

Schur convexity of difference of means

Sandeep Kumar and Sangeetha B. K.

Communicated by V. Lokesha

MSC 2020 Classifications: 26D15

Keywords and phrases: Schur convexity, Special means, harmonic convexity.

The authors would like to thank the Editor and the anonymous referees for their valuable comments and suggestions, which helped us to improve the quality of this manuscript.

Abstract. In this paper, we study Schur convexity and concavity, its properties like Schur, Schur harmonic convexity(concave) on the ratio of difference of means obtained by arithmetic mean, geometric mean, harmonic mean, contraharmonic mean, heron mean, root square mean, subsidiary means.

1 Introduction

In [15] I. J. Taneja refinement of inequalities among means we have the following inequality chain.

$$H(\Phi, \Psi) \leq G(\Phi, \Psi) \leq N_1(\Phi, \Psi) \leq H_e(\Phi, \Psi) \leq N_2(\Phi, \Psi) \leq A(\Phi, \Psi) \leq S(\Phi, \Psi)$$

For the real numbers $\Phi, \Psi > 0$,

$$A = A(\Phi, \Psi) = \frac{\Phi + \Psi}{2}$$

$$G = G(\Phi, \Psi) = \sqrt{\Phi\Psi}$$

$$H = H(\Phi, \Psi) = \frac{2}{\frac{1}{\Phi} + \frac{1}{\Psi}}$$

$$H_e = H_e(\Phi, \Psi) = \frac{\Phi + \sqrt{\Phi * \Psi} + \Psi}{3}$$

$$S = S(\Phi, \Psi) = \sqrt{\frac{\Phi^2 + \Psi^2}{2}}$$

$$C = C(\Phi, \Psi) = \frac{\Phi^2 + \Psi^2}{\Phi + \Psi}$$

$$N_1 = N_1(\Phi, \Psi) = \left(\frac{\sqrt{\Phi} + \sqrt{\Psi}}{2} \right)^2$$

$$N_2 = N_2(\Phi, \Psi) = \left(\frac{\sqrt{\Phi} + \sqrt{\Psi}}{2} \right) \left(\sqrt{\frac{\Phi + \Psi}{2}} \right)$$

are respectively called arithmetic mean, geometric mean, harmonic mean, root-square mean, contra-harmonic mean. N_1, N_2 are called subsidiary means which are discussed in Enormously. With the reference of these well known means, few difference of means are given as below.

$$\begin{aligned}
M_{SA}(\Phi, \Psi) &= S(\Phi, \Psi) - A(\Phi, \Psi) \\
M_{SN_2}(\Phi, \Psi) &= S(\Phi, \Psi) - N_2(\Phi, \Psi) \\
M_{SH_e}(\Phi, \Psi) &= S(\Phi, \Psi) - H_e(\Phi, \Psi) \\
M_{SN_1}(\Phi, \Psi) &= S(\Phi, \Psi) - N_1(\Phi, \Psi) \\
M_{SG}(\Phi, \Psi) &= S(\Phi, \Psi) - G(\Phi, \Psi) \\
M_{HN_1}(\Phi, \Psi) &= H(\Phi, \Psi) - N_1(\Phi, \Psi) \\
\\
M_{GA}(\Phi, \Psi) &= G(\Phi, \Psi) - A(\Phi, \Psi) \\
M_{AS}(\Phi, \Psi) &= A(\Phi, \Psi) - S(\Phi, \Psi) \\
M_{N_1N_2}(\Phi, \Psi) &= N_1(\Phi, \Psi) - N_2(\Phi, \Psi) \\
M_{HN_1}(\Phi, \Psi) &= H(\Phi, \Psi) - N_1(\Phi, \Psi) \\
M_{N_2N_1}(\Phi, \Psi) &= N_2(\Phi, \Psi) - N_1(\Phi, \Psi) \\
M_{N_1G}(\Phi, \Psi) &= N_1(\Phi, \Psi) - G(\Phi, \Psi)
\end{aligned}$$

In this paper, we discuss convexity of difference of means list above.

2 Preliminaries

This section provides the required definitions and lemmas.

Lemma 2.1. In [21] Jamal Roojin and Mehdi Hassni introduced the homogeneous functions $f(x)$ and $g(x)$, where

$$\begin{aligned}
f(x) &= \frac{\Phi^x - \Psi^x}{\zeta^x - \eta^x}, \\
g(x) &= \ln \frac{\Phi^x - \Psi^x}{\zeta^x - \eta^x}
\end{aligned}$$

for $\Phi \in (-\infty, +\infty)$

- convexity, if $\Phi\Psi - \zeta\eta \geq 0$
- concavity, if $\Phi\eta - \Psi\zeta \leq 0$ and
- equalities holds, if $\Phi\eta - \Psi\zeta = 0$ for $\Phi > \Psi \geq \zeta > \eta > 0$

The notion of Schur-convex function was introduced by I. Schur in 1923 and has interesting applications in quantum theory, analytic inequalities and quantum mechanics. We now present the basic lemmas required to prove the main theorems.

