HYPERBOLIC VALUED METRIC SPACE

Chinmay Ghosh, Anirban Bandyopadhyay and Soumen Mondal

Communicated by Harikrishnan Panackal

Abstract In this paper, our main aim is to develop some basic properties of hyperbolic valued metric spaces. We also establish the hyperbolic version of the Banach contraction principle. Further, we construct a hyperbolic valued metric on the space of all hyperbolic valued continuous functions and prove some results.

1 Introductions and Preliminaries

Corrado Segre [8] introduced bicomplex numbers in 1892. The set of bicomplex numbers [1], [7] is defined as

$$\mathbb{BC} = \{w = w_1 + \mathbf{j}w_2 : w_1, w_2 \in \mathbb{C}(\mathbf{i})\},$$

where $\mathbf{i}$ and $\mathbf{j}$ are imaginary units such that $\mathbf{ij} = \mathbf{ji}$, $\mathbf{i}^2 = \mathbf{j}^2 = -1$, and $\mathbb{C}(\mathbf{i})$ is the set of complex numbers with imaginary unit $\mathbf{i}$. Also, $\mathbb{C}(\mathbf{j})$ is the set of complex numbers with imaginary unit $\mathbf{j}$. Throughout this paper, we take $\mathbb{R}$ and $\mathbb{C}$ to be fields of real and complex numbers respectively and $\mathbb{N}$ to be the set of natural numbers. Note that the three sets $\mathbb{C}$, $\mathbb{C}(\mathbf{i})$, $\mathbb{C}(\mathbf{j})$ are the same. As $\mathbb{C}$ consists of two copies of $\mathbb{R}$, called the real and imaginary line, $\mathbb{BC}$ also consists of two copies of $\mathbb{C}$, called $\mathbb{C}(\mathbf{i})$ and $\mathbb{C}(\mathbf{j})$. The set $\mathbb{BC}$ forms a commutative ring with unity and with zero divisors under the usual addition and multiplication of bicomplex numbers. The product of imaginary units $\mathbf{i}$ and $\mathbf{j}$ defines a hyperbolic unit $\mathbf{k}$ such that $\mathbf{k}^2 = 1$. The three units satisfy

$$\mathbf{ij} = \mathbf{ji} = \mathbf{k}, \ \mathbf{ik} = \mathbf{ki} = -\mathbf{j} \text{ and } \mathbf{jk} = \mathbf{kj} = -\mathbf{i}.$$

Along with two complex planes $\mathbb{C}(\mathbf{i})$ and $\mathbb{C}(\mathbf{j})$, $\mathbb{BC}$ contains a third interesting plane called the hyperbolic plane. The elements of the hyperbolic plane are the bicomplex numbers like $\alpha = a_1 + \mathbf{k}a_2$ where $a_1, a_2 \in \mathbb{R}$. The set of hyperbolic numbers is denoted by $\mathbb{D}$ i.e.,

$$\mathbb{D} = \{\alpha = a_1 + \mathbf{k}a_2 : a_1, a_2 \in \mathbb{R}\}.$$

Note that $\mathbb{D}$ is a commutative ring with unity and with zero divisors under the usual addition and multiplication of $\mathbb{BC}$.

The two zero divisors

$$\mathbf{e}_1 = \frac{1 + \mathbf{k}}{2} \text{ and } \mathbf{e}_2 = \frac{1 - \mathbf{k}}{2}$$

form an idempotent basis in $\mathbb{BC}$.

If $\alpha = a_1 + \mathbf{k}a_2 \in \mathbb{D}$, then the idempotent representation of α is

$$\alpha = (a_1 + a_2)\mathbf{e}_1 + (a_1 - a_2)\mathbf{e}_2.$$

The sum and the product of two hyperbolic numbers can be defined pointwise with the idempotent basis.

There exists a bijection between the Euclidean plane $\mathbb{R}^2$ and $\mathbb{D}$, in which every hyperbolic number $\alpha = \alpha_1 \mathbf{e}_1 + \alpha_2 \mathbf{e}_2$ is mapped to $(\alpha_1, \alpha_2) \in \mathbb{R}^2$ and vice versa.

Also, we have the two projection maps $p_1, p_2 : \mathbb{D} \to \mathbb{R}$, given by

$$p_i(\alpha) = \alpha_i, \text{ where } \alpha = \alpha_1 \mathbf{e}_1 + \alpha_2 \mathbf{e}_2, \ i = 1, 2.$$

We say that a hyperbolic number $\alpha = \alpha_1 \mathbf{e}_1 + \alpha_2 \mathbf{e}_2 \in \mathbb{D}$ is a positive hyperbolic number if $\alpha_1, \alpha_2 > 0$. Thus the set of positive hyperbolic numbers $\mathbb{D}^+$ is given by

$$\mathbb{D}^+ = \{\alpha = \alpha_1 \mathbf{e}_1 + \alpha_2 \mathbf{e}_2 : \alpha_1, \alpha_2 > 0\},$$

and we denote the set of all zero divisors in $\mathbb{D}$ by $\mathbb{O}$. Set $\mathbb{O}_0 = \mathbb{O} \cup \{0\}$. And we use the notation $\mathbb{D}_0^+ = \mathbb{D}^+ \cup \mathbb{O}_0$.

For $\alpha, \beta \in \mathbb{D}$, define a relation $\preceq$ on $\mathbb{D}$ [5] by $\alpha \preceq \beta$ whenever $\beta - \alpha \in \mathbb{D}_0^+$. This relation is

reflexive, anti-symmetric as well as transitive and hence defines a partial order on $\mathbb{D}$. If we write the hyperbolic numbers α, β in idempotent representation as $\alpha = \alpha_1 \mathbf{e}_1 + \alpha_2 \mathbf{e}_2$ and $\beta = \beta_1 \mathbf{e}_1 + \beta_2 \mathbf{e}_2$, then $\alpha \preceq \beta$ implies that $\alpha_1 \leq \beta_1$ and $\alpha_2 \leq \beta_2$. And by $\alpha \prec \beta$ we mean $\alpha_1 < \beta_1$ and $\alpha_2 < \beta_2$.

For $A \subset \mathbb{D}$, define [9],

$$
\begin{aligned}
A_{\mathbf{e}_1} &= \{x \in \mathbb{R} : \exists y \in \mathbb{R} \text{ such that } x\mathbf{e}_1 + y\mathbf{e}_2 \in A\}, \\
A_{\mathbf{e}_2} &= \{y \in \mathbb{R} : \exists x \in \mathbb{R} \text{ such that } x\mathbf{e}_1 + y\mathbf{e}_2 \in A\}.
\end{aligned}
$$

and we will consider the supremum and infimum of A defined as follows

$$
\begin{aligned}
\sup A &= \sup A_{\mathbf{e}_1} \mathbf{e}_1 + \sup A_{\mathbf{e}_2} \mathbf{e}_2, \\
\inf A &= \inf A_{\mathbf{e}_1} \mathbf{e}_1 + \inf A_{\mathbf{e}_2} \mathbf{e}_2.
\end{aligned}
$$

Moreover, any bicomplex number $w = w_1 + jw_2$ can also be written as

$$w = z_1 \mathbf{e}_1 + z_2 \mathbf{e}_2 \text{ where } z_1 = w_1 - \mathbf{i}w_2, \ z_2 = w_1 + \mathbf{i}w_2 \in \mathbb{C}(\mathbf{i}).$$

The hyperbolic modulus of a bicomplex number is defined by

$$|w|_k = |z_1 \mathbf{e}_1 + z_2 \mathbf{e}_2|_k = |z_1| \, \mathbf{e}_1 + |z_2| \, \mathbf{e}_2 \in \mathbb{D}_0^+$$

where $|.|$ is the real modulus of a complex number.

In this paper, our main aim is to develop some basic properties of hyperbolic valued metric spaces. We also establish the hyperbolic version of the Banach contraction principle. Further, we construct a hyperbolic valued metric on the space of all hyperbolic valued continuous functions and prove some results.

2   Basic definitions and lemmas

In this section, we give some basic definitions and lemmas. We rename some definitions already known.

Definition 2.1 ([5]). Let X be a nonempty set. A function $d_{\mathbb{D}} : X \times X \longrightarrow \mathbb{D}_0^+$ satisfying the following:

(i) $d_{\mathbb{D}}(x, y) \succeq 0$ for all $x, y \in X$ and $d_{\mathbb{D}}(x, y) = 0$ if and only if $x = y$,

(ii) $d_{\mathbb{D}}(x, y) = d_{\mathbb{D}}(y, x)$ for all $x, y \in X$,

(iii) $d_{\mathbb{D}}(x, y) \preceq d_{\mathbb{D}}(x, z) + d_{\mathbb{D}}(z, y)$ for all $x, y, z \in X$,

is defined to be a hyperbolic valued metric or $\mathbb{D}$-metric on X and $(X, d_{\mathbb{D}})$ is called a hyperbolic valued metric space or $\mathbb{D}$-metric space.

