Palestine Journal of Mathematics

Vol. 12(4)(2023) , 360–367

Locally symmetric condition on four dimensional strict Walker metrics

Mohamed Ayatola Dramé, Babacar Mbodj and Abdoul Salam Diallo

Communicated by Zafar Ahsan

MSC 2010 Classifications: Primary 53B30, 53C35; Secondary 53B50, 53C80.

Keywords and phrases: Locally symmetric; Pseudo-Riemannian manifolds; Walker metrics.

The authors would like to thank the referee for his/her valuable suggestions and comments that helped them improve the paper.

Abstract The main purpose of the present paper is to study four dimensional strict Walker manifolds. Conditions for four dimensional strict Walker manifolds to be locally symmetric are given..

1 Introduction

The study of the curvature properties of a given class of pseudo-Riemannian manifolds is important to our knowledge of these spaces. They are used to exemplify some of the main differences between the geometry of Riemannian manifolds and the geometry of pseudo-Riemannian manifolds and thereby illustrate phenomena in pseudo-Riemannian geometry that are quite different from those which occur in Riemannian geometry.

Walker m-manifolds are pseudo-Riemannian manifolds which admit a non-trivial parallel null r-plane field with $r \leq m$. Walker m-manifolds are applicable in physics. Lorentzian Walker manifolds have been studied extensively in physics since they constitute the background metric of the pp-wave models. A pp-wave spacetime admits a covariantly constant null vector field U. Brinkmann spacetimes are those Lorentzian manifolds admitting a global parallel null vector field. A particular class consists of plane fronted waves (PFW) with obvious geometric and physical interest. Walker manifolds represent a special subclass of plane fronted waves.

A Walker manifold is a triple $(M, g, \mathcal{D})$, where M is an m-dimensional manifold, g an indefinite metric and $\mathcal{D}$ an r-dimensional parallel null distribution. Of special interest are manifolds of even dimensions admitting a field of null planes of maximum dimensional ($r = \frac{m}{2}$). Since the dimension of a null plane is $r \leq \frac{m}{2}$, the lowest possible case of a Walker metric is that of $(++--)$ manifolds admitting a field of parallel null 2-planes [12]. Curvature properties and a complete characterization of locally symmetric or locally conformally flat three-dimensional Walker manifolds have been studied in [3]. Recently, the conditions for a restricted four-dimensional Walker manifold to be Einstein, locally symmetric Einstein and locally conformally flat are given in [8]. A lot of examples of Walker structures have appeared, which proved to be important in differential geometry and general relativity as well [4, 5, 6, 7, 9, 11].

A field of r-plane $\mathcal{D}$ is said to be strictly parallel if each vector in the plane at a point $p \in M$ is carried by a parallel transport by a vector in the plane at another point $q \in M$, the latter vector being the same for all paths from p to q [12]. A four-dimensional Walker manifold is a strict Walker manifold if and only if $\mathcal{D}$ admits two null parallel spanning vector fields or, equivalently, if we can choose a coordinate system so $g_{ij}(x_1, x_2, x_3, x_4) = g_{ij}(x_3, x_4), i, j = 3, 4$. In this paper, we study a strict four-dimensional Walker manifold. We derive the $(0, 4)$-curvature tensor, the Ricci tensor, and study some of the properties associated with a class of strict four-dimensional Walker manifolds. We establish a theorem for the metric to be locally symmetric.

The paper is organized in the following way. In Section 2, we describe the curvature of four-dimensional strict Walker metrics. In Section 3, we recall some basic notions on symmetric spaces. In Section 4, locally symmetric condition will be studied on four-dimensional strict Walker metrics.

2 Description of the metric

A four-dimensional pseudo-Riemannian M of signature $(2,2)$ is said to be a Walker manifold if it admits a parallel totally isotropic 2-plane field. Such a manifold is locally isometric to (U,g) where U is an open subset of $\mathbb{R}^4$ and the g is given with respect to the coordinate vector field $\partial_i := \frac{\partial}{\partial x_i}$ by

$$g(\partial_1,\partial_3) = g(\partial_2,\partial_4) = 1 \text{ and } g(\partial_i,\partial_j) = g_{ij}(x_1,x_2,x_3,x_4) \text{ for } i,j = 3,4.$$