Lemma 2.2. Let $\Omega \subseteq R^n$ be symmetric with non empty interior geometrically convex set and let $\phi : \Omega \rightarrow R_+$ be continuous on Ω and differentiable in Ω^0 . If ϕ is symmetric on Ω and

$$(\ln\Phi_1 - \ln\Phi_2) \left(\Phi_1 \frac{\partial\phi}{\partial\Phi_1} - \Phi_2 \frac{\partial\phi}{\partial\Phi_2} \right) \geq 0. \quad (2.1)$$

where $\Phi = (\Phi_1, \Phi_2, \dots, \Phi_n) \in \Omega^0$, then ϕ is Schur-geometrically convex(concave)function.

Lemma 2.3. Let $\Omega \subseteq R^n$ be symmetric with non empty interior harmonically convex set and let $\phi : \Omega \rightarrow R_+$ be continuous on Ω and differentiable in Ω^0 . If ϕ is symmetric on Ω and

$$(\Phi_1 - \Phi_2) \left(\Phi_1^2 \frac{\partial \phi}{\partial \Phi_1} - \Phi_2^2 \frac{\partial \phi}{\partial \Phi_2} \right) \geq 0. \tag{2.2}$$

where $\Phi = (\Phi_1, \Phi_2, \dots, \Phi_n) \in \Omega^0$, then ϕ is Schur-harmonically convex(concave)function.

An important source of Schur-convex functions can be found in Merkle [16]. Other families of Schur-convex functions are studied in [[11]-[25]].

3 Main Results

In this section we set the difference of means which are defined in the equations in the above section.

$$\begin{aligned} M_{SN_1}(\Phi, \Psi) - M_{SG}(\Phi, \Psi) &= G - N_1 \\ M_{SH_e}(\Phi, \Psi) - M_{SN_1}(\Phi, \Psi) &= N_1 - H_e \\ M_{SG}(\Phi, \Psi) - M_{SS}(\Phi, \Psi) &= S - G \\ M_{SA}(\Phi, \Psi) - M_{SG}(\Phi, \Psi) &= G - A \\ M_{SS}(\Phi, \Psi) - M_{SN_1}(\Phi, \Psi) &= N_1 - S \end{aligned}$$

The above difference of the means are convex for all positive values. Now, we establish the ratio of difference of above means as follows:

$$\frac{M_{SN_2} - M_{SA}}{M_{SH} - M_{SN_2}} = \frac{A - N_2}{N_2 - H}$$

$$\frac{M_{SG} - M_{SA}}{M_{SG} - M_{SN_1}} = \frac{A - G}{G - N_1}$$

$$\frac{M_{SG} - M_{SN_1}}{M_{SG} - M_{SH_e}} = \frac{N_1 - G}{G - H_e}$$

$$\frac{M_{SS} - M_{SN_1}}{M_{SG} - M_{SS}} = \frac{N_1 - S}{S - G}$$

$$\frac{M_{SS} - M_{SA}}{M_{SG} - M_{SS}} = \frac{A - S}{S - G}$$

$$\frac{M_{SS} - M_{SA}}{M_{SH} - M_{SS}} = \frac{A - S}{S - H}$$

Theorem 3.1. For $\Phi > \Psi > 0$, the ratio of difference of mean

$$\frac{M_{SN_2} - M_{SA}}{M_{SH} - M_{SN_2}} = \frac{A - N_2}{N_2 - H}$$

is convex for all positive real values of 'r'

Proof.

$$\frac{M_{SN_2}(p, 1) - M_{SA}(p, 1)}{M_{SH}(p, 1) - M_{SN_2}(p, 1)} = \frac{A(p, 1) - N_2(p, 1)}{N_2(p, 1) - H(p, 1)}$$

from 2.1)

$$\begin{aligned}
 f(p, 1) &= AH - N_2^2 \\
 &= \left(\frac{p+1}{2}\right)\left(\frac{2p}{p+1}\right) - \left(\frac{\sqrt{p}+1}{2}\right)^2 \left(\sqrt{\frac{p+1}{2}}\right)^2 \\
 &= \frac{1}{8} [8p - (p(p+1+2\sqrt{p}) + p+1+2\sqrt{p})] \\
 &= \frac{1}{8} [8p - ((p+1)^2 + 2\sqrt{p}(p+1))] \\
 &= \frac{8p - (p+1)^2 - 2\sqrt{p}(p+1)}{8} \\
 f(p, 1) &= \frac{6p - p^2 + 1 - 2\sqrt{p}(p+1)}{8} \geq 0
 \end{aligned}$$