Observe that a $\mathbb{D}$-metric $d_{\mathbb{D}}$ on X can be decomposed as $d_{\mathbb{D}}(x,y) = d_1(x,y)\mathbf{e}_1 + d_2(x,y)\mathbf{e}_2$ where $d_1(x,y)$ and $d_2(x,y)$ are two real metrics on X.

Definition 2.2. Let $(X,d_{\mathbb{D}})$ and $(Y,\rho_{\mathbb{D}})$ be two $\mathbb{D}$-metric spaces. A $\mathbb{D}$-isometry between $(X,d_{\mathbb{D}})$ to $(Y,\rho_{\mathbb{D}})$ is a bijection $i_{\mathbb{D}} : X \longrightarrow Y$ such that $d_{\mathbb{D}}(x,y) = \rho_{\mathbb{D}}(i_{\mathbb{D}}(x),i_{\mathbb{D}}(y))$ for all $x,y \in X$. We say that $(X,d_{\mathbb{D}})$ and $(Y,\rho_{\mathbb{D}})$ are $\mathbb{D}$-isometric if there exists an isometry from $(X,d_{\mathbb{D}})$ to $(Y,\rho_{\mathbb{D}})$.

We may think that two hyperbolic valued metric spaces as the same if they are $\mathbb{D}$-isometric. Note that if $i_{\mathbb{D}}$ is a $\mathbb{D}$-isometry from $(X,d_{\mathbb{D}})$ and $(Y,\rho_{\mathbb{D}})$, then the inverse $i_{\mathbb{D}}^{-1}$ is also a $\mathbb{D}$-isometry from $(Y,\rho_{\mathbb{D}})$ to $(X,d_{\mathbb{D}})$, and hence being $\mathbb{D}$-isometric is a symmetric relation.

Definition 2.3. Let $(X,d_{\mathbb{D}})$ and $(Y,\rho_{\mathbb{D}})$ be two $\mathbb{D}$-metric spaces. An embedding of $(X,d_{\mathbb{D}})$ into $(Y,\rho_{\mathbb{D}})$ is an injection $i_{\mathbb{D}} : X \longrightarrow Y$ such that $d_{\mathbb{D}}(x,y) = \rho_{\mathbb{D}}(i_{\mathbb{D}}(x),i_{\mathbb{D}}(y))$ for all $x,y \in X$.

Note that an embedding $i_{\mathbb{D}}$ can be regarded as a $\mathbb{D}$-isometry between X and its image $i_{\mathbb{D}}(X)$.

Definition 2.4 ([7]). Let $(X,d_{\mathbb{D}})$ be a $\mathbb{D}$-metric space. A sequence $\{x_n\}$ in X converges to a point α if for every $\epsilon_{\mathbb{D}} \in \mathbb{D}^+$ there exists an $N \in \mathbb{N}$ such that $d_{\mathbb{D}}(x_n,\alpha) \prec \epsilon_{\mathbb{D}}$ for all $n \geq N$. We write $\lim_{n\to\infty} x_n = \alpha$ or $x_n \to \alpha$ and $\{x_n\}$ is called a $\mathbb{D}$-convergent sequence.

The following lemma can be easily proved and can be considered as an alternative definition of $\mathbb{D}$-convergence:

Lemma 2.5. *A sequence $\{x_n\}$ in a $\mathbb{D}$-metric space $(X,d_{\mathbb{D}})$ converges to α if and only if* $\lim_{n\to\infty} d_{\mathbb{D}}(x_n,\alpha) = 0$.

Definition 2.6. Let $(X,d_{\mathbb{D}})$ and $(Y,\rho_{\mathbb{D}})$ be two $\mathbb{D}$-metric spaces. A function $f : X \longrightarrow Y$ is $\mathbb{D}$-continuous at a point $\alpha \in X$ if for every $\epsilon_{\mathbb{D}} \in \mathbb{D}^+$ there exists a $\delta_{\mathbb{D}} \in \mathbb{D}^+$ such that $\rho_{\mathbb{D}}(f(x),f(y)) \prec \epsilon_{\mathbb{D}}$ whenever $d_{\mathbb{D}}(x,\alpha) \prec \delta_{\mathbb{D}}$. The function f is called $\mathbb{D}$-continuous on X if it is $\mathbb{D}$-continuous at all points in X.

The following lemma is easy to prove:

Lemma 2.7. *Let $(X,d_{\mathbb{D}})$ and $(Y,\rho_{\mathbb{D}})$ be two $\mathbb{D}$-metric spaces and $f : X \longrightarrow Y$. Then the following are equivalent:*
i) f is $\mathbb{D}$-continuous at a point $\alpha \in X$.
ii) For every sequence $\{x_n\}$ in the $\mathbb{D}$-metric space $(X,d_{\mathbb{D}})$ converging to $\alpha \in X$, the sequence $\{f(x_n)\}$ in the $\mathbb{D}$-metric space $(Y,\rho_{\mathbb{D}})$ converges to $f(\alpha)$.

Definition 2.8 ([7]). Let a be a point in a $\mathbb{D}$-metric space $(X,d_{\mathbb{D}})$, and assume that $r \in \mathbb{D}^+$. The open $\mathbb{D}$-ball centered at a with hyperbolic radius r is the set

$$B_{\mathbb{D}}(a;r) = \{x \in X : d_{\mathbb{D}}(x,a) \prec r\}.$$

The closed $\mathbb{D}$-ball centered at a with hyperbolic radius r is the set

$$\overline{B_{\mathbb{D}}(a;r)} = \{x \in X : d_{\mathbb{D}}(x,a) \preceq r\}.$$

And the $\mathbb{D}$-sphere centered at a with hyperbolic radius r is the set

$$S_{\mathbb{D}}(a;r) = \{x \in X : d_{\mathbb{D}}(x,a) = r\}.$$

Definition 2.9 ([7]). Let $(X,d_{\mathbb{D}})$ be a $\mathbb{D}$-metric space and $A \subset X$ $(A \neq \phi)$. Then
i) a point $\alpha \in A$ is said to be a $\mathbb{D}$-interior point of A if there exists a $B_{\mathbb{D}}(\alpha;r)$ for some $r \in \mathbb{D}^+$ such that $B_{\mathbb{D}}(\alpha;r) \subset A$.
ii) a point $\alpha \in X$ is said to be a $\mathbb{D}$-limit point of A if for all $B_{\mathbb{D}}(\alpha;r)$ (i.e. for all $r \in \mathbb{D}^+$), $A \cap (B_{\mathbb{D}}(\alpha;r) - \{\alpha\}) \neq \phi$.

The set of all $\mathbb{D}$-interior points of A is denoted by $A_{\mathbb{D}}^0$ and the set of all $\mathbb{D}$-limit points of A is denoted by $A_{\mathbb{D}}'$. We say the set $\overline{A_{\mathbb{D}}} = A \cup A_{\mathbb{D}}'$, the $\mathbb{D}$-closure of A.

Definition 2.10 ([7]). Let $(X, d_{\mathbb{D}})$ be a $\mathbb{D}$-metric space and $A \subset X$ $(A \neq \phi)$. Then

    i) A is said to be a $\mathbb{D}$-open subset of X if $A = A_{\mathbb{D}}^0$.

    ii) A is said to be a $\mathbb{D}$-closed subset of X if $A = \overline{A_{\mathbb{D}}}$.

Definition 2.11 ([7]). A sequence $\{x_n\}$ in a $\mathbb{D}$-metric space $(X, d_{\mathbb{D}})$ is a $\mathbb{D}$-Cauchy sequence if for each $\epsilon_{\mathbb{D}} \succ 0$ there exists an $N \in \mathbb{N}$ such that $d_{\mathbb{D}}(x_n, x_m) \prec \epsilon_{\mathbb{D}}$ whenever $n, m \geq N$.

Remark 2.12. If we write $d_{\mathbb{D}}(x, y) = d_1(x, y)\mathbf{e}_1 + d_2(x, y)\mathbf{e}_2$, for all $x, y \in X$, then the sequence $\{x_n\}$ in a $\mathbb{D}$-metric space $(X, d_{\mathbb{D}})$ is a $\mathbb{D}$-Cauchy sequence iff $\{x_n\}$ is Cauchy sequence in the real metric spaces (X, d_p) for $p = 1, 2$.

Definition 2.13 ([7]). A $\mathbb{D}$-metric space $(X, d_{\mathbb{D}})$ is called $\mathbb{D}$-complete if every $\mathbb{D}$-Cauchy sequence converges to a point in $(X, d_{\mathbb{D}})$.