One says that M is a strict Walker manifold if $g(\partial_i,\partial_j) = g_{ij}(x_3,x_4)$ for $i,j = 3,4$. In this paper, we consider the family of metrics $g_{a,b,c}$ given by

$$\begin{aligned} g_{a,b,c} &= 2(dx_1 \circ dx_3 + dx_2 \circ dx_4) + a(x_3,x_4)dx_3 \circ dx_3 \\ &\quad + b(x_3,x_4)dx_4 \circ dx_4 + 2c(x_3,x_4)dx_3 \circ dx_4, \end{aligned} \tag{2.1}$$

where a, b and c are functions of the (x_3,x_4). The couple $(U, g_{a,b,c})$ is called four-dimensional strict Walker manifold and its matrix form is given by

$$(g_{a,b,c})_{ij} = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & a & c \\ 0 & 1 & c & b \end{pmatrix}. \tag{2.2}$$

Next, we denote by $\partial_i := \frac{\partial}{\partial x_i}$, $f_i := \frac{\partial f(x_3,x_3)}{\partial x_i}$ and $f_{ij} := \frac{\partial^2 f(x_3,x_3)}{\partial x_i \partial_j}$. A straightforward calculation from (2.2), shows that the non-zero components of the Levi-Civita connection of the metric (2.1) are given by:

$$\begin{aligned} \nabla_{\partial_3}\partial_3 &= \frac{1}{2}a_3\partial_1 + \frac{1}{2}(2c_3 - a_4)\partial_2; \\ \nabla_{\partial_3}\partial_4 &= \frac{1}{2}a_4\partial_1 + \frac{1}{2}b_3\partial_2; \\ \nabla_{\partial_4}\partial_4 &= \frac{1}{2}(2c_4 - b_3)\partial_1 + \frac{1}{2}b_4\partial_2. \end{aligned}$$

The non-zero components of the curvature tensor of $(U, g_{a,b,c})$ are given by:

$$R(\partial_3,\partial_4)\partial_3 = \frac{1}{2}(a_{44} + b_{33} - 2c_{34})\partial_2 \text{ and } R(\partial_3,\partial_4)\partial_4 = -\frac{1}{2}(a_{44} + b_{33} - 2c_{34})\partial_1.$$

The nonzero component of the $(0,4)$-curvature tensor of the metric (2.1) is given by:

$$R_{3434} = \frac{1}{2}(a_{44} + b_{33} - 2c_{34}).$$

We find that the Ricci tensor and the scalar curvature of $(U, g_{a,b,c})$ vanish. More precisely, we have:

$$\rho_{ij} = 0 \text{ and } \tau = 0 \ \forall i,j = 1,2,3,4.$$

The nonzero components of the Einstein tensor $G_{ij} = \rho_{ij} - \frac{\tau}{4}g_{ij}$ are given by $G_{ij} = 0 \ \forall i,j = 1,2,3,4$.

Proposition 2.1. *Let $(M, g_{a,b,c})$ be a strict four-dimensional Walker manifolds. Then:*
(1) $(M, g_{a,b,c})$ is Ricci flat.
(2) $(M, g_{a,b,c})$ is Einstein.

We say that M is geodesically complete if all geodesics exist for all time. We say that M exhibits Ricci blowup if there exists a geodesic γ defined for $t \in [0,T)$ with $T < \infty$ and if $\lim_{t\to T} |\rho(\dot\gamma,\dot\gamma)| = \infty$. Clearly, if M exhibits Ricci blowup, then it is geodesically incomplete and it can not be isometrically embedded in a geodesically complete manifold.

Theorem 2.2. *([1]) A four-dimensional strict Walker manifold is geodesically complete.*

Proof. Let M be as in (2.1). Then the geodesic equations for $M$are given by:

$$\begin{aligned} 0 &= \ddot{x}_1 + \frac{1}{2}a_3\dot{x}_3\dot{x}_3 + a_4\dot{x}_3\dot{x}_4 + \frac{1}{2}(2c_4 - b_3)\dot{x}_4\dot{x}_4, \\ 0 &= \ddot{x}_2 + \frac{1}{2}(2c_3 - a_4)\dot{x}_3\dot{x}_3 + b_3\dot{x}_3\dot{x}_4 + \frac{1}{2}b_4\dot{x}_4\dot{x}_4, \\ 0 &= \ddot{x}_3, \quad 0 = \ddot{x}_4. \end{aligned}$$

