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Special Operation- Pro-addition (†) On Pythagorean triplets

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Abstract: The objective of this study is to introduce a novel operation called pro-addition (†) for Pythagorean triplets, which extends to various groups of Pythagorean triplets through a new generation process. This operation works in tandem with conventional addition and serves to establish either the closure property within a set or establish connections between sets belonging to different groups. It is our sincere aspiration that this article proves valuable to scholars, students, and math educators in their pursuit of exploring new avenues in this field.

1 Introduction

The Pythagorean theorem is a fundamental principle stating that in any right-angled triangle, the square of the hypotenuse is equal to the sum of the squares of the two shorter legs. This theorem can be represented by Pythagorean triplets, which are integer solutions (a, b, c) satisfying the polynomial expression $x^2 + y^2 = z^2$. Numerous proofs have been put forth for the Pythagorean theorem, and its converse has also been shown to hold true. Research in this field has explored various aspects of finding solutions to the polynomial expression $x^2 + y^2 = z^2$. For instance, around 300 BC, Euclid demonstrated that a triangle with sides a, b, and c satisfying the equation $a^2 + b^2 = c^2$ must be a right-angled triangle (refer to [1, 2]).

The generalization of the Pythagorean theorem by Dijkstra and the shorter proof by Bhaskara are among the numerous proofs documented in the literature (see [3]). The exploration of integer solutions to $x^2 + y^2 = z^2$ has led to various avenues of investigation. One direction involves dealing with polynomials of the form $x^2 + y^2 = z^2 \pm 1$, where the integer solutions were termed "almost Pythagorean triplets" or "nearly Pythagorean triplets" based on the sign $\pm$ (see [4]). Another aspect involves studying solutions of $x^2 + y^2 = z^2$ with special conditions. For example, a solution (a, b, c) is called isosceles if $a = b$. However, there are no isosceles integer solutions to $x^2 + y^2 = z^2$, leading to investigations of isosceles-like integer triples (a, b, c) with $|a - b| = 1$ (explored in [4]). Scholars in the literature (refer to [5, 6, 7, 8, 9]) have studied the concept of almost isosceles Pythagorean triplets by utilizing the Pell polynomial. In some articles, this type of triplet has been referred to as an "almost-isosceles right-angled triangle," emphasizing its relationship with Pythagorean triplets, almost Pythagorean triplets, and nearly Pythagorean triplets.

The concept of Pythagorean triplets and their connection to the Pythagorean theorem serves as a foundational principle in various fields of pure mathematics, encompassing number theory, elementary geometry, and applied mathematics. These triplets have significant implications and applications in different areas of study. Over the past decade, there has been a surge of interest in the generation of Pythagorean triplets, leading to a flourishing stream of literature. The relevance of this topic has been recognized in fields like cryptography and random number generation algorithms. As a result, researchers and mathematicians have delved into exploring novel methods for generating Pythagorean triplets and uncovering their diverse properties and applications. This ongoing research has contributed valuable insights and advancements in the study of these integer solutions and their significance in various mathematical contexts.

Euclid's formula is a fundamental method for generating Pythagorean triplets using an arbitrary pair of non-zero natural numbers m and n. According to this formula, all primitive Pythagorean triplets (a, b, c) can be derived from the following equations [3]:

$$a = m^2 - n^2, b = 2mn, c = m^2 + n^2$$

Here, $m > n$, and m and n are pairs of relatively prime numbers. An essential property to observe is that one of the numbers m or n is even, while the other is odd. Notably, each primitive Pythagorean triplet (a, b, c) where b is even can be obtained uniquely using this formula. This powerful method has proven to be instrumental in generating various Pythagorean triplets and has been extensively studied and applied in the field of number theory.