Definition 2.14. A subset K of a $\mathbb{D}$-metric space $(X, d_{\mathbb{D}})$ is called a $\mathbb{D}$-compact set if every sequence in K has a subsequence converging to a point in K. The space $(X, d_{\mathbb{D}})$ is $\mathbb{D}$-compact if X is $\mathbb{D}$-compact set, i.e., if each sequence in X has a $\mathbb{D}$-convergent subsequence.

Definition 2.15. A subset A of a $\mathbb{D}$-metric space $(X, d_{\mathbb{D}})$ is $\mathbb{D}$-bounded if there is a number $M \in \mathbb{D}$ such that $d_{\mathbb{D}}(a, b) \preceq M$ for all $a, b \in A$.

Definition 2.16. A subset A of a $\mathbb{D}$-metric space X is called totally $\mathbb{D}$-bounded if for each $\epsilon_{\mathbb{D}} \in \mathbb{D}^+$ there exists a finite number of open $\mathbb{D}$-balls $B_{\mathbb{D}}(a_1, \epsilon_{\mathbb{D}})$, $B_{\mathbb{D}}(a_2, \epsilon_{\mathbb{D}})$, $\ldots$, $B_{\mathbb{D}}(a_n, \epsilon_{\mathbb{D}})$ with centers in A and hyperbolic radius $\epsilon_{\mathbb{D}}$ that cover A i.e.,

$$A \subseteq B_{\mathbb{D}}(a_1, \epsilon_{\mathbb{D}}) \cup B_{\mathbb{D}}(a_2, \epsilon_{\mathbb{D}}) \cup ... \cup B_{\mathbb{D}}(a_n, \epsilon_{\mathbb{D}}).$$

Now we give an alternative description of $\mathbb{D}$-compactness which is sometimes more useful if we want to extend the concept of $\mathbb{D}$-compactness to even more general spaces where sequences are not always an efficient tool, and it is better to have a description of $\mathbb{D}$-compactness in terms of covering by $\mathbb{D}$-open sets which will be called $\mathbb{D}$-open cover.

Definition 2.17. A subset K of a $\mathbb{D}$-metric space $(X, d_{\mathbb{D}})$ is called a $\mathbb{D}$-compact set if for every $\mathbb{D}$-open covering $\mathcal{O}$ of K, i.e.,

$$K \subset \cup\{O : O \in \mathcal{O}\}$$

where $\mathcal{O}$ is (finite or infinite) collection of $\mathbb{D}$-open sets of $(X, d_{\mathbb{D}})$, there exists a finite number of $\mathbb{D}$-open sets $O_1, O_2, \ldots, O_n \in \mathcal{O}$ such that

$$K \subset O_1 \cup O_2 \cup \ldots \cup O_n.$$

Definition 2.18. Let $(X, d_{\mathbb{D}})$ be $\mathbb{D}$-metric space. A mapping $T : X \to X$ is said to be $\mathbb{D}$-Lipschitzian if there exists a constant $k \in \mathbb{D}^+$ such that for all $x, y \in X$

$$d_{\mathbb{D}}(Tx, Ty) \preceq k d_{\mathbb{D}}(x, y).$$

The smallest number k satisfying the above is called the $\mathbb{D}$-Lipschitz constant of T.

Definition 2.19. A $\mathbb{D}$-Lipschitzian mapping $T : X \to X$ with $\mathbb{D}$-Lipschitz constant $0 \prec k \prec 1$ is said to be a $\mathbb{D}$-contraction mapping.

Remark 2.20. Clearly every $\mathbb{D}$-Lipschitzian mapping on $\mathbb{D}$-metric space $(X, d_{\mathbb{D}})$ is $\mathbb{D}$-continuous.

Definition 2.21. A mapping $T : X \to X$ is said to be $\mathbb{D}$-contractive if

$$d_{\mathbb{D}}(T(x), T(y)) \prec d_{\mathbb{D}}(x, y).$$

3 Main results

In this section, we prove our main results. We check the validity of some of the results by examples given in the next section.

Proposition 3.1. *Assume that $(X, d_{\mathbb{D}})$ and $(Y, \rho_{\mathbb{D}})$ be two $\mathbb{D}$-metric spaces and $f : X \longrightarrow Y$. For a point $\alpha \in X$, the following two conditions are equivalent:*

i) f is $\mathbb{D}$-continuous at α.

ii) For all $B_{\mathbb{D}}(f(\alpha); r) \subset Y$ $(r \in \mathbb{D}^+)$, there exists a $B_{\mathbb{D}}(\alpha; r') \subset X$ $(r' \in \mathbb{D}^+)$ such that $f\left(B_{\mathbb{D}}(\alpha; r')\right) \subset B_{\mathbb{D}}(f(\alpha); r)$.

Proof. i)$\Longrightarrow$ii): Assume that f is $\mathbb{D}$-continuous at α. Clearly, $B_{\mathbb{D}}(f(\alpha); r)$ is an open $\mathbb{D}$-ball centered at $f(\alpha)$ in $(Y, \rho_{\mathbb{D}})$. Since f is $\mathbb{D}$-continuous at α, there is an $r' \succ 0$ such that

$$\rho_{\mathbb{D}}(f(x), f(\alpha)) \prec r \text{ whenever } d_{\mathbb{D}}(x, \alpha) \prec r'.$$

This means that $f\left(B_{\mathbb{D}}(\alpha; r')\right) \subseteq B_{\mathbb{D}}(f(\alpha); r)$.

ii)$\Longrightarrow$i): Now $B_{\mathbb{D}}(f(\alpha); \epsilon_{\mathbb{D}})$ $(\epsilon_{\mathbb{D}} \in \mathbb{D}^+)$ is an open $\mathbb{D}$-ball centered at $f(\alpha)$, then by ii), there exists an open $\mathbb{D}$-ball $B_{\mathbb{D}}(\alpha; \delta_{\mathbb{D}}) \subset X$ $(\delta_{\mathbb{D}} \in \mathbb{D}^+)$ such that

$$f\left(B_{\mathbb{D}}(\alpha; \delta_{\mathbb{D}})\right) \subset B_{\mathbb{D}}(f(\alpha); \epsilon_{\mathbb{D}})$$

which implies that $\rho_{\mathbb{D}}(f(x), f(\alpha)) \prec \epsilon_{\mathbb{D}}$ whenever $d_{\mathbb{D}}(x, \alpha) \prec \delta_{\mathbb{D}}$. So f is $\mathbb{D}$-continuous at α. $\square$

Proposition 3.2. *Assume that $(X, d_{\mathbb{D}})$ and $(Y, \rho_{\mathbb{D}})$ be two $\mathbb{D}$-metric spaces and $f : X \longrightarrow Y$. Then the following two conditions are equivalent:*

i) f is $\mathbb{D}$-continuous.

ii) Whenever V is a $\mathbb{D}$-open subset of Y, the inverse image $f^{-1}(V)$ is a $\mathbb{D}$-open set in X.

Proof. i)$\Longrightarrow$ii): Assume that f is $\mathbb{D}$-continuous and $V \subset Y$ is $\mathbb{D}$-open. We prove that $f^{-1}(V)$ is $\mathbb{D}$-open. For any $\alpha \in f^{-1}(V)$, $f(\alpha) \in V$, and we know that there is a $\mathbb{D}$-open ball U centered at α such that $f(U) \subset V$. So $U \subset f^{-1}(V)$, and since α is arbitrary, all the points of $f^{-1}(V)$ are interior points. Hence, $f^{-1}(V)$ is $\mathbb{D}$-open.

ii)$\Longrightarrow$i): Assume that the inverse images of $\mathbb{D}$-open sets are $\mathbb{D}$-open. To prove that f is $\mathbb{D}$-continuous at an arbitrary point α, Proposition 3.1 tells us that it suffices to show that for any $\mathbb{D}$-open ball $B_{\mathbb{D}}(f(\alpha); r) \subset Y$ $(r \in \mathbb{D}^+)$, there is a $\mathbb{D}$-open ball $B_{\mathbb{D}}(\alpha; r') \subset X$ $(r' \in \mathbb{D}^+)$ such that $f\left(B_{\mathbb{D}}(\alpha; r')\right) \subset B_{\mathbb{D}}(f(\alpha); r)$. But this is easy: Since the inverse image of an open set is open, we can simply choose $U = f^{-1}(B_{\mathbb{D}}(f(\alpha); r))$. $\square$

Note that, the above proposition still holds if we replace $\mathbb{D}$-open sets with $\mathbb{D}$-closed subsets since the inverse image commutes with complements and the $\mathbb{D}$-closed sets are the complements of $\mathbb{D}$-open sets.

Proposition 3.3. *Let $\{x_n\}$ be a sequence in a $\mathbb{D}$-metric space $(X, d_{\mathbb{D}})$ for which*

$$\sum_{i=1}^{\infty} d_{\mathbb{D}}(x_i, x_{i+1}) \prec \infty_{\mathbb{D}},$$

where $\infty_{\mathbb{D}} = \infty \mathbf{e}_1 + \infty \mathbf{e}_2$ [3]. Then $\{x_n\}$ is a $\mathbb{D}$-Cauchy sequence in $(X, d_{\mathbb{D}})$.