The last two equations have the following trivial solutions:

$$x_3(t) = At + B; \quad x_4(t) = A't + B',$$

where $A, B, A', B' \in \mathbb{R}$. The first two equations then have the form $\ddot{x}_1 = f_1(t)$ and $\ddot{x}_2 = f_2(t)$ which can be solved. More precisely, we obtain:

$$\begin{aligned} x_1(t) &= -\frac{1}{2}\Big[\frac{1}{2}a_3A^2 + a_4AA' + \frac{1}{2}(2c_4 - b_3)A'^2\Big]t^2 \\ x_2(t) &= -\frac{1}{2}\Big[\frac{1}{2}(2c_3 - a_4)A^2 + b_3AA' + \frac{1}{2}b_4A'^2\Big]t^2. \end{aligned}$$

Thus the geodesics extend for infinite time and M is geodesically complete. The proof is complete. □

Example 2.3. Let $(M, g_{x_4^2,x_3^2,x_3^2})$ be a strict Walker manifold of dimension 4 of the form

$$g_{(x_4^2,x_3^2,x_3^2)} = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & x_4^2 & x_3^2 \\ 0 & 1 & x_3^2 & x_3^2 \end{pmatrix}.$$

The nonzero components of the Christoffel symbols of $(M, g_{x_4^2,x_3^2,x_3^2})$ are given by : $\Gamma_{33}^2 = (2x_3 - x_4),\ \Gamma_{34}^1 = x_4,\ \Gamma_{34}^2 = x_3,\ \Gamma_{34}^2 = x_3,\ \Gamma_{44}^1 = -x_3$. The geodesic equations of $(M, g_{x_4^2,x_3^2,x_3^2})$ are given by :

$$\begin{aligned} 0 &= \ddot{x}_1 + x_4\dot{x}_3\dot{x}_4 - x_3\dot{x}_4\dot{x}_4; \\ 0 &= \ddot{x}_2 + (2x_3 - x_4)\dot{x}_3\dot{x}_3 + x_3\dot{x}_3\dot{x}_4; \\ 0 &= \ddot{x}_3; \quad 0 = \ddot{x}_4. \end{aligned}$$

We obtain: $x_3(t) = At + B$, $x_4(t) = A't + B'$, $x_1(t) = \frac{1}{2}(x_3A'^2 - x_4AA')t^2$ and $x_2(t) = \frac{1}{2}\big[-(2x_3 - x_4)A^2 - x_3AA'\big]t^2$. Hence $(M, g_{x_4^2,x_3^2,x_3^2})$ is geodesically complete.

Example 2.4. Let $(M, g_{x_4^2,x_3^2,x_4^2})$ be a strict Walker manifold of dimension 4 of the form

$$g_{(x_4^2,x_3^2,x_4^2)} = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & x_4^2 & x_4^2 \\ 0 & 1 & x_4^2 & x_3^2 \end{pmatrix}.$$

The nonzero components of the Christoffel symbols of $(M, g_{x_4^2,x_3^2,x_4^2})$ are given : $\Gamma_{33}^2 = -x_4,\ \Gamma_{34}^1 = x_4,\ \Gamma_{34}^2 = x_3,\ \Gamma_{34}^2 = x_3,\ \Gamma_{44}^1 = (2x_4 - x_3)$. The geodesic equations of $(M, g_{x_4^2,x_3^2,x_4^2})$ are given by :

$$\begin{aligned} 0 &= \ddot{x}_1 + x_4\dot{x}_3\dot{x}_4 - (2x_4 - x_3)\dot{x}_4\dot{x}_4; \\ 0 &= \ddot{x}_2 + x_4\dot{x}_3\dot{x}_3 + x_3\dot{x}_3\dot{x}_4; \\ 0 &= \ddot{x}_3; \quad 0 = \ddot{x}_4. \end{aligned}$$

We obtain $x_3(t) = At + B$, $x_4(t) = A't + B'$, $x_1(t) = \frac{1}{2}(x_4A'A - (2x_4 - x_3)A'^2)t^2$ and $x_2(t) = -\frac{1}{2}(-x_4A^2 - x_3AA')t^2$. Hence $(M, g_{x_4^2,x_3^2,x_4^2})$ is geodesically complete.

3 Symmetric spaces

We say that a parametrized curve $c(t)$ in M is a geodesic if it satisfies the geodesic equation

$$\nabla_{\dot{\gamma}}\dot{\gamma} = 0,$$

or, equivalently, in a system of coordinates where $c = (c^1, \ldots, c^m)$ we have:

$$\ddot{c}_i + \Gamma^i_{jk}\dot{c}_j\dot{c}_k = 0.$$

This means that c locally minimizes the distance between points on c. An affine connection ∇ is said to be geodesically complete if every geodesic extends to the parameter range $(-\infty, +\infty)$.