Thomas Harriot, a prominent English mathematician and scientist, played a pioneering role in proposing the existence of Pythagorean triples in series (see [10]). While classical formulae available in the literature are known to generate all primitive Pythagorean triplets, they do not cover all possible triplets, particularly the non-primitive ones. In more recent times, Bhanotar et al. have explored methods for generating primitive Pythagorean triplets (refer to [11, 12]). Their work introduced an intriguing algebraic approach, extending the field $Q^+ \cup \{0\}$ to explore the dual of given triplets, resulting in the identification of important sequences and interconnectivity. Agrawal (in [13]) delved into the history of Pythagorean triples both before and after the time of Pythagoras, shedding light on the rich historical context of this mathematical concept. Furthermore, many mathematicians and experts from diverse branches have contributed to the development and communication of various aspects related to the construction, characterization, and geometrical representation of Primitive Pythagorean triples. Numerous works in the literature (see [14, 15, 16, 17, 18, 19, 20, 21, 22, 23]) have tackled this fascinating area, offering valuable insights and enhancing our understanding of Pythagorean triplets in different mathematical contexts.

To ensure accessibility, we introduce the following notations. The set of natural numbers $\mathbb{N}$ has been divided into two infinite sets: $\mathbb{N}_1$ – a set of all odd integers and $\mathbb{N}_2$ –a set of all even integers such that,

$$\mathbb{N} = \mathbb{N}_1 \cup \mathbb{N}_2 \text{ and } \mathbb{N}_1 \cap \mathbb{N}_2 = \phi. \tag{1.1}$$

In the context of this concept, two commonly used terms are "Pythagorean triplets (PTs)" and "Primitive Pythagorean Triplet (PPTs)."

Definition:1.1 Pythagorean triplet (PT): A triplet (a, b, h), with a, b, and h belonging to some natural numbers is said to be a Pythagorean triplet (PT) if it satisfies the condition:

$$a^2 + b^2 = h^2 \tag{1.2}$$

Definition 1.2:Primitive Pythagorean triplet (PPT): A triplet (a, b, h), with a, b, and h belonging to some natural numbers satisfying the following properties are known as primitive triplets (PT)/primitive Pythagorean triplets (PPTs):

(a) G.C.D. of a and $b = 1$.

(b) $a < b < h$ and

(c) $a^2 + b^2 = h^2$.

Indeed, the conditions of primitive triplets distinguish them from regular triplets and offer a unique perspective on Pythagorean triplets. Mathematicians have explored various methods to construct these triplets, each representing their distinct ideas and approaches. In an exciting and enjoyable manner, we present a different perspective on Pythagorean triplets (PTs) for odd and even integers, as follows [11]:

$$(a, b, h) = (a, \frac{a^2 - i^2}{2i}, \frac{a^2 - i^2}{2i}) \tag{1.3}$$

For $a \in \mathbb{N}_1 - \{1\}$, we can find b by selecting an integer i from $\mathbb{N}_1$ that satisfies the condition $i = 1^2, 3^2, 5^2, ... \in \mathbb{N}_1$ and is less than $a/2$. Then, we can take $h = b + i$, which fulfills the condition $a^2 + b^2 = h^2$, forming a Pythagorean triplet (PT). We classify a triplet as an odd primitive triplet if the smallest integer among the three, i.e., a, is an odd number. Interestingly, it is possible to have more than one primitive triplet satisfying these conditions (see [2, 3, 8]). This unique approach allows for the discovery of multiple odd primitive Pythagorean triplets through suitable choices of a, b, and h, enriching our understanding of these fascinating integer solutions.

Similarly, for $a \in \mathbb{N}_2 - \{2\}$, we need to choose a corresponding value i from $\mathbb{N}_2$ that satisfies the condition $i = 2n^2 = 2, 8, 18, 32, ... < a/2$. Then, we can take the hypotenuse $h = b + i$, which fulfills the condition $a^2 + b^2 = h^2$, forming a Pythagorean triplet (PT). We classify a triplet as an even primitive triplet (EPT) if the smallest integer among the three, i.e., a, is an even number. This approach allows us to consider feasible values for i and potentially predict additional primitive triplets if they exist. For example, $(3, 4, 5), (5, 12, 17), \ldots$ are odd PPTs, whereas $(8, 15, 17), (12, 35, 37), \ldots$ are even PPTs. Now, a simple question arises: How many primitive Pythagorean triplets are possible to correspond to a particular value of a?