Therefore, it satisfies convexity property \forall non negative values of ' p '. □

Theorem 3.2. For $\Phi > \Psi > 0$, the ratio of difference of mean

$$\frac{M_{SG} - M_{SA}}{M_{SN_1} - M_{SG}} = \frac{A - G}{G - N_1}$$

is convex for all positive real values of ' t '

Proof. Let

$$f(p, 1) = \frac{M_{SG}(p, 1) - M_{SA}(p, 1)}{M_{SN_1}(p, 1) - M_{SG}(p, 1)} = \frac{A(p, 1) - G(p, 1)}{G(p, 1) - N_1(p, 1)}$$

$$\begin{aligned}
 f(p, 1) &= AN_1 - G^2 \\
 &= \left(\frac{p+1}{2}\right)\left(\frac{\sqrt{p}+1}{2}\right)^2 - p \\
 &= \frac{1}{8} [p(p+1+2\sqrt{p}) + (p+1+2\sqrt{p})] - p \\
 &= \frac{1}{8} [p^2 + p + 2p\sqrt{p} + p+1+2\sqrt{p}] - p \\
 &= \frac{1}{8} [(p+1)^2 + 2\sqrt{p}(p+1)] - p \\
 &= \frac{1}{8} [(p+1)(p+1+2\sqrt{p}) - 8p] \\
 &= \frac{1}{8} [p^2 + p + 2\sqrt{p}(p+1) + 2p - 8p] \\
 f(p, 1) &= \frac{p^2 + 1 - 6p + 2\sqrt{p}(p+1)}{8} \geq 0
 \end{aligned}$$

Therefore, it satisfies convexity property for all non negative values of ' p '. □

Theorem 3.3. For $\Phi > \Psi > 0$, the ratio of difference of mean

$$\frac{M_{SG} - M_{SN_1}}{M_{SH_e} - M_{SG}} = \frac{N_1 - G}{G - H_e}$$

is convex for all positive real values of ' t '

Proof.

$$\begin{aligned} \frac{M_{SG}(p, 1) - M_{SN_1}(p, 1)}{M_{SH_e}(p, 1) - M_{SG}(p, 1)} &= \frac{N_1(p, 1) - G(p, 1)}{G(p, 1) - H_e(p, 1)} \\ f(p, 1) &= N_1 H_e - G^2 \\ &= \frac{1}{12} \left(\frac{\sqrt{p} + 1}{2} \right)^2 \left(\frac{p + 1 + \sqrt{p}}{3} \right)^{-p} \\ &= \frac{1}{12} \left(\frac{p + 1 + 2\sqrt{p}}{4} \right) \left(\frac{p + 1 + \sqrt{p}}{3} \right)^{-p} \\ &= \frac{1}{12} \left[p(p + 1 + 2\sqrt{p}) + (p + 1 + 2\sqrt{p}) + 2\sqrt{p}(p + 1 + 2\sqrt{p}) - 12p \right] \\ &= \frac{1}{12} [p^2 + 1 + 2\sqrt{p}(p + 1) + \sqrt{p}(p + 1) + 4p - 12p] \\ f(p, 1) &= \frac{1}{12} [p^2 + 1 - 8p + 3\sqrt{p}(p + 1)] \geq 0 \end{aligned}$$

Therefore, it satisfies convexity property \forall non negative values of 'p'. □

Theorem 3.4. For $\Phi > \Psi > 0$, the ratio of difference of mean

$$\frac{M_{SS} - M_{SN_1}}{M_{SG} - M_{SS}} = \frac{N_1 - S}{S - G}$$

is convex for all positive real values of 't'

Proof.

$$\begin{aligned} \frac{M_{SS}(p, 1) - M_{SN_1}(p, 1)}{M_{SG}(p, 1) - M_{SS}(p, 1)} &= \frac{N_1(p, 1) - S(p, 1)}{S(p, 1) - G(p, 1)} \\ f &= N_1 G - S^2 \\ &= \left(\frac{\sqrt{p} + 1}{2} \right)^2 \sqrt{p} - \left(\sqrt{\frac{p^2 + 1}{2}} \right)^2 \\ &= \left(\frac{(p + 1 + 2\sqrt{p})\sqrt{p} - 2(p + 1)^2 - 2p}{4} \right) \\ f(p, 1) &= \left(\frac{(p + 1)\sqrt{p} + 6p - 2(p + 1)^2}{4} \right) \geq 0 \end{aligned}$$

Therefore, it satisfies convexity property \forall non negative values of 'p'. □

4 Schur convex and combined type properties of difference of means

In this component, we get the Schur convexity(concavity), Schur geometrically convexity(concavity), Schur harmonically convexity(concavity) for the difference of ratio of means in 2 variables.