The exponential map $\exp_p : T_pM \to M$ is a local diffeomorphism from a neighborhood of the origin $0 \in T_pM$ to a neighborhood of p in M. It is characterized by the fact that the curves $c_v(s) := \exp_p(sv)$ are geodesics in M with initial velocity $v \in T_pM$. One says that (M, ∇) is complete if $\exp_p$ is defined for all T_pM. Conjugate points arise where $\exp_p$ fails to be a local diffeomorphism. Furthermore, since there can be different geodesics joining two points, $\exp_p$ can fail to be globally one-to-one.

Let (M, g) be a pseudo-Riemannian manifold and $p \in M$. Let $\exp_p$ be an exponential map; this is the local diffeomorphism from a neighborhood of 0 in T_pM to a neighborhood of p in M. It is characterized by the fact that the curves $s \mapsto \exp_p(sv)$ are geodesics starting at p with initial direction v for $v \in T_pM$. The geodesic symmetry at p is then defined on a suitable neighborhood of p by setting:

$$S_p(Q) = \exp_p\{-\exp_p^{-1}(Q)\}.$$

We may use $\exp_p$ to identify a neighborhood of 0 in T_pM with a neighborhood of p in M and to regard T_pM as a pseudo-Riemannian manifold locally isometric to M; under this identification, $S_p(v) = -v$ is simply multiplication by -1 and the straight lines through the origin are geodesics. One has the following result:

Lemma 3.1. *Let (M, g) be a connected pseudo-Riemannian manifold.*

(a) The following assertions are equivalent and, if either is satisfied, then (M, g) is said to be locally symmetric or to be a local symmetric space:

(i) The geodesic symmetry S_p is an isometry for all $p \in M$.

(ii)$\nabla R = 0$.

(b) If (M, g) is locally symmetric, then (M, g) is locally homogeneous, i.e., given any two points $p, q \in M$, there is an isometry $\varphi_{p,q}$ from some neighborhood of p in M to some neighborhood of q in M.

Remark 3.2. If (M, g) is a local symmetric space that is complete and simply connected, then (M, g) is said to be globally symmetric or to be a symmetric space. In this setting, the geodesic symmetry extends to a global isometry of (M, g), and (M, g) is homogeneous. If G_0 is the isotropy subgroup of the group of isometries G of (M, g), then $M = G/G_0$ with the induced metric. Global symmetric spaces form a very special class of pseudo-Riemannian manifolds and techniques of group theory are used to study them. The associated Lie algebras $\mathcal{G}$ and $\mathcal{G}_0$ of G and of G_0 play a central role. We refer to Helgason [10] for further details.

4 Locally symmetric strict Walker manifolds

A symmetric space is a connected pseudo-Riemannian manifold whose geodesic symmetries are isometries. A manifold is said to be locally symmetric if it is isometric to a symmetric space. A well-known characterization states that a pseudo-Riemannian manifold (M, g) is locally symmetric if and only if $\nabla R = 0$, where R is the Riemann curvature tensor. In particular, a locally symmetric space is Ricci-parallel.

Theorem 4.1. *A strict four dimensional Walker of the form (2.2) for some functions $a(x_3, x_4)$, $b(x_3, x_4)$ and $c(x_3, x_4)$ is locally symmetric if and only if the following equations are satisfied*

$$\frac{\partial^3 a}{\partial x_4^2 \partial x_3} + \frac{\partial^3 b}{\partial x_3^3} - \frac{2\partial^3 c}{\partial x_3^2 \partial x_4} = 0 \textit{ and } \frac{\partial^3 b}{\partial x_3^2 \partial x_4} + \frac{\partial^3 a}{\partial x_4^3} - \frac{2\partial^3 c}{\partial x_4^2 \partial x_3} = 0. \tag{4.1}$$

Proof. The only nonzero component of the Riemann curvature tensor is:

$$R_{3434} = \frac{1}{2}(b_{33} + a_{44} - 2c_{34}).$$

Recall the locally symmetric condition is equivalent to $\nabla R = 0$. A straightforward calculation is that for such a metric $g_{a,b,c}$, the possibly non-vanishing components of the covariant derivative of R, i.e. $\nabla_k R_{ijlm} = (\nabla_{\partial k} R)(\partial_i, \partial_j, \partial_l, \partial_m)$ are given by