Proof. We write, $d_{\mathbb{D}}(x, y) = d_1(x, y)\mathbf{e}_1 + d_2(x, y)\mathbf{e}_2$, then $d_1(x, y)$ and $d_2(x, y)$ are real metrices on X..

Now $\sum_{i=1}^{\infty} d_{\mathbb{D}}(x_i, x_{i+1}) \prec \infty_{\mathbb{D}} \Longrightarrow \sum_{i=1}^{\infty} d_1(x_i, x_{i+1}) < \infty$ and $\sum_{i=1}^{\infty} d_2(x_i, x_{i+1}) < \infty$.

Then $\{x_n\}$ is Cauchy sequence in the real metric space (X, d_p) for $p = 1, 2$ [[4], Exercise 2.14] and hence by Remark 2.12, $\{x_n\}$ is $\mathbb{D}$-Cauchy sequence in $(X, d_{\mathbb{D}})$. $\square$

Proposition 3.4. *Assume that A is a subset of a $\mathbb{D}$-complete metric space $(X, d_{\mathbb{D}})$. Then the subspace $(A, d_{\mathbb{D}}|A \times A)$ is $\mathbb{D}$-complete if and only if A is a $\mathbb{D}$-closed subset of X.*

Proposition 3.5. *Let K be a $\mathbb{D}$-compact subset of a $\mathbb{D}$-metric space $(X, d_{\mathbb{D}})$. Then K is totally $\mathbb{D}$-bounded.*

Proof. Assume that K is not totally $\mathbb{D}$-bounded.

There is an $\epsilon_{\mathbb{D}} \in \mathbb{D}^+$ such that no finite collection of $\mathbb{D}$-open balls with hyperbolic radius $\epsilon_{\mathbb{D}}$ covers K.

Now we construct a sequence $\{x_n\}$ in K that does not have a $\mathbb{D}$-convergent subsequence.

Choose an arbitrary element $x_1 \in K$. Since $B_{\mathbb{D}}(x_1, \epsilon_{\mathbb{D}})$ does not cover K, we can choose $x_2 \in K - B_{\mathbb{D}}(x_1, \epsilon_{\mathbb{D}})$.

Again since $B_{\mathbb{D}}(x_1, \epsilon_{\mathbb{D}})$ and $B_{\mathbb{D}}(x_2, \epsilon_{\mathbb{D}})$ do not cover K, we can choose $x_2 \in K - (B_{\mathbb{D}}(x_1, \epsilon_{\mathbb{D}}) \cup B_{\mathbb{D}}(x_2, \epsilon_{\mathbb{D}}))$

Continuing this way, we get a sequence $\{x_n\}$ such that

$$x_n \in K - (B_{\mathbb{D}}(x_1, \epsilon_{\mathbb{D}}) \cup B_{\mathbb{D}}(x_2, \epsilon_{\mathbb{D}}) \cup \cdots \cup B_{\mathbb{D}}(x_{n-1}, \epsilon_{\mathbb{D}})).$$

This means that $d_{\mathbb{D}}(x_n, x_m) \succeq \epsilon_{\mathbb{D}}$ for all $n, m \in \mathbb{N}, n \neq m$, and hence $\{x_n\}$ has no $\mathbb{D}$-convergent subsequence. $\square$

Theorem 3.6 (Extreme Value Theorem). *Let K be a nonempty $\mathbb{D}$-compact subset of $(X, d_{\mathbb{D}})$ and $f : K \to \mathbb{D}$ be a $\mathbb{D}$-continuous function. Let $\sup f(K) = M = M_1 \mathbf{e}_1 + M_2 \mathbf{e}_2$ and $\inf f(K) = m = m_1 \mathbf{e}_1 + m_2 \mathbf{e}_2$, then there exist $a, b, c, d \in K$ such that $p_1(f(a)) = M_1$, $p_2(f(b)) = M_2$, $p_1(f(c)) = m_1$, $p_2(f(d)) = m_2$.*

Proof. Since we have $\sup f(K) = \sup f(K)_{\mathbf{e}_1} \mathbf{e}_1 + \sup f(K)_{\mathbf{e}_2} \mathbf{e}_2 = M_1 \mathbf{e}_1 + M_2 \mathbf{e}_2$, we have $M_1 = \sup f(K)_{\mathbf{e}_1}$, where

$$f(K)_{\mathbf{e}_1} = \{x \in \mathbb{R} : \text{ there exists } y \in \mathbb{R} \text{ such that } x\mathbf{e}_1 + y\mathbf{e}_2 \in f(K)\}.$$

Thus, there exists a sequence $\{x_n\}$ in $f(K)_{\mathbf{e}_1}$ which converges to M_1.

Again for each x_n in $f(K)_{\mathbf{e}_1}$ there exists $y_n \in f(K)_{\mathbf{e}_2}$ such that $x_n \mathbf{e}_1 + y_n \mathbf{e}_2 \in f(K)$, for all $n \in \mathbb{N}$.

Again since $f(K)$ is a $\mathbb{D}$-continuous image of a $\mathbb{D}$-compact set, $f(K)$ is $\mathbb{D}$-compact.

Thus the sequence $\{x_n \mathbf{e}_1 + y_n \mathbf{e}_2\}$ in $f(K)$ has a $\mathbb{D}$-convergent subsequence, say $\{x_{n_k} \mathbf{e}_1 + y_{n_k} \mathbf{e}_2\}$ converging to $M_1 \mathbf{e}_1 + Y \mathbf{e}_2$ (say).

Now, let $\alpha_{n_k} \in K$ be such that $f(\alpha_{n_k}) = x_{n_k} \mathbf{e}_1 + y_{n_k} \mathbf{e}_2$, for all n.

Then also $\{\alpha_{n_k}\}$ has a $\mathbb{D}$-convergent subsquence, say $\{\alpha_{n_j}\}$, since K is $\mathbb{D}$-compact.

Let $\{\alpha_{n_j}\}$ converges to a.

Thus, $f(a) = M_1 \mathbf{e}_1 + Y \mathbf{e}_2$.

Which implies $p_1(f(a)) = M_1$.

Again since, $M_2 = \sup f(K)_{\mathbf{e}_2}$, where

$$f(K)_{\mathbf{e}_2} = \{y' \in \mathbb{R} : \text{there exists } x' \in \mathbb{R} \text{ such that } x'\mathbf{e}_1 + y'\mathbf{e}_2 \in f(K)\}.$$

Thus, there exists a sequence $\{y'_n\}$ in $f(K)_{\mathbf{e}_2}$ which converges to M_2.

Again for each y'_n in $f(K)_{\mathbf{e}_2}$ there exists $x'_n \in f(K)_{\mathbf{e}_1}$ such that $x'_n \mathbf{e}_1 + y'_n \mathbf{e}_2 \in f(K)$, $\forall n$.

Again since $f(K)$ is a $\mathbb{D}$-continuous image of a $\mathbb{D}$-compact set, $f(K)$ is $\mathbb{D}$-compact.

Thus the sequence $\{x'_n \mathbf{e}_1 + y'_n \mathbf{e}_2\}$ in $f(K)$ has a $\mathbb{D}$-convergent subsquence, say $\{x'_{n_k} \mathbf{e}_1 + y'_{n_k} \mathbf{e}_2\}$ converging to $X \mathbf{e}_1 + M_2 \mathbf{e}_2$ (say).

Now, let $b_{n_k} \in K$ be such that $f(b_{n_k}) = x'_{n_k} \mathbf{e}_1 + y'_{n_k} \mathbf{e}_2$, $\forall n$.

Then also $\{b_{n_k}\}$ has a $\mathbb{D}$-convergent subsquence, say $\{b_{n_j}\}$, since K is $\mathbb{D}$-compact.

Let $\{b_{n_j}\}$ converges to b.

Thus, $f(b) = X \mathbf{e}_1 + M_2 \mathbf{e}_2$.

Which implies $p_1(f(b)) = M_2$.

Similarly, we can find $c, d \in K$ such that $p_1(f(c)) = m_1$ and $p_2(f(d)) = m_2$.

This completes the proof. $\square$

Remark 3.7. Note that, in the above theorem, we may not be able to find points $a, b \in K$ such that $f(a) = \sup f(K)$ and $f(b) = \inf f(K)$.

For example, take $K = [0, 1]_{\mathbb{D}} \subset \mathbb{D}$ equipped with $\mathbb{D}$-metric as in Example 4.1. Take $f : [0, 1]_{\mathbb{D}} \to \mathbb{D}$, given by $f(z) = z_1 \mathbf{e}_1 + (1 - z_1) \mathbf{e}_2$, where $z = z_1 \mathbf{e}_1 + z_2 \mathbf{e}_2$.