Theorem 4.1. The ratio of difference of mean

$$\frac{M_{SN_2} - M_{SA}}{M_{SH} - M_{SN_2}} = \frac{A - N_2}{N_2 - H}$$

- Schur convexity
- Schur geometrical convexity
- Schur harmonical convexity, $\forall \Phi \geq \Psi$.

Proof.

$$\begin{aligned} \text{Let } f(\Phi, \Psi) &= \frac{M_{SN_2}(\Phi, \Psi) - M_{SA}(\Phi, \Psi)}{M_{SH} - M_{SN_2}} \\ &= \frac{A(\Phi, \Psi) - N_2(\Phi, \Psi)}{N_2(\Phi, \Psi) - H(\Phi, \Psi)} \\ f(\Phi, \Psi) &= AH - N_2^2 \\ f(\Phi, \Psi) &= \frac{6\Phi\Psi - \Phi^2 - \Psi^2 - 2\sqrt{\Phi\Psi}(\Phi + \Psi)}{8} \end{aligned}$$

The partial derivatives of $f(\Phi, \Psi)$ after simple calculations gives.

$$\begin{aligned} \frac{\partial f}{\partial \Phi} &= \frac{1}{8} [6\Psi - 2\Phi - 2\sqrt{\Phi\Psi} - \sqrt{\frac{\Psi}{\Phi}}(\Phi + \Psi)] \\ \frac{\partial f}{\partial \Psi} &= \frac{1}{8} [6\Phi - 2\Psi - 2\sqrt{\Phi\Psi} - \sqrt{\frac{\Phi}{\Psi}}(\Phi + \Psi)] \end{aligned}$$

□

Proof. Case-(1) Schur convexity(concavity) Then

$$\begin{aligned} s &= \frac{\partial f}{\partial \Phi} - \frac{\partial f}{\partial \Psi} \\ \frac{\partial f}{\partial \Phi} - \frac{\partial f}{\partial \Psi} &= \frac{1}{8} [6\Psi - 2\Phi - 2\sqrt{\Phi\Psi} - \sqrt{\frac{\Psi}{\Phi}}(\Phi + \Psi)] - \frac{1}{8} [6\Phi - 2\Psi - 2\sqrt{\Phi\Psi} - \sqrt{\frac{\Phi}{\Psi}}(\Phi + \Psi)] \\ &= \frac{(\Phi - \Psi)}{8} \left[\frac{(\Phi + \Psi) - 8\sqrt{\Phi\Psi}}{\sqrt{\Phi\Psi}} \right] \\ \frac{\partial f}{\partial \Phi} - \frac{\partial f}{\partial \Psi} &= \frac{(\Phi - \Psi)}{8} \left[\frac{2A - 8G}{G} \right] \geq 0 \end{aligned}$$

for $\Phi \geq \Psi$. This proves the condition for Schur convexity. □

Proof. Case-(2) Schur geometrically convexity(concavity)

Then

$$\begin{aligned} s &= \Phi \frac{\partial f}{\partial \Phi} - \Psi \frac{\partial f}{\partial \Psi} \\ \Phi \frac{\partial f}{\partial \Phi} - \Psi \frac{\partial f}{\partial \Psi} &= \frac{\Phi}{8} [6\Psi - 2\Phi - 2\sqrt{\Phi\Psi} - \sqrt{\frac{\Psi}{\Phi}}(\Phi + \Psi)] - \frac{\Psi}{8} [6\Phi - 2\Psi - 2\sqrt{\Phi\Psi} - \sqrt{\frac{\Phi}{\Psi}}(\Phi + \Psi)] \\ &= -\frac{(\Phi - \Psi)}{4} [(\Phi + \Psi) + \sqrt{\Phi\Psi}] \\ \Phi \frac{\partial f}{\partial \Phi} - \Psi \frac{\partial f}{\partial \Psi} &= -\frac{(\Phi - \Psi)}{4} [2A + G] \leq 0 \end{aligned}$$

for $\Phi \geq \Psi$. This proves the condition for Schur geometrically concavity. □

Proof. Case-(3) Schur harmonically convexity(concavity)