$$(\nabla_{\partial_3} R)(\partial_3, \partial_4, \partial_3, \partial_4) = \frac{1}{2}\frac{\partial}{\partial x_3}\frac{\partial^2 a}{\partial x_4^2} + \frac{1}{2}\frac{\partial}{\partial x_3}\frac{\partial^2 b}{\partial x_3^2} - \frac{\partial}{\partial x_3}\frac{\partial^2 c}{\partial x_3 \partial x_4}, \tag{4.2}$$

and

$$(\nabla_{\partial_4} R)(\partial_3, \partial_4, \partial_3, \partial_4) = \frac{1}{2}\frac{\partial}{\partial x_4}\frac{\partial^2 a}{\partial x_4^2} + \frac{1}{2}\frac{\partial}{\partial x_4}\frac{\partial^2 b}{\partial x_3^2} - \frac{\partial}{\partial x_4}\frac{\partial^2 c}{\partial x_4 \partial x_3}. \tag{4.3}$$

Therefore, for the strict Walker metric $g_{a,b,c}$ to be locally symmetric, it is necessary that:

$$\frac{1}{2}\frac{\partial^3 a}{\partial x_4^2 \partial x_3} + \frac{1}{2}\frac{\partial^3 b}{\partial x_3^3} - \frac{\partial^3 c}{\partial x_3^2 \partial x_4} = 0 \text{ and } \frac{1}{2}\frac{\partial^3 b}{\partial x_3^2 \partial x_4} + \frac{1}{2}\frac{\partial^3 a}{\partial x_4^3} - \frac{\partial^3 c}{\partial x_4^2 \partial x_3} = 0.$$

The proof is complete. □

Using both (4.2) and (4.3), by standard calculations, we obtain the following:

Theorem 4.2. *A strict four-dimensional Walker of the form (2.2) is locally symmetric if and only the functions a, b and c satisfying the following forms:*

$$\begin{aligned} a(x_3, x_4) &= \frac{K}{2}x_4^2 + G_1(x_3, x_4) + x_4 H_1(x_3) + F_1(x_3), \\ b(x_3, x_4) &= \frac{K}{2}x_3^2 + G_2(x_3, x_4) + x_3 H_2(x_4) + F_2(x_4), \\ c(x_3, x_4) &= G_3(x_3, x_4) - \frac{K}{2}x_3 x_4 + x_4 H_3(x_4) + F_3(x_3). \end{aligned}$$

where $K \in \mathbb{R}$ and $G_1, G_2, G_3, F_1, F_2, F_3, H_1, H_2$ and H_3 are smooth functions .

Proof. From (4.2), we have :

$$(\nabla_{\partial_3} R)(\partial_3, \partial_4, \partial_3, \partial_4) = \frac{1}{2}\frac{\partial}{\partial x_3}\frac{\partial^2 a}{\partial x_4^2} + \frac{1}{2}\frac{\partial}{\partial x_3}\frac{\partial^2 b}{\partial x_3^2} - \frac{\partial}{\partial x_3}\frac{\partial^2 c}{\partial x_3 \partial x_4} = 0.$$

That means:

$$\frac{\partial^2 a}{\partial x_4^2} + \frac{\partial^2 b}{\partial x_3^2} - 2\frac{\partial^2 c}{\partial x_4 \partial x_3} = A(x_4). \tag{4.4}$$

From (4.3), we et:

$$\frac{\partial^2 a}{\partial x_4^2} + \frac{\partial^2 b}{\partial x_3^2} - 2\frac{\partial^2 c}{\partial x_3 \partial x_4} = B(x_3). \tag{4.5}$$

By (4.4) and (4.5), we have: $A(x_4) = B(x_3) = K \in \mathbb{R}$. Hence, by (4.4):

$$\frac{\partial^2 a}{\partial x_4^2} = K + \Big(\frac{2\partial^2 c}{\partial x_3 \partial x_4} - \frac{\partial^2 b}{\partial x_3^2}\Big),$$

this implies

$$\frac{\partial a}{\partial x_4} = K x_4 + \int \Big(\frac{2\partial^2 c}{\partial x_3 \partial x_4} - \frac{\partial^2 b}{\partial x_3^2}\Big) dx_4 + H_1(x_3),$$

and so

$$a(x_3,x_4) = Kx_4^2 + \int\Big(\int\Big(\frac{2\partial^2 c}{\partial x_3\partial x_4} - \frac{\partial^2 b}{\partial x_3^2}\Big)dx_4\Big)dx_4 + x_4H_1(x_3) + F_1(x_3).$$