Indeed, as described above, we can obtain multiple primitive Pythagorean triplets corresponding to a given positive integer a, whether it is even or odd. The approach of choosing different values for i from the appropriate set $\mathbb{N}_1$ or $\mathbb{N}_2$ ensures that we can generate various Pythagorean triplets, each with a as one of its components. This flexibility in the selection of i allows us to explore different combinations and find distinct primitive triplets for a given value of a. Consequently, we can observe a rich variety of Pythagorean triplets with unique properties and characteristics, adding to the intrigue and fascination of this mathematical concept.

For example,

- For $a = 3$, which is an odd integer, we choose i from the set of squares of odd integers, i.e., $i = 1^2, 3^2, 5^2, ... < a/2 = 3/2$. Thus, the possible value of i is 1. Using this value, we can calculate the integer b as follows:

$$b = \frac{a^2 - i^2}{2i} = \frac{3^2 - 1^2}{2 \cdot 1} = 4$$

  The next highest integer in the sequence, which we call the hypotenuse, is calculated as:

$$h = \frac{a^2 + i^2}{2i} = \frac{3^2 + 1^2}{2 \cdot 1} = 5$$

  This triplet $(3, 4, 5)$ satisfies the condition of a Pythagorean triplet: $3^2 + 4^2 = 5^2$. It is the only odd primitive Pythagorean triplet corresponding to the smallest leg $a = 3$. This example demonstrates how the approach described earlier can be used to find specific primitive Pythagorean triplets for a given value of a.

- For $a = 8$, which is an even integer, we choose i from the set of even squares, i.e., $i = 2n^2 = 2, 8, 18, 32, ... < a/2 = 8/2 = 4$. The possible value of i is 2. Using this value of i, we can calculate the integer b as follows:

$$b = \frac{a^2 - i^2}{2i} = \frac{8^2 - 2^2}{2 \cdot 2} = 15$$

  Next, we can calculate the hypotenuse h in two ways:

$$h = \frac{a^2 + i^2}{2i} = \frac{8^2 + 2^2}{2 \cdot 2} = 17 \text{ or } h = b + i = 15 + 2 = 17$$

  Hence, there is only one even primitive Pythagorean triplet $(8, 15, 17)$ corresponding to the value of $a = 8$. This example illustrates how the approach presented earlier can be applied to find specific primitive Pythagorean triplets for a given value of a.

At this juncture, it is essential to inform the readers that there are infinitely many odd integers that possess multiple Pythagorean primitive triplets (PPTs). These special Pythagorean triplets are known as "Multiple Primitive Pythagorean Triplets" (MPPTs). For such odd integers, it is possible to find not just one, but two, three, or even more distinct primitive triplets that satisfy the Pythagorean theorem. This remarkable property of MPPTs makes them particularly fascinating and adds to the intriguing nature of Pythagorean triplets. The existence of MPPTs showcases the richness and complexity of the world of Pythagorean triplets, making them a captivating subject of exploration in mathematics.

Let's consider two more examples, which are as follows:

- Let's consider the case of $a = 20$, an even integer. We can choose different values of i from the set of even squares, i.e., $i = 2n^2 = 2, 8, 18, ... < a/2 = 20/2 = 10$. The possible values of i are 2 and 8.

  For $i = 2$, we can calculate the integer b as follows:

  $$b = \frac{a^2 - i^2}{2i} = \frac{20^2 - 2^2}{2 \cdot 2} = 99$$

  The hypotenuse h is then:

  $$h = b + i = 99 + 2 = 101$$

  Thus, we have the first primitive Pythagorean triplet as $(20, 99, 101)$.

  For $i = 8$, we calculate the integer b:

  $$b = \frac{a^2 - i^2}{2i} = \frac{20^2 - 8^2}{2 \cdot 8} = 21$$

  The hypotenuse h is then:

  $$h = b + i = 21 + 8 = 29$$

  Thus, we have the second primitive Pythagorean triplet as $(20, 21, 29)$.