Then

$$\begin{aligned}
 s &= \Phi^2 \frac{\partial f}{\partial \Phi} - \Psi^2 \frac{\partial f}{\partial \Psi} \\
 &= \frac{\Phi^2}{8} [6\Psi - 2\Phi - 2\sqrt{\Phi\Psi} - \sqrt{\frac{\Psi}{\Phi}}(\Phi + \Psi)] - \frac{\Psi^2}{8} [6\Phi - 2\Psi - 2\sqrt{\Phi\Psi} - \sqrt{\frac{\Phi}{\Psi}}(\Phi + \Psi)] \\
 &= \frac{(\Phi - \Psi)}{8} [8\Phi\Psi - 2(\Phi + \Psi)^2 - 3(\Phi + \Psi)\sqrt{\Phi\Psi}]
 \end{aligned}$$

$$\Phi^2 \frac{\partial f}{\partial \Phi} - \Psi^2 \frac{\partial f}{\partial \Psi} = \frac{(\Phi - \Psi)}{4} [4(G^2 - A^2) - 3AG] \geq 0$$

for $\Phi \geq \Psi$. This proves the condition for Schur harmonically convexity. □

Theorem 4.2. For $\Phi > \Psi > 0$, the difference of ratio of mean

$$\frac{M_{SG} - M_{SA}}{M_{SN_1} - M_{SG}} = \frac{A - G}{G - N_1}$$

is

- (1) Schur convexity
- (2) Schur geometrical convexity
- (3) Schur harmonical convexity, $\forall \Phi \geq \Psi$.

Proof. Let

$$f(\Phi, \Psi) = \frac{M_{SN_1}(\Phi, \Psi) - M_{SG}(\Phi, \Psi)}{M_{SH_e}(\Phi, \Psi) - M_{SN_1}(\Phi, \Psi)} = \frac{G(\Phi, \Psi) - N_1(\Phi, \Psi)}{N_1(\Phi, \Psi) - H_e(\Phi, \Psi)}$$

Hence,

$$\begin{aligned}
 f(\Phi, \Psi) &= AN_1 - G^2 \\
 f(\Phi, \Psi) &= \left(\frac{\Phi + \Psi}{2}\right) \left(\frac{\Phi + \Psi + 2\sqrt{\Phi\Psi}}{4}\right) - \Phi\Psi \\
 f(\Phi, \Psi) &= \frac{\Phi^2 + \Psi^2 - 6\Phi\Psi + 2\sqrt{\Phi\Psi}(\Phi + \Psi)}{8}
 \end{aligned}$$

the partial derivatives of $f(\Phi, \Psi)$ and simple computations gives us,

$$\begin{aligned}
 \frac{\partial f}{\partial \Phi} &= \frac{1}{8} \left[2\Phi - 6\Psi + 2\sqrt{\Phi\Psi} + \sqrt{\frac{\Psi}{\Phi}}(\Phi + \Psi) \right] \\
 \frac{\partial f}{\partial \Psi} &= \frac{1}{8} \left[2\Psi - 6\Phi + 2\sqrt{\Phi\Psi} + \sqrt{\frac{\Phi}{\Psi}}(\Phi + \Psi) \right]
 \end{aligned}$$

□

Proof. Case-(1) Schur convex Then

$$\begin{aligned}
 s &= \frac{\partial f}{\partial \Phi} - \frac{\partial f}{\partial \Psi} \\
 &= \frac{\Phi - \Psi}{8} \left[\frac{8\sqrt{\Phi\Psi} + (\Phi + \Psi)}{\sqrt{\Phi\Psi}} \right] \geq 0 \\
 \frac{\partial f}{\partial \Phi} - \frac{\partial f}{\partial \Psi} &= \frac{\Phi - \Psi}{8} \left[\frac{8G + 2A}{G} \right] \geq 0.
 \end{aligned}$$

for $\Phi \geq \Psi$. This verifies the condition for Schur convex from 5 □

Proof. Case-(2) Schur geometrically convex. we have

$$\begin{aligned}\Phi \frac{\partial f}{\partial \Phi} &= \frac{\Phi}{8} \left[2\Phi - 6\Psi + 2\sqrt{\Phi\Psi} + \sqrt{\frac{\Psi}{\Phi}}(\Phi + \Psi) \right] \\ \Psi \frac{\partial f}{\partial \Psi} &= \frac{\Psi}{8} \left[2\Psi - 6\Phi + 2\sqrt{\Phi\Psi} + \sqrt{\frac{\Phi}{\Psi}}(\Phi + \Psi) \right] \\ \Phi \frac{\partial f}{\partial \Phi} &= \frac{1}{8} \left[2\Phi^2 - 6\Phi\Psi + 2\Phi\sqrt{\Phi\Psi} + \Phi(\Phi + \Psi)\sqrt{\frac{\Psi}{\Phi}} \right] \\ \Psi \frac{\partial f}{\partial \Psi} &= \frac{1}{8} \left[2\Psi^2 - 6\Phi\Psi + 2\Psi\sqrt{\Phi\Psi} + \Psi(\Phi + \Psi)\sqrt{\frac{\Phi}{\Psi}} \right] \\ \Phi \frac{\partial f}{\partial \Phi} - \Psi \frac{\partial f}{\partial \Psi} &= \frac{\Phi - \Psi}{4} [(\Phi + \Psi) + \sqrt{\Phi\Psi}] \\ \Phi \frac{\partial f}{\partial \Phi} - \Psi \frac{\partial f}{\partial \Psi} &= \frac{\Phi - \Psi}{4} [2A + G] \geq 0\end{aligned}$$