We set $G_1(x_3,x_4) = \int\Big(\int\Big(\frac{2\partial^2 c}{\partial x_4\partial x_3} - \frac{\partial^2 b}{\partial x_3^2}\Big)dx_4\Big)dx_4$. We obtain:

$$a(x_3,x_4) = Kx_4^2 + G_1(x_3,x_4) + x_4H_1(x_3) + F_1(x_3).$$

By analogously, with the same routine, we get:

$$b(x_3,x_4) = Kx_3^2 + G_2(x_3,x_4) + x_3H_2(x_4) + F_2(x_4),$$

where $G_2(x_3,x_4) = \int\Big(\int\Big(\frac{2\partial^2 c}{\partial x_4\partial x_3} - \frac{\partial^2 a}{\partial x_4^2}\Big)dx_3\Big)dx_3$.
Using (4.5), we have:

$$\frac{\partial^2 a}{\partial x_4^2} + \frac{\partial^2 b}{\partial x_3^2} - 2\frac{\partial^2 c}{\partial x_4\partial x_3} = B(x_3) = K.$$

Hence:

$$\begin{aligned}\frac{\partial^2 c}{\partial x_4\partial x_3} &= \frac{1}{2}\frac{\partial^2 a}{\partial x_4^2} + \frac{1}{2}\frac{\partial^2 b}{\partial x_3^2} - \frac{K}{2}\\ \frac{\partial}{\partial x_3}(\frac{\partial c}{\partial x_4}) &= \frac{1}{2}\frac{\partial^2 a}{\partial x_4^2} + \frac{1}{2}\frac{\partial^2 b}{\partial x_3^2} - \frac{K}{2}.\end{aligned}$$

By a first integration with respect to x_3, we get:

$$\frac{\partial c}{\partial x_4} = \int\Big(\frac{1}{2}\frac{\partial^2 a}{\partial x_4^2} + \frac{1}{2}\frac{\partial^2 b}{\partial x_3^2}\Big)dx_3 - \frac{K}{2}x_3 + H(x_4).$$

By a second integration with respect to x_4, we get:

$$c(x_3,x_4) = \int\Big(\int\Big(\frac{1}{2}\frac{\partial^2 a}{\partial x_4^2} + \frac{1}{2}\frac{\partial^2 b}{\partial x_3^2}\Big)dx_3\Big)dx_4 - \frac{K}{2}x_3x_4 + x_4H(x_4) + F(x_3).$$

Put, $G_2(x_3,x_4) = \int\Big(\int\Big(\frac{1}{2}\frac{\partial^2 a}{\partial x_4^2} + \frac{1}{2}\frac{\partial^2 b}{\partial x_3^2}\Big)dx_3\Big)dx_4$, then, we obtain:

$$c(x_3,x_4) = G_2(x_3,x_4) - \frac{K}{2}x_3x_4 + x_4H(x_4) + F(x_3). \tag{4.6}$$

The proof is complete. □

Corollary 4.3. *Let M be as in (2.1). With the following choices of a,b and c , M is locally symmetric:*

(i) $a(x_4) = \alpha x_4^2 + \beta x_4 + \eta$, $b(x_3) = \alpha' x_3^2 + \beta' x_3 + \eta'$ *and* $c(x_3) = \alpha'' x_3^2 + \beta'' x_3 + \eta''$, *where* $\alpha, \beta, \eta, \alpha', \beta', \eta', \alpha'', \beta'', \eta''$ *are constants.*

(ii) $a(x_4) = \alpha x_4^2 + \beta x_4 + \eta$, $b(x_3) = \alpha' x_3^2 + \beta' x_3 + \eta'$ *and* $c(x_4) = \alpha'' x_4^2 + \beta'' x_4 + \eta''$, *where* $\alpha, \beta, \eta, \alpha', \beta', \eta', \alpha'', \beta'', \eta''$ *are constants.*

(iii) $a(x_3,x_4) = \alpha x_4^2 + A(x_3)x_4 + B(x_3)$, $b(x_3,x_4) = \alpha x_3^2 + A(x_4)x_3 + B(x_4)$ *and* $c(x_3,x_4) = 0$, *where A and B smooth functions.*

