex power series,

$$f(\xi) = \sum_{n=0}^{\infty} a_n \xi^n = \sum_{n=0}^{\infty} b_n \xi_1^n e_1 + \sum_{n=0}^{\infty} c_n \xi_2^n e_2 = f_1(\xi_1) e_1 + f_2(\xi_2) e_2, \tag{2.25}$$

where $f_1(\xi_1) = \sum_{n=0}^{\infty} b_n \xi_1^n$ and $f_2(\xi_2) = \sum_{n=0}^{\infty} c_n \xi_2^n$ are complex power series. Assume that the radius of convergence of the bicomplex power series (2.25) is unity and

$$|\arg a_n|_j \preccurlyeq \alpha < \frac{\pi}{2}, \tag{2.26}$$

$n \geq 1$ for some constant $\alpha = \alpha e_1 + \alpha e_2 > 0$.

From equations (1.11) and (1.12), we can write $|\arg b_n| \leq \alpha < \frac{\pi}{2}$ and $|\arg c_n| \leq \alpha < \frac{\pi}{2}$, $n \geq 1$ for some constant $\alpha > 0$.

Since the radius of convergence of $f(\xi) = f_1(\xi_1)e_1 + f_2(\xi_2)e_2$ is unity hence the radii of convergence of both $f_1(\xi_1)$ and $f_2(\xi_2)$ are also unity (see, the theorem [21, p.141]). By the Theorem 1.13, 1 is a singular point of both series $f_1(\xi_1)$ and $f_2(\xi_2)$.
By the application of the decomposition theorem of Ringleb, 1 is the singular point of the series $f(\xi) = f_1(\xi_1)e_1 + f_2(\xi_2)e_2$.

□

## 3 Gap Theorems in Bicomplex Space

As an application of the bicomplex Vivanti-Pringsheim theorem, we illustrate the natural boundary of a bicomplex function by the following example:

Consider the bicomplex Series

$$f(\xi) = \sum_{n=0}^{\infty} \xi^{2^n}, \quad \|\xi\| < 1. \tag{3.1}$$

Then $f(\xi)$ is a function with the natural boundary for $\|\xi\| = 1$.
Writing (3.1) in idempotent form

$$\begin{aligned} f(\xi) &= \sum_{n=0}^{\infty} \xi^{2^n} \\ &= \left(\sum_{n=0}^{\infty} \xi_1^{2^n}\right) e_1 + \left(\sum_{n=0}^{\infty} \xi_2^{2^n}\right) e_2 \\ &= f_1(\xi_1)e_1 + f_2(\xi_2)e_2, \quad |\xi_1| < 1,\ |\xi_2| < 1, \end{aligned} \tag{3.2}$$

where $f_1(\xi_1),\ f_2(\xi_2)$ are in complex domain. From Example-5 [11, p.38], $f_1(\xi_1)$ and $f_2(\xi_2)$ have natural boundary for $|\xi_1| = 1$ and $|\xi_2| = 1$ respectively.
Since

$$\|\xi\| = \sqrt{\frac{|\xi_1|^2 + |\xi_2|^2}{2}} = 1. \tag{3.3}$$

Hence $\|\xi\| = 1$ is also a natural boundary for $f(\xi)$.

In this section, we have derived a bicomplex version of some theorems. Initially, we derive Hadamard's gap theorem in bicomplex space.

**Theorem 3.1** (Bicomplex Hadamard's Gap Theorem). $f(\xi) = \sum_{n=0}^{\infty} a_n \xi^n$, $\xi \in \mathbb{T}$ *with a finite radius of convergence, can not be continued beyond its circle of convergence if* $a_n = 0$ *except for* $n = n_k$*, where*

$$\frac{n_{k+1}}{n_k} \geq \lambda > 1. \tag{3.4}$$

*Proof.* Consider the bicomplex series with a finite radius of convergence

$$f(\xi) = \sum_{n=0}^{\infty} a_n \xi^n = \left(\sum_{n=0}^{\infty} b_n \xi_1^n\right) e_1 + \left(\sum_{n=0}^{\infty} c_n \xi_2^n\right) e_2 = f_1(\xi_1)e_1 + f_2(\xi_2)e_2. \tag{3.5}$$