for $\Phi \geq \Psi$. This proves Schur geometrically convex. \square

Proof. Case-(3) Schur harmonic convexity. we have

$$\begin{aligned}s &= \Phi^2 \frac{\partial f}{\partial \Phi} - \Psi^2 \frac{\partial f}{\partial \Psi} \\ \Phi^2 \frac{\partial f}{\partial \Phi} &= \frac{\Phi^2}{8} \left[2\Phi - 6\Psi + 2\sqrt{\Phi\Psi} + \sqrt{\frac{\Psi}{\Phi}}(\Phi + \Psi) \right] \\ \Psi^2 \frac{\partial f}{\partial \Psi} &= \frac{\Psi^2}{8} \left[2\Psi - 6\Phi + 2\sqrt{\Phi\Psi} + \sqrt{\frac{\Phi}{\Psi}}(\Phi + \Psi) \right] \\ &= \frac{\Phi - \Psi}{4} \left[(\Phi + \Psi)^2 - 4\Phi\Psi + \sqrt{\Phi\Psi}(\Phi + \Psi) + \sqrt{\Phi\Psi} \left(\frac{\Phi + \Psi}{2} \right) \right] \\ \Phi^2 \frac{\partial f}{\partial \Phi} - \Psi^2 \frac{\partial f}{\partial \Psi} &= \left(\frac{\Phi - \Psi}{4} \right) [4A^2 - 4G + 3GA] \geq 0\end{aligned}$$

for $\Phi \geq \Psi$. This verifies the condition for Schur harmonic convex. \square

Theorem 4.3. *The ratio of difference of mean*

$$\frac{M_{SG} - M_{SN_1}}{M_{SH_e} - M_{SG}} = \frac{N_1 - G}{G - H_e}$$

- (1) Schur convexity
- (2) Schur geometrical convexity
- (3) Schur harmonical convexity, for all $\Phi \geq \Psi$.

Proof.

$$\text{let } f(\Phi, \Psi) = \frac{M_{SG}(\Phi, \Psi) - M_{SN_1}(\Phi, \Psi)}{M_{SH_e}(\Phi, \Psi) - M_{SG}(\Phi, \Psi)} = \frac{N_1 - G}{G - H_e}$$

Hence,

$$\begin{aligned}
 f(\Phi, \Psi) &= N_1 H_e - G^2 \\
 f(\Phi, \Psi) &= \frac{1}{12} \left(\Phi^2 + \Psi^2 - 8\Phi\Psi + 3\sqrt{\Phi\Psi}(\Phi + \Psi) \right) \\
 \frac{\partial f}{\partial \Phi} &= \frac{1}{12} \left(2\Phi - 8\Psi + 3\sqrt{\Phi\Psi} + \frac{3\sqrt{\Psi}(\Phi + \Psi)}{2\sqrt{\Phi}} \right) \\
 \frac{\partial f}{\partial \Psi} &= \frac{1}{12} \left(2\Psi - 8\Phi + 3\sqrt{\Phi\Psi} + \frac{3\sqrt{\Phi}(\Phi + \Psi)}{2\sqrt{\Psi}} \right)
 \end{aligned}$$

□

Proof. Case-(1) Schur convexity.

Then

$$\begin{aligned}
 \frac{\partial f}{\partial \Phi} &= \frac{1}{12} \left(2\Phi - 8\Psi + 3\sqrt{\Phi\Psi} + \frac{3\sqrt{\Psi}(\Phi + \Psi)}{2\sqrt{\Phi}} \right) \\
 \frac{\partial f}{\partial \Psi} &= \frac{1}{12} \left(2\Psi - 8\Phi + 3\sqrt{\Phi\Psi} + \frac{3\sqrt{\Phi}(\Phi + \Psi)}{2\sqrt{\Psi}} \right) \\
 \frac{\partial f}{\partial \Phi} - \frac{\partial f}{\partial \Psi} &= \frac{(\Phi - \Psi)}{12} \left[\frac{10\sqrt{\Phi\Psi} - 3(\Phi + \Psi)}{\sqrt{\Phi\Psi}} \right] \\
 \frac{\partial f}{\partial \Phi} - \frac{\partial f}{\partial \Psi} &= \frac{(\Phi - \Psi)}{12} \left[\frac{10G - 3A}{G} \right] \geq 0
 \end{aligned}$$