  In conclusion, for $a = 20$, we find two distinct primitive Pythagorean triplets: $(20, 99, 101)$ and $(20, 21, 29)$. This example highlights the existence of multiple primitive triplets for a single value of a, which aligns with the concept of Multiple Primitive Pythagorean Triplets (MPPTs).

- Indeed! Let's consider the case of $a = 33$, an odd integer. We can choose different values of i from the set of squares of odd integers, i.e., $i = 1^2, 3^2, 5^2, ... = 1, 9, 25, ... < a/2 = 33/2 = 16.5$. The possible values of i are 1 and 9.

  For $i = 1$, we can calculate the integer b as follows:

  $$b = \frac{a^2 - i^2}{2i} = \frac{33^2 - 1^2}{2 \cdot 1} = 544$$

  The hypotenuse h is then:

  $$h = b + i = 544 + 1 = 545$$

  Thus, we have the first primitive Pythagorean triplet as $(33, 544, 545)$.

  For $i = 9$, we calculate the integer b:

  $$b = \frac{a^2 - i^2}{2i} = \frac{33^2 - 9^2}{2 \cdot 9} = 56$$

The hypotenuse h is then:

$$h = b + i = 56 + 9 = 65$$

Thus, we have the second primitive Pythagorean triplet as $(33, 56, 65)$.

In conclusion, for $a = 33$, we find two distinct primitive Pythagorean triplets: $(33, 544, 545)$ and $(33, 56, 65)$. As mentioned earlier, these are Multiple Primitive Pythagorean Triplets (MPPTs) for the given value of $a = 33$. The existence of MPPTs for odd integers demonstrates the fascinating and rich nature of Pythagorean triplets.

Let's elaborate on some known and obvious results related to primitive Pythagorean triplets:

- A triplet corresponding to the first leg from the set $\{1, 4\} \cup \{2(2n - 3) | n \in \mathbb{N}, n \geq 2\}$ cannot possess a primitive Pythagorean triplet (PPT).
- A triplet corresponding to the first leg which is a member of the set $\{4(2n + 3) | n \in \mathbb{N}\}$ possesses at least one primitive Pythagorean triplet.
- A triplet corresponding to the first leg which is a member of the set $\{3(2n+9)|n \in \mathbb{N} \cup 0\}$ possesses one or more than one primitive Pythagorean triplet.
- At least one of the integers a, b, and h in a primitive Pythagorean triplet is prime or divisible by 5.

This article is divided into three distinct sections. Section 2, introduces various groups of Pythagorean triplets (PTs), while Section 3 delves into a unique operation called Pro-addition $(\dagger)$ within the PT family. Finally, the last section 4, presents conclusions drawn from the study.

2 Different groups of PT

Pythagorean Triplets (PTs) are categorized into distinct groups based on their research and patterns, with each group named after a famous mathematician who made significant contributions to the field. In the context of a right triangle, the shorter leg is represented by the letter a, the second leg by the letter b, and the hypotenuse by the letter h. Thus, a Pythagorean triplet takes the form (a, b, h), where all three numbers are positive integers. It is common to consider $a < b$ with $(a, b) = 1$ and satisfying the relation $a^2 + b^2 = h^2$.

According to our theory, each class of Pythagorean triplets is a subclass of the Pythagorean family. Considering the aforementioned restrictions, members of Pythagorean triplets or triplets that are integer multiples of a certain basic triplet can be included. If (a, b, h) is a primitive triplet, then for some $k \in \mathbb{N}$, $k(a, b, h) = (ka, kb, kh)$ is also a Pythagorean triplet, but it is not a primitive Pythagorean triplet (PPT).