Then, f is $\mathbb{D}$-continuous and K is a $\mathbb{D}$-compact subset.

Now, $\sup f(K) = \mathbf{e}_1 + \mathbf{e}_2 = 1$ and $\inf f(K) = 0.\mathbf{e}_1 + 0.\mathbf{e}_2 = 0$, but there do not exists $a, b \in [0, 1]_{\mathbb{D}}$ such that $f(a) = 1$ and $f(b) = 0$.

Theorem 3.8 (Banach's Contraction Mapping Principle). *Let* $(X, d_{\mathbb{D}})$ *be* $\mathbb{D}$-*complete metric space and* $T : X \to X$ *be a* $\mathbb{D}$-*contraction mapping having* $\mathbb{D}$-*Lipschitz constant* k. *Then* T *has a unique fixed point* x_0, *i.e.* $T(x_0) = x_0$, *and for each* $x \in X$, $\lim\limits_{n \to \infty} T^n(x) = x_0$. *Moreover*

$$d_{\mathbb{D}}(T^n(x), x_0) \preceq \frac{k^n}{1-k} d_{\mathbb{D}}(x, T(x)).$$

Proof. Since T has $\mathbb{D}$-Lipschitz constant k, for each $x \in X$,

$$d_{\mathbb{D}}(T(x), T^2(x)) \preceq k d_{\mathbb{D}}(x, T(x)).$$

Adding $d_{\mathbb{D}}(x, T(x))$ to both sides of the above gives

$$d_{\mathbb{D}}(x, T(x)) + d_{\mathbb{D}}(T(x), T^2(x)) \preceq d_{\mathbb{D}}(x, T(x)) + k d_{\mathbb{D}}(x, T(x)).$$

Then we have,

$$d_{\mathbb{D}}(x, T(x)) \preceq \frac{1}{1-k} \left(d_{\mathbb{D}}(x, T(x)) - d_{\mathbb{D}}(T(x), T^2(x)) \right).$$

Now define the function $\phi_{\mathbb{D}} : X - \mathbb{D}^+$ by setting $\phi_{\mathbb{D}}(x) = \frac{1}{1-k} d_{\mathbb{D}}(x, T(x))$ for $x \in X$. Thus

$$d_{\mathbb{D}}(x, T(x)) \preceq \phi_{\mathbb{D}}(x) - \phi_{\mathbb{D}}(T(x)), \ x \in X.$$

Therefore if $x \in X$ and $m, n \in \mathbb{N}$ with $n < m$,

$$d_{\mathbb{D}}(T^n(x), T^{m+1}(x)) \preceq \sum_{i=n}^{m} d_{\mathbb{D}}(T^i(x), T^{i+1}(x)) \preceq \phi_{\mathbb{D}}(T^n(x)) - \phi_{\mathbb{D}}(T^{m+1}(x)).$$

In particular, taking $n = 1$ and letting $m \to \infty$ we have

$$\sum_{i=1}^{\infty} d_{\mathbb{D}}(T^i(x), T^{i+1}(x)) \preceq \phi_{\mathbb{D}}(T(x)) \prec \infty_{\mathbb{D}}.$$

By proposition(3.3), $\{T^n(x)\}$ is a $\mathbb{D}$-cauchy sequence.
Since X is $\mathbb{D}$-complete, there exists $x_0 \in X$ such that

$$\lim_{n \to \infty} T^n(x) = x_0.$$

Now using remark(2.20), we have

$$x_0 = \lim_{n \to \infty} T^n(x) = \lim_{n \to \infty} T^{n+1}(x) = T(x_0).$$

Thus x_0 is a fixed point of T. To prove the uniqueness, let y be another fixed point of T. Then from the previous result we have

$$x_0 = \lim_{n \to \infty} T^n(y) = y.$$

Now we have

$$d_{\mathbb{D}}(T^n(x), T^{m+1}(x)) \preceq \phi_{\mathbb{D}}(T^n(x)) - \phi_{\mathbb{D}}(T^{m+1}(x)),$$

letting $m \to \infty$, we get

$$d_{\mathbb{D}}(T^n(x), x_0) \preceq \phi_{\mathbb{D}}(T^n(x)) = \frac{1}{1-k} d_{\mathbb{D}}(T^n(x), T^{n+1}(x)).$$

Since,

$$\frac{1}{1-k} d_{\mathbb{D}}(T^n(x), T^{n+1}(x)) \preceq \frac{k^n}{1-k} d_{\mathbb{D}}(x, T(x)),$$

we have

$$d_{\mathbb{D}}(T^n(x), x_0) \preceq \frac{k^n}{1-k} d_{\mathbb{D}}(x, T(x)).$$

<div style="text-align: right;">□</div>

Now a natural question arises from the previous theorem. If we replace the sequence $\{T^n(x)\}$ with $\{y_n\}$ where $y_0 = x$ and y_{n+1} is approximately $T(y_n)$, then under what condition it still be the case that $\lim_{n \to \infty} y_n = x_0$? The answer to this question is given in the next theorem. We only state the theorem as it can be proved using the idempotent decomposition of hyperbolic number in the same lines as in [[4], Theorem 3.2]

Theorem 3.9. *Let $(X, d_{\mathbb{D}})$ be $\mathbb{D}$-complete metric space, let $T : X \to X$ be a $\mathbb{D}$-contraction mapping with $\mathbb{D}$-Lipschitz constant k and suppose $x_0 \in X$ is the fixed point of T. Let $\{\epsilon_{\mathbb{D}n}\}$ be a sequence in $\mathbb{D}^+$ for which $\lim_{n \to \infty} \epsilon_{\mathbb{D}n} = 0$, let $y_0 \in X$ and suppose $\{y_n\} \subseteq X$ satisfying $d_{\mathbb{D}}(y_{n+1}, T(y_n)) \preceq \epsilon_{\mathbb{D}n}$. Then*

$$\lim_{n \to \infty} y_n = x_0.$$

Theorem 3.10. *Suppose $(X, d_{\mathbb{D}})$ be $\mathbb{D}$-complete metric space and $T : X \to X$ be a mapping for which T^N is a $\mathbb{D}$-contraction mapping for some positive integer N. Then T has a unique fixed point.*

Proof. Proof is along the similar lines as in [[4], Theorem 3.3]   □

The next result shows that a $\mathbb{D}$-contractive mapping can also have a fixed point in a $\mathbb{D}$-compact subspace of $\mathbb{D}$, induced with $\mathbb{D}$-metric $d_{\mathbb{D}}$ as in Example 4.1.

Theorem 3.11. *Let $(X, d_{\mathbb{D}})$ be a $\mathbb{D}$-compact metric space such that $X \subseteq \mathbb{D}$ and $T : X \to X$ be a $\mathbb{D}$-contractive mapping. Then T has a unique fixed point x_0, and for each $x \in X$, $\lim_{n \to \infty} T^n(x) = x_0$.*

Proof. Let us consider the mapping $\psi_{\mathbb{D}} : X \to \mathbb{D}^+$ given by

$$\psi_{\mathbb{D}}(x) = d_{\mathbb{D}}(x, T(x)), \ x \in X.$$

Let $\psi_{\mathbb{D}}(x) = \psi_1(x)\mathbf{e}_1 + \psi_2(x)\mathbf{e}_2$. Then $\psi_i : X \to R^+$ and $\psi_i(x) = d_i(x, T(x))$, $x \in X$, for every $i = 1, 2$.

Since $d_{\mathbb{D}} = d_1\mathbf{e}_1 + d_2\mathbf{e}_2$ is $\mathbb{D}$-contractive, each d_i is contractive mapping on X.

Now for each $i = 1, 2$, ψ_i is continuous and bounded below, so ψ_i assumes its minimum value at some point $y_{i0} \in X$. Since $y_{i0} \neq T(y_{i0})$ implies

$$\psi_i(T(y_{i0})) = d_i(T(y_{i0}), T^2(y_{i0})) < d_i(y_{i0}T(y_{i0})) = \psi_i(y_{i0}),$$

which is a contradiction. Hence $y_{i0} = T(y_{i0})$ and so y_{i0} is a solution of $\psi_i(x) = 0$.

Take

$$x_{i0} = \min\{p_i(x) : \psi_i(x) = 0\}, \ i = 1, 2.$$

Now putting $x_0 = x_{10}\mathbf{e}_1 + x_{20}\mathbf{e}_2$, we get $\psi_{\mathbb{D}}(x_0) = 0$.

Hence $T(x_0) = x_0$.

Let y_0 be another fixed point of T. Then $T(y_0) = y_0$. Hence

$$d_{\mathbb{D}}(x_0, y_0) = d_{\mathbb{D}}(T(x_0), T(y_0)) \prec d_{\mathbb{D}}(x_0, y_0),$$

which is a contradiction.