Example 4.4. Let M be as in (2.1). With the following choices of $a = x_4^2, b = x_3^2$ and $c = x_3^2$. Then M is a locally symmetric strict Walker manifold. Indeed, the nonzero components of the christoffel symbols of M are given by : $\Gamma_{33}^2 = (2x_3 - x_4)$, $\Gamma_{34}^1 = x_4$, $\Gamma_{34}^2 = x_3$, $\Gamma_{34}^2 = x_3$, $\Gamma_{44}^1 = -x_3$. The nonzero components of the Levi-Civita connection of M are given by : $\nabla_{\partial_3}\partial_3 = (2x_3 - x_4)\partial_2$, $\nabla_{\partial_3}\partial_4 = x_4\partial_1 + x_3\partial_2$, $\nabla_{\partial_4}\partial_4 = -x_3\partial_1$. The nonzero components of the curvature operator of M is: $R(\partial_3, \partial_4)\partial_3 = 2\partial_2$ and the nonzero Riemann tensor components of M are given by : $R_{3434} = 2$. The components of the Ricci tensor and the scalar curvature of M are given by : $\rho_{33} = 0$, $\rho_{34} = 0$ and $\tau = 0$. Since the Riemann curvature tensor of M is constant, so M is locally symmetric.

Example 4.5. Let M be as in (2.1). With the following choices of $a = x_4^2, b = x_3^2$ and $c = x_4^2$. Then M is a locally symmetric strict Walker manifold. Indeed, the nonzero components of the christoffel symbols of M are given by: $\Gamma_{33}^2 = -x_4$, $\Gamma_{34}^1 = x_4$, $\Gamma_{34}^2 = x_3$, $\Gamma_{34}^2 = x_3$, $\Gamma_{44}^1 = (2x_4 - x_3)$. The nonzero components of the Levi-Civita connection of M are given by: $\nabla_{\partial_3}\partial_3 = -x_4\partial_2$, $\nabla_{\partial_3}\partial_4 = x_4\partial_1 + x_3\partial_2$, $\nabla_{\partial_4}\partial_4 = (2x_4 - x_3)\partial_1$. The nonzero components of the curvature operator of M is: $R(\partial_3, \partial_4)\partial_3 = 2\partial_2$ and the nonzero Riemann tensor components of M are given by : $R_{3434} = 2$. The components of the Ricci tensor and the scalar curvature of M are given by : $\rho_{33} = 0$, $\rho_{34} = 0$ and $\tau = 0$. Since the curvature tensor of M is constant, so M is locally symmetric.

Corollary 4.6. *Let M be as in (2.1). With the following choices of $a = b = c = f(x_3, x_4)$, then M is locally symmetric if and only if the function a is solution of the following system of partial differential equations:*

$$\frac{1}{2}\frac{\partial^3 a}{\partial x_4^2 \partial x_3} + \frac{1}{2}\frac{\partial^3 a}{\partial x_3^3} - \frac{\partial^3 a}{\partial x_3^2 \partial x_4} = 0 \text{ and } \frac{1}{2}\frac{\partial^3 a}{\partial x_3^2 \partial x_4} + \frac{1}{2}\frac{\partial^3 a}{\partial x_4^3} - \frac{\partial^3 a}{\partial x_4^2 \partial x_3} = 0.$$

Corollary 4.7. *Let M be as in (2.1). With the following choices of $a = f(x_3, x_4), b = f(x_3, x_4)$ and $c \equiv 0$, then M is locally symmetric if and only if the functions a and b are solutions of the following system of partial differential equations:*

$$\frac{1}{2}\frac{\partial^3 a}{\partial x_4^2 \partial x_3} + \frac{1}{2}\frac{\partial^3 b}{\partial x_3^3} = 0 \text{ and } \frac{1}{2}\frac{\partial^3 b}{\partial x_3^2 \partial x_4} + \frac{1}{2}\frac{\partial^3 a}{\partial x_4^3} = 0.$$

Corollary 4.8. *Let M be as in (2.1). With the following choices of $a = b = f(x_3, x_4)$ and $c \equiv 0$, then M is locally symmetric if and only if the function a is solution of the following system of partial differential equations:*

$$\frac{1}{2}\frac{\partial^3 a}{\partial x_4^2 \partial x_3} + \frac{1}{2}\frac{\partial^3 a}{\partial x_3^3} = 0 \text{ and } \frac{1}{2}\frac{\partial^3 a}{\partial x_3^2 \partial x_4} + \frac{1}{2}\frac{\partial^3 a}{\partial x_4^3} = 0.$$

Corollary 4.9. *Let M be as in (2.1). With the following choices of $a = b \equiv 0$ and $c = f(x_3, x_4)$, then M is locally symmetric if and only if the function c is solution of the following system of partial differential equations:*

$$\frac{\partial^3 c}{\partial x_4^2 \partial x_3} = 0 \text{ and } \frac{\partial^3 c}{\partial x_3^2 \partial x_4} = 0.$$

The Walker metric appear in several specific pseudo-Riemannian structures like 2-step nilpotent Lie groups with degenerate centers, parakahler and hypersymplectic structures, hypersurfaces with nilpotent shape operators and some four-dimensional Osserman manifolds. Indecomposable metrics of neutral signature (playing a distinguished role in the investigated holonomy of indefinite metrics) are also equipped with a Walker structure. This clearly motivates the study of pseudo-Riemannian manifolds carrying a parallel degenerate plane field (see [2] for more information).