Let's use the symbol $\mathfrak{P}$ to represent the entire set of triples with the form (a, b, h) as follows:

$$\mathfrak{P} = \{(a, b, h)|a, b, h \in \mathbb{N}, a^2 + b^2 = h^2\}. \tag{2.1}$$

Definition 2.1: Plato Family $\mathfrak{P}_1$: We can define the infinite set referred to as the PTs of the Plato family in the following manner:

$$\mathfrak{P}_1 = \{(a, b, h)|a, b, h \in \mathbb{N}, a < b, (a, b) = 1, |h - b| = 1\} \tag{2.2}$$

Odd triplets like $(3, 4, 5)$ and $(7, 24, 25)$ are part of $\mathfrak{P}_1$, a collection which is recognized as the Plato family.

Definition 2.2: Pythagorean Sub-family $\mathfrak{P}_2$: We define the infinite set referred to as a Pythagorean subfamily $\mathfrak{P}_2$ in the following manner:

$$\mathfrak{P}_2 = \{(a, b, h)|a, b, h \in \mathbb{N}, a < b < h, (a, b) = 1, |h - b| = 2\} \tag{2.3}$$

As an illustration, $(8,15,17)$ and $(12,35,37)$ are included in $\mathfrak{P}_2$, a group designated as the Pythagorean Sub-family $\mathfrak{P}_2$.

Definition 2.3: Fermat Family $\mathfrak{P}_3$: We define the infinite set known as the Fermat Family of Pythagorean Triplets as follows :

$$\mathfrak{P}_3 = \{(a,b,h)|a,b,h \in \mathbb{N}, a < b, (a,b) = 1, |b-a| = 1\} \tag{2.4}$$

As an instance, $(3,4,5)$ and $(20,21,29)$ are part of $\mathfrak{P}_3$, a collection known as the Fermat family.

At this juncture, it is essential to note that the set $\mathfrak{P}^{'} = \mathfrak{P} - \mathfrak{P}_1 \cup \mathfrak{P}_2 \cup \mathfrak{P}_3$ is an infinite set. This infinite set $\mathfrak{P}^{'}$ includes triplets such as $(20,45,53), (39,80,89)$, and so on. Now, we shall proceed to define the addition of Pythagorean triplets based on this split family, which is as presented in the follwing section.

3 Special Operation - Pro-addition $(\dagger)$ on PTs

Within the member triples of the Plato family $\mathfrak{P}_1$ and the Pythagorean sub-family $\mathfrak{P}_2$, we define a set of special operations. These operations work in parallel to regular addition and serve two main purposes: either establishing a closure property within a set or creating a connection between the sets $\mathfrak{P}_1$ and $\mathfrak{P}_2$. To facilitate understanding and convenience, we introduce the notation for the addition of Pythagorean triplets as "pro-addition."

Definition-3.1 Pro-addition in $(\dagger)$ $\mathfrak{P}_1$: Let $a_2 \geq a_1$, and consider two members of the group $\mathfrak{P}_1$, denoted as $t_1 = (a_1,b_1,h_1)$ and $t_2 = (a_2,b_2,h_2)$. We introduce a unique and special operation for the addition of Pythagorean triplets, known as the 'pro-addition,' defined as follows:

$$t_1 \dagger t_2 = (a_1 + a_2, b_1 + b_2 - l^2, h_1 + h_2 - l^2) \tag{3.1}$$

where, $l = |\frac{(a_2-a_1)}{2}|, b_j = \frac{a_j^2-1}{2}, h_j = \frac{a_j^2+1}{2} = b_j + 1, j = 1,2.$

Theorem 3.1. *If $t_1 = (a_1,b_1,h_1)$ and $t_2 = (a_2,b_2,h_2)$ belong to the set $\mathfrak{P}_1$ with $a_2 \geq a_1$, then their pro-addition is given by $t_1 \dagger t_2 = (a_1 + a_2, b_1 + b_2 - l^2, h_1 + h_2 - l^2)$, where $l = \left|\frac{(a_2-a_1)}{2}\right|$, and this results in a new Pythagorean triplet.*

Proof.