Hence T has a unique fixed point.

Now let $x \in X$ and consider the sequence $\{d_{\mathbb{D}}(T^n(x), x_0)\}$. If $T^n(x) \neq x_0$,

$$d_{\mathbb{D}}(T^{n+1}(x), x_0) = d_{\mathbb{D}}(T^{n+1}(x), T(x_0)) \prec d_{\mathbb{D}}(T^n(x), x_0),$$

so $\{d_{\mathbb{D}}(T^n(x), x_0)\}$ is strictly decreasing. Consequently the limit

$$r = \lim_{n \to \infty} d_{\mathbb{D}}(T^n(x), x_0)$$

exists and $r \in \mathbb{D}^+$. Also since X is compact, the sequence $\{T^n(x)\}$ has a $\mathbb{D}$-convergent subsequence $\{T^{n_k}(x)\}$, say $\lim_{k \to \infty} T^{n_k}(x) = z \in X$. Since $\{T^n(x)\}$ is decreasing,

$$r = d_{\mathbb{D}}(z, x_0) = \lim_{k \to \infty} d_{\mathbb{D}}(T^{n_k}(x), x_0) = \lim_{k \to \infty} d_{\mathbb{D}}(T^{n_k+1}(x), x_0) = \lim_{k \to \infty} d_{\mathbb{D}}(T(z), x_0).$$

But if $z \neq x_0$, then $d_{\mathbb{D}}(T(z), x_0) = d_{\mathbb{D}}(T(z), T(x_0)) \prec d_{\mathbb{D}}(z, x_0)$. This proves that any $\mathbb{D}$-convergent subsequence of $\{T^n(x)\}$ $\mathbb{D}$-converges to x_0, so it must be the case that $\lim_{n \to \infty} T^n(x) = x_0$.   □

Our next task is to construct a $\mathbb{D}$-metric on the space of $\mathbb{D}$-continuous functions. To proceed further, first, we consider the space of $\mathbb{D}$-bounded functions.

Let $(X, d_\mathbb{D})$ and $(Y, \rho_\mathbb{D})$ be two $\mathbb{D}$-metric spaces. A function $f : X \to Y$ is $\mathbb{D}$-bounded if the range set $f(X) = \{f(x) : x \in X\} \subset Y$ is $\mathbb{D}$-bounded with respect to the $\mathbb{D}$-metric $\rho_\mathbb{D}$, i.e. for all $u, v \in X$, there exists $M \in \mathbb{D}^+$, such that $\rho_\mathbb{D}(f(u), f(v)) \preceq M$.

Let
$$B_\mathbb{D}(X, Y) = \{f : X \to Y \mid f \text{ is } \mathbb{D} - \text{bounded}\}$$

be the collection of all $\mathbb{D}$-bounded functions from X to Y. We shall turn $B_\mathbb{D}(X, Y)$ into a $\mathbb{D}$-metric space by introducing a $\mathbb{D}$-metric $\sigma_\mathbb{D}$.

Proposition 3.12. *Let $(X, d_\mathbb{D})$ and $(Y, \rho_\mathbb{D})$ be two $\mathbb{D}$-metric spaces. Then*

$$\sigma_\mathbb{D}(f, g) = \sup\{\rho_\mathbb{D}(f(x), g(x)) \mid x \in X\}$$

defines a $\mathbb{D}$-metric $\sigma_\mathbb{D}$ on $B_\mathbb{D}(X, Y)$.

Proof. It is clear form the definition $\sigma_\mathbb{D}(f, g) \succeq 0$ and $\sigma_\mathbb{D}(f, g) = 0$ if and only if $f = g$.

Also $\sigma_\mathbb{D}(f, g) = \sigma_\mathbb{D}(g, f)$, for all $f, g \in B_\mathbb{D}(X, Y)$

Again for all $x \in X$,

$$\rho_\mathbb{D}(f(x), g(x)) \preceq \rho_\mathbb{D}(f(x), h(x)) + \rho_\mathbb{D}(h(x), g(x)) \preceq \sigma_\mathbb{D}(f, h) + \sigma_\mathbb{D}(h, g),$$

and taking supremum over all $x \in X$, we get

$$\sigma_\mathbb{D}(f, g) \preceq \sigma_\mathbb{D}(f, h) + \sigma_\mathbb{D}(h, g).$$

Thus, $\sigma_\mathbb{D}$ defines a $\mathbb{D}$-metric on $B_\mathbb{D}(X, Y)$. □

Theorem 3.13. *Let $(X, d_\mathbb{D})$ and $(Y, \rho_\mathbb{D})$ be two $\mathbb{D}$-metric spaces and assume that $(Y, \rho_\mathbb{D})$ is $\mathbb{D}$-complete. Then $(B_\mathbb{D}(X, Y), \sigma_\mathbb{D})$ is also $\mathbb{D}$-complete.*

Now we shall turn to the spaces of $\mathbb{D}$-continuous functions.

As before we assume that $(X, d_\mathbb{D})$ and $(Y, \rho_\mathbb{D})$ be two $\mathbb{D}$-metric spaces. We define

$$Cb_\mathbb{D}(X, Y) = \{f : X \to Y \mid f \text{ is } \mathbb{D}\text{-continuous and } \mathbb{D}\text{-bounded}\}$$

to be the collection of all $\mathbb{D}$-bounded $\mathbb{D}$-continuous functions from X to Y.

Since $Cb_\mathbb{D}(X, Y)$ is a subset of $B_\mathbb{D}(X, Y)$, the metric

$$\sigma_\mathbb{D}(f, g) = \sup\{\rho_\mathbb{D}(f(x), g(x)) \mid x \in X\}$$

that we introduced on $B_\mathbb{D}(X, Y)$ is also a $\mathbb{D}$-metric on $Cb_\mathbb{D}(X, Y)$.

Proposition 3.14. *$Cb_\mathbb{D}(X, Y)$ is a $\mathbb{D}$-closed subset of $B_\mathbb{D}(X, Y)$.*

Proof. It suffices to show that if $\{f_n\}$ is a sequence in $Cb_\mathbb{D}(X, Y)$ that $\mathbb{D}$-converges to an element $f \in B(X, Y)$, then $f \in Cb_\mathbb{D}(X, Y)$.

By the definition of the $\mathbb{D}$-metric $\sigma_\mathbb{D}$, it is clear that $\{f_n\}$ $\mathbb{D}$-converges uniformly to f. Again since f_n is $\mathbb{D}$-bounded $\mathbb{D}$-continuous for all $n \in \mathbb{N}$, it can be shown that f is also $\mathbb{D}$-bounded $\mathbb{D}$-continuous and hence $f \in Cb_\mathbb{D}(X, Y)$. □

Theorem 3.15. *Let $(X, d_\mathbb{D})$ and $(Y, \rho_\mathbb{D})$ be two $\mathbb{D}$-metric spaces and assume that $(Y, \rho_\mathbb{D})$ is $\mathbb{D}$-complete. Then $(Cb_\mathbb{D}(X, Y), \sigma_\mathbb{D})$ is also $\mathbb{D}$-complete.*

Proof. Recall from Proposition 3.4 that a $\mathbb{D}$-closed subspace of a $\mathbb{D}$-complete space is itself $\mathbb{D}$-complete. Since $B_\mathbb{D}(X, Y)$ is $\mathbb{D}$-complete by Theorem 3.13, and $Cb_\mathbb{D}(X, Y)$ is a $\mathbb{D}$-closed subset of $B_\mathbb{D}(X, Y)$ by the above proposition, it follows that $Cb_\mathbb{D}(X, Y)$ is $\mathbb{D}$-complete. □

Proposition 3.16. *Let $(X, d_\mathbb{D})$ and $(Y, \rho_\mathbb{D})$ be two $\mathbb{D}$-metric spaces and assume that X is $\mathbb{D}$-compact. Then all $\mathbb{D}$-continuous functions from X to Y are $\mathbb{D}$-bounded.*

Proof. Assume that $f : X \longrightarrow Y$ is $\mathbb{D}$-continuous, and pick a point $a \in X$. It suffices to prove that the function

$$h(x) = \rho_\mathbb{D}(f(x), f(a))$$

is $\mathbb{D}$-bounded, and this will follow from the Extreme Value Theorem 3.6 if we can show that it is $\mathbb{D}$-continuous.

Since

$$\delta_\mathbb{D}\left(h(x), h(y)\right) = \delta_\mathbb{D}\left(\rho_\mathbb{D}(f(x), f(a)), \rho_\mathbb{D}(f(y), f(a))\right) \preceq \rho_\mathbb{D}(f(x), f(y)),$$

(where $\delta_\mathbb{D}$ is the $\mathbb{D}$-metric on $\mathbb{D}$)

And since f is $\mathbb{D}$-continuous, so is h.   □

Now if we define

$$C_\mathbb{D}(X, Y) = \{f : X \to Y \mid f \text{ is } \mathbb{D} - \text{continuous}\}$$

where X is $\mathbb{D}$-compact, the above proposition tells us that $Cb_\mathbb{D}(X, Y)$ and $C_\mathbb{D}(X, Y)$ coincide.