5 Conclusion

Various geometric quantities are computed explicitly in terms of metrics coefficients, including the Christoffel symbols, curvature operator, Ricci curvature, and Weyl tensor. Using these formulas, we have obtained a description of four-dimensional strict Walker metrics which are locally symmetric.

References

[1] M. Brozos-Vázquez, E. García-Río, P. Gilkey, and R. Vázquez-Lorenzo, Completeness, Ricci blowup, the Osserman and the conformal Osserman condition for Walker signature $(2,2)$ manifolds, *Proceedings of XV International Workshop on Geometry and Physics, Publ. de la RSME* **10**, 57–66 (2007).

[2] M. Brozos-Vázquez, E. García-Río, P. Gilkey, S. Nikević and R. Vázquez-Lorenzo. *The Geometry of Walker Manifolds*. Synthesis Lectures on Mathematics and Statistics, 5. Morgan and Claypool Publishers, Williston, VT, (2009).

[3] M. Chaichi, E. García-Río and M. E. Vázquez-Abal, Three-dimensional Lorentz manifolds admitting a parallel null vector field, *J. Phys. A, Math. Gen.* **38**, (4), 841–850 (2005).

[4] A. S. Diallo, Examples of conformally 2-nilpotent Osserman manifolds of signature $(2,2)$, *Afr. Diaspora J. Math. (N.S.)* **9**, (1), 96–103 (2010).

[5] A. S. Diallo and M. Hassirou, Examples of Osserman metrics of $(3,3)$-signature, *J. Math. Sci. Adv. Appl.* **7**, (2), 95–103 (2011).

[6] A. S. Diallo, M. Hassirou and O. T. Issa, Walker Osserman metric of signature $(3,3)$, *Bull. Math. Anal. Appl.* **9**, (4), 21–30 (2017).

[7] A. S. Diallo, S. Longwap and F. Massamba, Almost Kähler eight-dimensional Walker manifold, *Novi Sad J. Math.* **48**, (1), 129–141 (2018).

[8] A. S. Diallo and F. Massamba, Some properties of four-dimensional Walker manifolds, *New Trends in Mathematical Sciences*, **5**, (3), 253–261 (2017).

[9] A. S. Diallo, A. Ndiaye and A. Niang, Minimal graphs on three-dimensional Walker manifolds, *Nonlinear analysis, geometry and applications. Trends Math., Birkhauser/Springer, Cham*, 425–438, (2020).

[10] S. Helgason, *Differential geometry and symmetric spaces*, Pure and Applied Mathematics, vol. XII, Academic Press, New York-London, (1962).

[11] A. Niang, A. Ndiaye and A. S. Diallo, A classification of strict Walker 3-manifold, *Konuralp J. Math.* **9**, (1), 148–153 (2021).

[12] A. G. Walker, Canonical form for a Riemannian space with a parallel field of null planes, *Quart J Math Oxford* **1**, (2), 69–79 (1950).

Author information

Mohamed Ayatola Dramé, Université Alioune Diop de Bambey, UFR SATIC, Département de Mathématiques, Équipe de Recherce en Analyse Non Linéaire et Géométrie (ANLG), B. P. 30, Bambey, Sénégal.
E-mail: rabanyayatoulah@gmail.com

Babacar Mbodj, Université Alioune Diop de Bambey, UFR SATIC, Département de Mathématiques, Équipe de Recherce en Analyse Non Linéaire et Géométrie (ANLG), B. P. 30, Bambey, Sénégal.
E-mail: babacarmbodj35@gmail.com

Abdoul Salam Diallo, Université Alioune Diop de Bambey, UFR SATIC, Département de Mathématiques, Équipe de Recherce en Analyse Non Linéaire et Géométrie (ANLG), B. P. 30, Bambey, Sénégal.
E-mail: abdoulsalam.diallo@uadb.edu.sn

Received: 2023-02-10
Accepted: 2023-05-18