$$\begin{aligned}
&(h_1 + h_2 - l^2)^2 - (b_1 + b_2 - l^2)^2 \\
&\quad = (h_1 + h_2)^2 - 2l^2(h_1 + h_2) + l^4 - (b_1 + b_2)^2 + 2l^2(b_1 + b_2) - l^4 \\
&\quad = \left(\frac{a_1^2 + a_2^2 + 2}{2}\right)^2 - \left(\frac{a_1^2 + a_2^2 - 2}{2}\right)^2 + 2l^2\left(\frac{a_1^2 + a_2^2 - 2}{2} - \frac{a_1^2 + a_2^2 + 2}{2}\right) \\
&\quad = \frac{1}{4}\left[\left(a_1^2 + a_2^2\right)^2 + 4\left(a_1^2 + a_2^2\right) + 4 - \left(a_1^2 + a_2^2\right)^2 + 4\left(a_1^2 + a_2^2\right) - 4\right] + 2l^2(-2) \\
&\quad = 2\left(a_1^2 + a_2\right) - 4\left(|\frac{a_2 - a_1}{2}|\right)^2 \\
&\quad = a_1^2 + a_2^2 + 2a_1a_2 \\
&\quad = (a_1 + a_2)^2
\end{aligned}$$

Hence, $(a_1 + a_2, b_1 + b_2 - l^2, h_1 + h_2 - l^2)$, form a PT. This completes the proof. □

Remark: 3.1. If $t_1 = (a_1,b_1,h_1)$ and $t_2 = (a_2,b_2,h_2)$ belong to the set $\mathfrak{P}_1$, with $a_2 \geq a_1$, then their pro-addition $t_1 \dagger t_2$ will result in an even Pythagorean triplet, and hence, $t_1 \dagger t_2 \in \mathfrak{P}_2$. Furthermore, when $t_1 = t_2$, the pro-addition $t_1 \dagger t_2$ becomes $t_1 \dagger t_1 = 2t_1 = (2a_1, 2b_1, 2h_1)$, resulting in a new Pythagorean triplet that is even. Therefore, $2t_1 \in \mathfrak{P}_2$.

Example:3.1. If $t_1 = (3,4,5)\,, t_2 = (7,24,25) \in \mathfrak{P}_1$, then their pro-addition,

$$\begin{aligned} t_1 \dagger t_2 &= \left(3+7, 4+24-\left(\frac{7-3}{2}\right)^2, 5+25-\left(\frac{7-3}{2}\right)^2\right) \\ &= (10,24,26) \\ &= 2\,(5,12,13) \in \mathfrak{P}_2. \end{aligned}$$

Example:3.2. If $t_1 = (5,12,13)\,, t_2 = (13,84,85) \in \mathfrak{P}_1$, then their pro-addition,

$$\begin{aligned} t_1 \dagger t_2 &= \left(5+13, 12+84-\left(\frac{13-5}{2}\right)^2, 13+85-\left(\frac{13-5}{2}\right)^2\right) \\ &= (18,80,82) \\ &= 2\,(9,40,41) \in \mathfrak{P}_2. \end{aligned}$$

Definition-3.2. Pro-addition $(\dagger)$ in $\mathfrak{P}_2$: If $a_2 \geq a_1$, and $t_1 = (a_1, b_1, h_1)$ and $t_2 = (a_2, b_2, h_2)$ are members of the family $\mathfrak{P}_2$, then their 'pro-addition' of Pythagorean triplets is defined as follows:

$$t_1 \dagger t_2 = \left(a_1 + a_2, b_1 + b_2 + \left(\frac{a_1 a_2 + 2}{2}\right), h_1 + h_2 + \left(\frac{a_1 a_2 - 2}{2}\right)\right) \tag{3.2}$$

where, $b_j = \frac{a_j^2 - 4}{4}, h_j = \frac{a_j^2 + 4}{4} = b_j + 2, j = 1, 2$

Remark: 3.2. If $t_1 = (a_1, b_1, h_1)$ and $t_2 = (a_2, b_2, h_2)$ belong to the family $\mathfrak{P}_2$ with $a_2 \geq a_1$, then their 'pro-addition' $t_1 \dagger t_2$ will result in a new Pythagorean triplet that is even, and hence, $t_1 \dagger t_2 \in \mathfrak{P}_2$.