Theorem 3.17. *Let $(X, d_\mathbb{D})$ and $(Y, \rho_\mathbb{D})$ be two $\mathbb{D}$-metric spaces and assume that X is $\mathbb{D}$-compact, then*

$$\sigma_\mathbb{D}(f, g) = \sup\{\rho_\mathbb{D}(f(x), g(x)) \mid x \in X\}$$

defines a $\mathbb{D}$-metric $\sigma_\mathbb{D}$ on $C_\mathbb{D}(X, Y)$. Also if $(Y, \rho_\mathbb{D})$ is $\mathbb{D}$-complete, so is $(C_\mathbb{D}(X, Y), \sigma_\mathbb{D})$.

4   Examples

In this section, we give some examples supporting our results.

Example 4.1. Let $X = \mathbb{D}$ and $d_\mathbb{D} : \mathbb{D} \times \mathbb{D} \longrightarrow \mathbb{D}_0^+$ be given by

$$d_\mathbb{D}(\alpha, \beta) = |\alpha_1 - \beta_1|\, \mathbf{e}_1 + |\alpha_2 - \beta_2|\, \mathbf{e}_2,$$

where $\alpha = \alpha_1\mathbf{e}_1 + \alpha_2\mathbf{e}_2$, $\beta = \beta_1\mathbf{e}_1 + \beta_2\mathbf{e}_2 \in \mathbb{D}$ and $|.|$ is the real modulus.

Then clearly $|\alpha_1 - \beta_1|, |\alpha_2 - \beta_2| \geq 0$ and thus $d_\mathbb{D}(\alpha, \beta) \succeq 0$, $\forall \alpha, \beta \in \mathbb{D}$.

Also

$$d_\mathbb{D}(\alpha, \beta) = 0 \Longleftrightarrow |\alpha_1 - \beta_1| = 0 = |\alpha_2 - \beta_2| \Longleftrightarrow \alpha_1 = \beta_1 \text{ and } \alpha_2 = \beta_2 \Longleftrightarrow \alpha = \beta.$$

Again, it is clear from the definition

$$d_\mathbb{D}(\alpha, \beta) = d_\mathbb{D}(\beta, \alpha), \ \forall \alpha, \beta \in \mathbb{D}.$$

Next let $\alpha = \alpha_1\mathbf{e}_1 + \alpha_2\mathbf{e}_2$, $\beta = \beta_1\mathbf{e}_1 + \beta_2\mathbf{e}_2, \gamma = \gamma_1\mathbf{e}_1 + \gamma_2\mathbf{e}_2 \in \mathbb{D}$.

Then,

$$
\begin{aligned}
d_\mathbb{D}(\alpha, \beta) &= |\alpha_1 - \beta_1|\, \mathbf{e}_1 + |\alpha_2 - \beta_2|\, \mathbf{e}_2 \\
&= |(\alpha_1 - \gamma_1) + (\gamma_1 - \beta_1)|\, \mathbf{e}_1 + |(\alpha_2 - \gamma_2) + (\gamma_2 - \beta_2)|\, \mathbf{e}_2 \\
&\preceq |\alpha_1 - \gamma_1|\, \mathbf{e}_1 + |\gamma_1 - \beta_1|\, \mathbf{e}_1 + |\alpha_2 - \gamma_2|\, \mathbf{e}_2 + |\gamma_2 - \beta_2|\, \mathbf{e}_2 \\
&= |\alpha_1 - \gamma_1|\, \mathbf{e}_1 + |\alpha_2 - \gamma_2|\, \mathbf{e}_2 + |\gamma_1 - \beta_1|\, \mathbf{e}_1 + |\alpha_2 - \gamma_2|\, \mathbf{e}_2 + |\gamma_2 - \beta_2|\, \mathbf{e}_2 \\
&= d_\mathbb{D}(\alpha, \gamma) + d_\mathbb{D}(\gamma, \alpha).
\end{aligned}
$$

Thus $d_\mathbb{D}(\alpha, \beta) = |\alpha_1 - \beta_1|\, \mathbf{e}_1 + |\alpha_2 - \beta_2|\, \mathbf{e}_2$ is a hyperbolic metric on $\mathbb{D}$.

Example 4.2. Let $X = \mathbb{BC}$, and $d_\mathbb{D} : \mathbb{BC} \times \mathbb{BC} \longrightarrow \mathbb{D}_0^+$ be given by $d_\mathbb{D}(\alpha, \beta) = |\alpha - \beta|_k$, where $\alpha, \beta \in \mathbb{BC}$ and $|.|_k$ is the hyperbolic modulus on bicomplex number.

Here we shall show the triangle Inequality for $d_\mathbb{D}$.

Let $\alpha, \beta, \gamma \in \mathbb{BC}$ with

$$
\begin{aligned}
\alpha &= \alpha_1\mathbf{e}_1 + \alpha_2\mathbf{e}_2 = (\alpha_{11} + i.\alpha_{12})\mathbf{e}_1 + (\alpha_{21} + i.\alpha_{22})\mathbf{e}_2, \\
\beta &= \beta_1\mathbf{e}_1 + \beta_2\mathbf{e}_2 = (\beta_{11} + i.\beta_{12})\mathbf{e}_1 + (\beta_{21} + i.\beta_{22})\mathbf{e}_2, \\
\gamma &= \gamma_1\mathbf{e}_1 + \gamma_2\mathbf{e}_2 = (\gamma_{11} + i.\gamma_{12})\mathbf{e}_1 + (\gamma_{21} + i.\gamma_{22})\mathbf{e}_2.
\end{aligned}
$$

be their idempotent representation. Then

$$d_{\mathbb{D}}(\alpha,\gamma) + d_{\mathbb{D}}(\gamma,\beta) = |\alpha - \gamma|_k + |\gamma - \beta|_k$$

$$= |\alpha_1\mathbf{e}_1 + \alpha_2\mathbf{e}_2 - \gamma_1\mathbf{e}_1 - \gamma_2\mathbf{e}_2|_k + |\gamma_1\mathbf{e}_1 + \gamma_2\mathbf{e}_2 - \beta_1 e_1 - \beta_2\mathbf{e}_2|_k$$

$$= |(\alpha_1 - \gamma_1)\mathbf{e}_1 + (\alpha_2 - \gamma_2)\mathbf{e}_2|_k + |(\gamma_1 - \beta_1)\mathbf{e}_1 + (\gamma_2 - \beta_2)\mathbf{e}_2|_k$$

$$= |\alpha_1 - \gamma_1|\,\mathbf{e}_1 + |\alpha_2 - \gamma_2|\,\mathbf{e}_2 + |\gamma_1 - \beta_1|\,\mathbf{e}_1 + |\gamma_2 - \beta_2|\,\mathbf{e}_2,$$

where $|.|$ is modulus over $\mathbb{C}(\mathbf{i})$

$$= (|\alpha_1 - \gamma_1| + |\gamma_1 - \beta_1|)\,\mathbf{e}_1 + (|\alpha_2 - \gamma_2| + |\gamma_2 - \beta_2|)\,\mathbf{e}_2$$

$$= \left(\sqrt{(\alpha_{11} - \gamma_{11})^2 + (\alpha_{12} - \gamma_{12})^2} + \sqrt{(\gamma_{11} - \beta_{11})^2 + (\gamma_{12} - \beta_{12})^2} \right)\mathbf{e}_1$$

$$+ \left(\sqrt{(\alpha_{21} - \gamma_{21})^2 + (\alpha_{22} - \gamma_{22})^2} + \sqrt{(\gamma_{21} - \beta_{21})^2 + (\gamma_{22} - \beta_{22})^2} \right)\mathbf{e}_2$$

$$\preceq \left(\sqrt{(\alpha_{11} - \beta_{11})^2 + (\alpha_{12} - \beta_{12})^2} \right)\mathbf{e}_1 + \left(\sqrt{(\alpha_{21} - \beta_{21})^2 + (\alpha_{22} - \beta_{22})^2} \right)\mathbf{e}_2$$

$$= |\alpha_1 - \beta_1|\,\mathbf{e}_1 + |\alpha_2 - \beta_2|\,\mathbf{e}_2$$

$$= |\alpha - \beta|_k$$

$$= d_{\mathbb{D}}(\alpha,\beta)$$

which shows the triangle inequality.