If $a_n = 0$ except for $n = n_k$, where $\frac{n_{k+1}}{n_k} \geq \lambda > 1$.

$$\text{For } n \neq n_k,\ \ a_n = b_n e_1 + c_n e_2 = 0 \Longrightarrow b_n = 0,\ c_n = 0. \tag{3.6}$$

$\because$ $f(\xi)$ has finite radius of convergence, the component series $f_i(\xi_i)$, $(i=1,2)$ also have finite radius of convergence (see, [21, p.141]).
Also, $b_n = 0$, $c_n = 0$ where $\dfrac{n_{k+1}}{n_k} \geq \lambda > 1$ by Theorem 1.14, $f_i(\xi_i)$, $(i=1,2)$ can not be continued beyond its circle of convergence.
By application of the Ringleb theorem

$$f(\xi) = f_1(\xi_1)e_1 + f_2(\xi_2)e_2, \tag{3.7}$$

can not be continued beyond its circle of convergence.

□

We derive the bicomplex version of Fabry's theorem, which is a much deeper result after Hadamard's theorem.

**Theorem 3.2** ( Bicomplex Fabry's Theorem). *If* $\dfrac{\lambda_n}{n} \to \infty$ *and the radius of convergence of*

$$f(\xi) = \sum_{n=0}^{\infty} a_n \xi^{\lambda_n}, \quad \xi \in \mathbb{T} \tag{3.8}$$

*is 1,* $f(\xi)$ *is not continued beyond the* $\|\xi\| = 1$.

*Proof.* The bicomplex series $f(\xi)$ can be written in idempotent components as

$$f(\xi) = \sum_{n=0}^{\infty} a_n \xi^{\lambda_n} = f_1(\xi_1)e_1 + f_2(\xi_2)e_2, \tag{3.9}$$

where $f_1(\xi_1) = \sum_{k=0}^{\infty} a_n \xi_1^{\lambda_n}$ and $f_2(\xi_2) = \sum_{k=0}^{\infty} a_n \xi_2^{\lambda_n}$ are complex power series.
Since the radius of convergence of $f(\xi) = f_1(\xi_1)e_1 + f_2(\xi_2)e_2$ is unity hence the radii of convergence of both $f_1(\xi_1)$ and $f_2(\xi_2)$ are also unity (see, [21, p.141]).

Also, $\dfrac{\lambda_n}{n} \to \infty$, by the Theorem 1.15, $f_1(\xi_1)$ and $f_2(\xi_2)$ are not continued beyond the lines $|\xi_1| = 1$ and $|\xi_2| = 1$ respectively.

By the application of the decomposition theorem of Ringleb, the function $f(\xi) = f_1(\xi_1)e_1 + f_2(\xi_2)e_2$ is not continued beyond $\|\xi\| = \sqrt{\frac{|\xi_1|^2+|\xi_2|^2}{2}} = 1$. □

**Theorem 3.3** (Bicomplex Hadamard's Theorem). *If* $\lim | \sqrt[\lambda_\nu]{a_{\lambda_\nu}}|_j = 1$ *and* $\lambda_{\nu+1} - \lambda_\nu > \theta\lambda_\nu$, $\theta > 0$, $\|\xi\| = 1$ *is a singular curve for* $f(\xi) = \sum a_{\lambda_\nu} \xi^{\lambda_\nu}$, $\xi \in \mathbb{T}$.