for $\Phi \geq \Psi$. This proves the condition of Schur convexity from 5

□

Proof. Case-(2) Schur geometrically convex. we have

$$\begin{aligned}
 s &= \Phi \frac{\partial f}{\partial \Phi} - \Psi \frac{\partial f}{\partial \Psi} \\
 \Phi \frac{\partial f}{\partial \Phi} - \Psi \frac{\partial f}{\partial \Psi} &= \frac{\Phi}{12} \left(2\Phi - 8\Psi + 3\sqrt{\Phi\Psi} + \frac{3\sqrt{\Psi}(\Phi + \Psi)}{2\sqrt{\Phi}} \right) \\
 &\quad - \frac{\Psi}{12} \left(2\Psi - 8\Phi + 3\sqrt{\Phi\Psi} + \frac{3\sqrt{\Phi}(\Phi + \Psi)}{2\sqrt{\Psi}} \right) \\
 &= \frac{(\Phi - \Psi)}{12} (2(\Phi + \Psi) + 3\sqrt{\Phi\Psi}) \\
 \Phi \frac{\partial f}{\partial \Phi} - \Psi \frac{\partial f}{\partial \Psi} &= \frac{(\Phi - \Psi)}{12} (4A + 3G) \geq 0
 \end{aligned}$$

for $\Phi \geq \Psi$. This proves the condition of Schur geometrically convex.

□

Proof. Case-(3) Schur harmonically convex. we have

$$\begin{aligned}
 s &= \Phi^2 \frac{\partial f}{\partial \Phi} - \Psi^2 \frac{\partial f}{\partial \Psi} \\
 \Phi^2 \frac{\partial f}{\partial \Phi} - \Psi^2 \frac{\partial f}{\partial \Psi} &= \frac{\Phi^2}{12} \left(2\Phi - 8\Psi + 3\sqrt{\Phi\Psi} + \frac{3\sqrt{\Psi}(\Phi + \Psi)}{2\sqrt{\Phi}} \right) \\
 &\quad - \frac{\Psi^2}{12} \left(2\Psi - 8\Phi + 3\sqrt{\Phi\Psi} + \frac{3\sqrt{\Phi}(\Phi + \Psi)}{2\sqrt{\Psi}} \right) \\
 &= \frac{\Phi^2}{12} \left[2((\Phi + \Psi)^2 - \Phi\Psi) - 8\Phi\Psi + 3\sqrt{\Phi\Psi}(\Phi + \Psi) \right. \\
 &\quad \left. + 3 \left(\frac{(\Phi + \Psi)}{2} \right) \sqrt{\Phi\Psi} \right] \\
 \Phi^2 \frac{\partial f}{\partial \Phi} - \Psi^2 \frac{\partial f}{\partial \Psi} &= \frac{(\Phi - \Psi)}{12} [8A^2 - 10G^2 + 9AG] \\
 (\Phi - \Psi) \left[\Phi^2 \frac{\partial f}{\partial \Phi} - \Psi^2 \frac{\partial f}{\partial \Psi} \right] &= \frac{(\Phi - \Psi)^2}{12} [8A^2 - 10G^2 + 9AG] \geq 0
 \end{aligned}$$

for $\Phi \geq \Psi$. This proves the condition of Schur harmonically convexity. \square

Theorem 4.4. *The ratio of difference of mean*

$$\frac{M_{SS} - M_{SN_1}}{M_{SG} - M_{SS}} = \frac{N_1 - S}{S - G}$$

- (1) Schur convexity/concavity.
- (2) Schur geometrically convexity/concavity.
- (3) Schur harmonically convexity/concavity. , $\forall \Phi \geq \Psi$.

Proof.