More generally we have the following example:

Example 4.3. Let $d_1,\ d_2 : \mathbb{C} \times \mathbb{C} \to [0,\infty)$ be two (real valued) metrics on $\mathbb{C}$, Take, $d_{\mathbb{D}} : \mathbb{BC} \times \mathbb{BC} \longrightarrow \mathbb{D}_0^+$ given by

$$d_{\mathbb{D}}(x,y) = d_1(x_1,y_1)\mathbf{e}_1 + d_2(x_2,y_2)\mathbf{e}_2,$$

where $x = x_1\mathbf{e}_1 + x_2\mathbf{e}_2,\ y = y_1\mathbf{e}_1 + y_2\mathbf{e}_2 \in \mathbb{D}$, then $d_{\mathbb{D}}$ is a hyperbolic metric on $\mathbb{BC}$.

Example 4.4. The space in Example 4.1 is $\mathbb{D}$-complete.

Example 4.5. Every $\mathbb{D}$-closed and $\mathbb{D}$-bounded subset of $\mathbb{D}$ is $\mathbb{D}$-compact with respect to the $\mathbb{D}$-metric $d_{\mathbb{D}}$ as in Example 4.1.

Remark 4.6. The situation is more interesting when we take $X = \mathbb{D}$ and choose $d_{\mathbb{D}}$ as in Example 4.1

Let $\alpha = \alpha_1\mathbf{e}_1 + \alpha_2\mathbf{e}_2 \in \mathbb{D}$ and $r = r_1\mathbf{e}_1 + r_2\mathbf{e}_2 \succ 0$ (i.e. $r \in \mathbb{D}^+$),
Then

$$B_{\mathbb{D}}(\alpha; r) = \{x = x_1\mathbf{e}_1 + x_2\mathbf{e}_2 \in \mathbb{D} : |\alpha_1 - x_1|\,\mathbf{e}_1 + |\alpha_2 - x_2|\,\mathbf{e}_2 \prec r_1\mathbf{e}_1 + r_2\mathbf{e}_2\}$$

$$= \{x = x_1\mathbf{e}_1 + x_2\mathbf{e}_2 \in \mathbb{D} : |\alpha_1 - x_1| < r_1, |\alpha_2 - x_2| < r_2\},$$

which is a open square with vertices at $(\alpha_1 + r_1)\,\mathbf{e}_1 + (\alpha_2 + r_2)\,\mathbf{e}_2,\ (\alpha_1 + r_1)\,\mathbf{e}_1 + (\alpha_2 - r_2)\,\mathbf{e}_2,\ (\alpha_1 - r_1)\,\mathbf{e}_1 + (\alpha_2 + r_2)\,\mathbf{e}_2$ and $(\alpha_1 - r_1)\,\mathbf{e}_1 + (\alpha_2 - r_2)\,\mathbf{e}_2$;

$$\overline{B_{\mathbb{D}}}(\alpha; r) = \{x = x_1\mathbf{e}_1 + x_2\mathbf{e}_2 \in \mathbb{D} : |\alpha_1 - x_1|\,\mathbf{e}_1 + |\alpha_2 - x_2|\,\mathbf{e}_2 \preceq r_1\mathbf{e}_1 + r_2\mathbf{e}_2\}$$

$$= \{x = x_1\mathbf{e}_1 + x_2\mathbf{e}_2 \in \mathbb{D} : |\alpha_1 - x_1| \le r_1, |\alpha_2 - x_2| \le r_2\},$$

which is a closed square with vertices at $(\alpha_1 + r_1)\,\mathbf{e}_1 + (\alpha_2 + r_2)\,\mathbf{e}_2,\ (\alpha_1 + r_1)\,\mathbf{e}_1 + (\alpha_2 - r_2)\,\mathbf{e}_2,\ (\alpha_1 - r_1)\,\mathbf{e}_1 + (\alpha_2 + r_2)\,\mathbf{e}_2$ and $(\alpha_1 - r_1)\,\mathbf{e}_1 + (\alpha_2 - r_2)\,\mathbf{e}_2$; and

$$S_{\mathbb{D}}(\alpha; r) = \{x = x_1\mathbf{e}_1 + x_2\mathbf{e}_2 \in \mathbb{D} : |\alpha_1 - x_1|\,\mathbf{e}_1 + |\alpha_2 - x_2|\,\mathbf{e}_2 = r_1\mathbf{e}_1 + r_2\mathbf{e}_2\}$$

$$= \{x = x_1\mathbf{e}_1 + x_2\mathbf{e}_2 \in \mathbb{D} : |\alpha_1 - x_1| = r_1, |\alpha_2 - x_2| = r_2\}.$$

$$= \{(\alpha_1 + r_1)\,\mathbf{e}_1 + (\alpha_2 + r_2)\,\mathbf{e}_2, (\alpha_1 + r_1)\,\mathbf{e}_1 + (\alpha_2 - r_2)\,\mathbf{e}_2, (\alpha_1 - r_1)\,\mathbf{e}_1$$

$$+ (\alpha_2 + r_2)\,\mathbf{e}_2, (\alpha_1 - r_1)\,\mathbf{e}_1 + (\alpha_2 - r_2)\,\mathbf{e}_2\}$$

which consists of only four points.

So, in this situation, unlike $\mathbb{R}^2$ equipped with the usual metric,

$$\overline{B_{\mathbb{D}}(\alpha; r)} - B_{\mathbb{D}}(\alpha; r) \neq S_{\mathbb{D}}(\alpha; r).$$

Remark 4.7. Let $\alpha, \beta \in \mathbb{D}$, with $\alpha = \alpha_1 \mathbf{e}_1 + \alpha_2 \mathbf{e}_2$ and $\beta = \alpha_1 \mathbf{e}_1 + \alpha_2 \mathbf{e}_2$ be their idempotent representation. Also let $\alpha \prec \beta$.

Then the open $\mathbb{D}$-interval $(\alpha, \beta)_{\mathbb{D}}$ is defined by

$$(\alpha, \beta)_{\mathbb{D}} = \{x : \alpha \prec x \prec \beta\}$$

and the closed $\mathbb{D}$-interval $[\alpha, \beta]_{\mathbb{D}}$ is defined by

$$[\alpha, \beta]_{\mathbb{D}} = \{x : \alpha \preceq x \preceq \beta\}.$$

Note that, unlike the real number system, if $\gamma, \delta \in (\alpha, \beta)_{\mathbb{D}}$ or $\gamma, \delta \in [\alpha, \beta]_{\mathbb{D}}$ that does not necessarily mean $\gamma \preceq \delta$ or $\gamma \succeq \delta$.

References

[1] D. Alpay, M. E. Luna-Elizarrarás, M. Shapiro and D. C. Struppa, *Basics of Functional Analysis with Bicomplex Scalars and Bicomplex Schur Analysis*, Springer International Publishing, Cham, Switzerland (2014).

[2] M. Bilgin and S. Ersoy, *Algebraic Properties of Bihyperbolic Numbers*, Adv. Appl. Clifford Algebras, **30**(13), 01-17 (2020).

[3] C.Ghosh, S. Biswas and T.Yasin, *Hyperbolic valued signed measure*, Int. J. Math. Trends Technol. **55**(7), 515–522 (2018).

[4] M. A. Khamsi and W. A. Kirk, *An Introduction to Metric Spaces and Fixed Point Theory*, Pure and Applied Mathematics, Wiley-Interscience, New York, NY, USA, 2001.

[5] R. Kumar and H. Saini, *Topological bicomplex modules*, Adv. Appl. Clifford Algebras, **26**(4), 1249–1270 (2016).

[6] T. L. Lindstrøm, *Spaces An Introduction to Real Analysis*, American Mathematical Society (2017).

[7] H. Saini, A. Sharma and R. Kumar, *Some Fundamental Theorems of Functional Analysis with Bicomplex and Hyperbolic Scalars*, Adv. Appl. Clifford Algebras, **30**(66), 01-23 (2020).

[8] C. Segre, *Le rappresentazioni reali delle forme complesse e gli enti iperalgebrici (The real representation of complex elements and hyperalgebraic entities)*, Math. Ann., **40**, 413-467 (1892).

[9] G.Y. Tellez-Sanchez and J. Bory-Reyes, *Generalized Iterated Function Systems on Hyperbolic Numbers Plane*, Fractals, **27**(04), doi:10.1142/S0218348X19500452 (2019).

Author information

Chinmay Ghosh, Department of Mathematics, Kazi Nazrul University, Nazrul Road, P.O.- Kalla C.H., Asansol-713340, West Bengal, India.
E-mail: `chinmayarp@gmail.com`

Anirban Bandyopadhyay, 53, Gopalpur Primary School, Raninagar-I, Murshidabad, Pin-742304, West Bengal, India.
E-mail: `anirbanbanerjee010@gmail.com`

Soumen Mondal, 28, Dolua Dakshinpara Haridas Primary School, Beldanga, Murshidabad, Pin-742133, West Bengal, India.
E-mail: `mondalsoumen79@gmail.com`