Theorem 3.2. *If $t_1 = (a_1, b_1, h_1)$ and $t_2 = (a_2, b_2, h_2)$ belong to the family $\mathfrak{P}_2$ with $a_2 \geq a_1$, then their 'pro-addition' is given by:*

$$t_1 \dagger t_2 = \left(a_1 + a_2, b_1 + b_2 + \left(\frac{a_1 a_2 + 2}{2}\right), h_1 + h_2 + \left(\frac{a_1 a_2 - 2}{2}\right)\right)$$

This results in a new Pythagorean triplet.

Proof.

$$\begin{aligned} &\left(h_1 + h_2 + \left(\frac{a_1 a_2 - 2}{2}\right)\right)^2 - \left(b_1 + b_2 + \left(\frac{a_1 a_2 + 2}{2}\right)\right)^2 \\ &= \left(b_1 + 2 + b_2 + 2 + \left(\frac{a_1 a_2 - 2}{2}\right)\right)^2 - \left(b_1 + b_2 + \left(\frac{a_1 a_2 + 2}{2}\right)\right)^2 \\ &= (b_1 + b_2 + 4)^2 + \left(\frac{a_1 a_2 - 2}{2}\right)^2 + (b_1 + b_2 + 4)(a_1 a_2 - 2) \\ &\quad - (b_1 + b_2)^2 - \left(\frac{a_1 a_2 + 2}{2}\right)^2 - (b_1 + b_2)(a_1 a_2 + 2) \\ &= 4(b_1 + b_2) + 2a_1 a_2 + 8 \\ &= 4\left(\frac{a_1^2 - 4}{4} + \frac{a_2^2 - 4}{4}\right) + 2a_1 a_2 + 8 \\ &= a_1^2 + a_2^2 + 2a_1 a_2 \\ &= (a_1 + a_2)^2 \end{aligned}$$

Hence, $\left(a_1 + a_2, b_1 + b_2 + \left(\frac{a_1 a_2 + 2}{2}\right), h_1 + h_2 + \left(\frac{a_1 a_2 - 2}{2}\right)\right)$, form a PT. This completes the proof. □

Example:3.3. If $t_1 = (8, 15, 17), t_2 = (12, 35, 37) \in \mathfrak{P}_2$, then their pro-addition,

$$\begin{aligned} t_1 \dagger t_2 &= (20, 50 + 49, 54 + 47) \\ &= (20, 99, 101) \in \mathfrak{P}_2. \end{aligned}$$

Example:3.4. If $t_1 = (12, 35, 37), t_2 = (20, 99, 191) \in \mathfrak{P}_2$, then their pro-addition,

$$\begin{aligned} t_1 \dagger t_2 &= (32, 134 + 121, 138 + 119) \\ &= (32, 255, 257) \in \mathfrak{P}_2. \end{aligned}$$

Definition-3.3. Pro-addition ($\dagger$) of members of group $\mathfrak{P}_1$ and $\mathfrak{P}_2$: Let $a_2 \geq a_1$ and let $t_1 = (a_1, b_1, h_1) \in \mathfrak{P}_1$ and $t_2 = (a_2, b_2, h_2) \in \mathfrak{P}_2$ be the two PTs, then their 'pro-addition' ($\dagger$) is defined as follow:

$$t_1 \dagger t_2 = (a_1 + a_2, b_1 + 2b_2 + a_1 a_2 + 2, h_1 + 2h_2 + a_1 a_2 - 2) \in \mathfrak{P}_1 \tag{3.3}$$

where, $b_1 = \frac{a_1^2 - 1}{2}, h_1 = \frac{a_1^2 + 1}{2} = b_1 + 1$, and $b_2 = \frac{a_2^2 - 4}{4}, h_2 = \frac{a_2^2 + 4}{2} = b_2 + 2$