*Proof.* Proceeding as (3.5), the bicomplex power series

$$f(\xi) = \sum_{\nu=0}^{\infty} a_{\lambda_\nu} \xi^{\lambda_\nu} = f_1(\xi_1)e_1 + f_2(\xi_2)e_2, \tag{3.10}$$

where $f_1(\xi_1) = \sum_{\nu=0}^{\infty} a_{1\lambda_\nu} \xi_1^{\lambda_\nu}$ and $f_2(\xi_2) = \sum_{\nu=0}^{\infty} a_{2\lambda_\nu} \xi_2^{\lambda_\nu}$ are complex power series.
Now, from equation (1.13), we get

$$\begin{aligned} \sqrt[\lambda_\nu]{a_{\lambda_\nu}} &= \sqrt[\lambda_\nu]{a_{1\lambda_\nu}}e_1 + \sqrt[\lambda_\nu]{a_{2\lambda_\nu}}e_2 \\ \Rightarrow | \sqrt[\lambda_\nu]{a_{\lambda_\nu}}|_j &= | \sqrt[\lambda_\nu]{a_{1\lambda_\nu}}e_1 + \sqrt[\lambda_\nu]{a_{2\lambda_\nu}}e_2|_j \\ \Rightarrow | \sqrt[\lambda_\nu]{a_{\lambda_\nu}}|_j &= | \sqrt[\lambda_\nu]{a_{1\lambda_\nu}}|e_1 + | \sqrt[\lambda_\nu]{a_{2\lambda_\nu}}|e_2. \end{aligned} \tag{3.11}$$

$$\because \lim | \sqrt[\lambda_\nu]{a_{\lambda_\nu}}|_j \to 1 = e_1 + e_2, \tag{3.12}$$

$$\Rightarrow \lim(|\sqrt[\lambda_\nu]{a_{1\lambda_\nu}}| \to 1 \text{ and } \lim(|\sqrt[\lambda_\nu]{a_{2\lambda_\nu}}| \to 1, \text{ (see, e.g. [15, p.69])}, \tag{3.13}$$

and $\lambda_{\nu+1} - \lambda_\nu > \theta\lambda_\nu,\ \theta > 0$, by the Theorem 1.16, $|\xi_1| = 1$ and $|\xi_2| = 1$ are a singular lines for $f_1(\xi_1) = \sum a_{1\lambda_\nu}\xi_1^{\lambda_\nu}$, and $f_2(\xi_2) = \sum a_{2\lambda_\nu}\xi_2^{\lambda_\nu}$ respectively.

Hence, by the application of the decomposition theorem of Ringleb, $\|\xi\| = \sqrt{\frac{|\xi_1|^2+|\xi_2|^2}{2}} = 1$ is a singular plane for the function $f(\xi) = f_1(\xi_1)e_1 + f_2(\xi_2)e_2$. □

**Remark 3.4.** The analytic prolongation requires knowledge of the natural boundary [8, p.80]. If $f(\xi)$ is extended into the bicomplex space, then $f(\xi) = f(\xi_1)e_1 + f(\xi_2)e_2$, and since for $\omega$ complex, $\xi_1 = \xi_2 = \omega$ the projections $\Gamma_1$ and $\Gamma_2$ of the curve $\Gamma$ in the $\xi_1$ and $\xi_2$ planes, respectively, are curves congruent to $\Gamma$, and thus the natural boundaries of the component functions $f(\xi_1)$ and $f(\xi_2)$, respectively.

Continuous curves joining pairs of points similarly placed with regard to $\Gamma_1$ and $\Gamma_2$, thereby crossing $\Gamma_1$ and $\Gamma_2$, are the projections of a continuous curve $C$ in the bicomplex space joining a point within $\Gamma$ in the complex plane to a point outside $C$ in the complex plane. As a result, analytic continuation along $C$ is ruled out, and that analytic continuation along $C$ is impossible [21, p.159].

## 4 Conclusion

The bicomplex Hadamard's gap theorem, bicomplex Fabry's, and Vivanti- Pringsheim theorem have been proved, which are generalizations of the respective complex theorems. The decomposition theorem of Ringleb plays a vital role in these generalizations. The idempotent representation has a significant application in proving the above theorems. Hadamard's gap theorem can be used to prove the Fatou-Polya theorem, saying that most of the power series with positive radii of convergence have a natural boundary [11]. For future scope, we can generalize the Szász theorem for the Dirichlet series in the bicomplex space.

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