Theorem 3.3. *For $a_2 \geq a_1$ and let $t_1 = (a_1, b_1, h_1) \in \mathfrak{P}_1$ and $t_2 = (a_2, b_2, h_2) \in \mathfrak{P}_2$ be the two PTs, then their 'pro-addition' defined as follows form a PT*

$$t_1 \dagger t_2 = (a_1 + a_2, b_1 + 2b_2 + a_1 a_2 + 2, h_1 + 2h_2 + a_1 a_2 - 2) \in \mathfrak{P}_1.$$

where, $b_1 = \frac{a_1^2 - 1}{2}, h_1 = \frac{a_1^2 + 1}{2} = b_1 + 1$, *and* $b_2 = \frac{a_2^2 - 4}{4}, h_2 = \frac{a_2^2 + 4}{2} = b_2 + 2$

Proof.

$$\begin{aligned} &(h_1 + 2h_2 + a_1 a_2 - 2)^2 - (b_1 + 2b_2 + a_1 a_2 + 2)^2 \\ &\qquad = (b_1 + 1 + 2b_2 + 4 + a_1 a_2 - 2)^2 - (b_1 + 2b_2 + a_1 a_2 + 2)^2 \\ &\qquad = (b_1 + 2b_2 + a_1 a_2 + 3)^2 - (b_1 + 2b_2 + a_1 a_2 + 2)^2 \\ &\text{Let } b_1 + 2b_2 + a_1 a_2 = A \\ &\qquad = (A + 3)^2 - (A - 2)^2 \\ &\qquad = 2A + 5 \\ &\qquad = 2(a_1^2 + a_2^2 + 2a_1 a_2) + 5 \\ &\qquad = \left(a_1^2 - 1 + a_2^2 - 4 + 2a_1 a_2\right) + 5 \\ &\qquad = (a_1 + a_2)^2 \end{aligned}$$

This shows that, $(a_1 + a_2, b_1 + 2b_2 + a_1 a_2 + 2, h_1 + 2h_2 + a_1 a_2 - 2) \in \mathfrak{P}_1$, form a PT. This completes the proof. □

Example:3.5. If $t_1 = (5, 12, 13) \in \mathfrak{P}_1$, and $t_2 = (12, 35, 37) \in \mathfrak{P}_2$, then their pro-addition is ,

$$\begin{aligned} t_1 \dagger t_2 &= (5 + 12, 12 + 2(35) + 60 + 2, 13 + 2(37) + 60 - 2) \\ &= (17, 144, 145) \in \mathfrak{P}_1. \end{aligned}$$

Example:3.6. If $t_1 = (7, 24, 25) \in \mathfrak{P}_1$, and $t_2 = (16, 63, 65) \in \mathfrak{P}_2$, then their pro-addition is ,

$$\begin{aligned} t_1 \dagger t_2 &= (7 + 16, 24 + 2(63) + 112 + 2, 25 + 2(65) + 112 - 2) \\ &= (23, 264, 265) \in \mathfrak{P}_1. \end{aligned}$$

4 Conclusion

In consequence of the previous article [11], this study presents a novel method for generating all Pythagorean triplets, both primitive and non-primitive. The special operation of Pro-addition (†) within the Pythagorean triplet families $\mathfrak{P}_1$ and $\mathfrak{P}_2$ has demonstrated closure properties for addition, providing a coherent and elegant approach to generating new triplets from existing ones. The potential extension of research into fields such as Field theory and Linear algebra holds promise for uncovering deeper connections and applications of this newly proposed method. The discovery of multiplicative closure property within the defined framework further amplifies the potential for future investigations and opens exciting avenues for exploring Pythagorean triplets in novel contexts. The method presented in this article not only expands our understanding of the structure and properties of Pythagorean triplets but also contributes significantly to the broader field of mathematics. It offers valuable insights and fresh perspectives, fostering interdisciplinary research possibilities and igniting interest in the study of Pythagorean triplets. As researchers delve deeper into the implications and applications of this approach, it is expected to enrich various mathematical disciplines and inspire new discoveries.

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