$$\begin{aligned}
 f(\Phi, \Psi) &= \frac{M_{SS}(\Phi, \Psi) - M_{SN_1}(\Phi, \Psi)}{M_{SG}(\Phi, \Psi) - M_{SS}(\Phi, \Psi)} = \frac{N_1 - S}{S - G} \\
 f &= \left(\frac{\sqrt{\Phi} + \sqrt{\Psi}}{2} \right)^2 \sqrt{\Phi\Psi} - \left(\sqrt{\frac{\Phi^2 + \Psi^2}{2}} \right)^2 \\
 &= \left(\frac{\Phi + \Psi + 2\sqrt{\Phi\Psi}}{4} \right) \sqrt{\Phi\Psi} - \frac{\Phi^2 + \Psi^2}{2} \\
 &= \frac{(\Phi + \Psi + 2\sqrt{\Phi\Psi})\sqrt{\Phi\Psi} - 2(\Phi^2 + \Psi^2 + 2\Phi\Psi - 2\Phi\Psi)}{4} \\
 &= \frac{(\Phi + \Psi + 2\sqrt{\Phi\Psi})\sqrt{\Phi\Psi} - 2((\Phi + \Psi)^2 - 2\Phi\Psi)}{4} \\
 &= \frac{(\Phi + \Psi)\sqrt{\Phi\Psi} + 2\Phi\Psi - 2(\Phi + \Psi)^2 + 4\Phi\Psi}{4} \\
 f &= \frac{(\Phi + \Psi)\sqrt{\Phi\Psi} - 2(\Phi + \Psi)^2 + 6\Phi\Psi}{4} \\
 \frac{\partial f}{\partial \Phi} &= \frac{1}{4} \left[\sqrt{\Phi\Psi} + \frac{\sqrt{\Psi}(\Phi + \Psi)}{2\sqrt{\Phi}} + 6\Psi - 4(\Phi + \Psi) \right] \\
 \frac{\partial f}{\partial \Psi} &= \frac{1}{4} \left[\sqrt{\Phi\Psi} + \frac{\sqrt{\Phi}(\Phi + \Psi)}{2\sqrt{\Psi}} + 6\Phi - 4(\Phi + \Psi) \right]
 \end{aligned}$$

Case-(1) Schur convexity.

$$\begin{aligned}
 \frac{\partial f}{\partial \Phi} - \frac{\partial f}{\partial \Psi} &= \frac{1}{4} \left[\frac{\sqrt{\Psi}(\Phi + \Psi)}{2\sqrt{\Phi}} - \frac{\sqrt{\Phi}(\Phi + \Psi)}{2\sqrt{\Psi}} + 6\Psi - 6\Phi \right] \\
 &= \frac{1}{4} \left[\frac{(\Phi + \Psi)(\Psi - \Phi) + 6(\Psi - \Phi)}{2\sqrt{\Phi\Psi}} \right] \\
 &= \frac{1}{4} \left[\frac{-(\Phi + \Psi)(\Phi - \Psi)}{2\sqrt{\Phi\Psi}} - 6(\Phi - \Psi) \right] \\
 &= \frac{-(\Phi - \Psi)}{4} \left[\frac{(\Phi + \Psi)}{2\sqrt{(\Phi\Psi)}} + 6 \right] \\
 \frac{\partial f}{\partial \Phi} - \frac{\partial f}{\partial \Psi} &= \frac{-(\Phi - \Psi)}{4} \left[\frac{A + 6G}{G} \right] \leq 0. \\
 (\Phi - \Psi) \frac{\partial f}{\partial \Phi} - \frac{\partial f}{\partial \Psi} &= \frac{-(\Phi - \Psi)^2}{4} \left[\frac{A + 6G}{G} \right] \leq 0.
 \end{aligned}$$

This proves f is Schur concave.

Case-(2) Schur geometrically convexity.

$$\begin{aligned}
 \Phi \frac{\partial f}{\partial \Phi} - \Psi \frac{\partial f}{\partial \Psi} &= \frac{\Phi}{4} \left[\sqrt{\Phi\Psi} + \frac{\sqrt{\Psi}(\Phi + \Psi)}{2\sqrt{\Phi}} + 6\Psi - 4(\Phi + \Psi) \right] \\
 &\quad - \frac{\Psi}{4} \left[\sqrt{\Phi\Psi} + \frac{\sqrt{\Phi}(\Phi + \Psi)}{2\sqrt{\Psi}} + 6\Phi - 4(\Phi + \Psi) \right] \\
 &= \frac{1}{4} \left[\Phi\sqrt{\Phi\Psi} + \Phi \frac{\sqrt{\Psi}(\Phi + \Psi)}{2\sqrt{\Phi}} + 6\Phi\Psi - 4\Phi(\Phi + \Psi) \right] \\
 &\quad - \frac{1}{4} \left[\Psi\sqrt{\Phi\Psi} + \Psi \frac{\sqrt{\Phi}(\Phi + \Psi)}{2\sqrt{\Psi}} + 6\Phi\Psi - 4\Psi(\Phi + \Psi) \right] \\
 &= \frac{1}{4} \left[\Phi\sqrt{\Phi\Psi} + \frac{\sqrt{\Phi\Psi}(\Phi + \Psi)}{2} + 6\Phi\Psi - 4\Phi(\Phi + \Psi) \right] \\
 &\quad - \frac{1}{4} \left[\Psi\sqrt{\Phi\Psi} + \frac{\sqrt{\Phi\Psi}(\Phi + \Psi)}{2} + 6\Phi\Psi - 4\Psi(\Phi + \Psi) \right] \\
 &= \frac{1}{4} \left[\Phi\sqrt{\Phi\Psi} - 4\Phi(\Phi + \Psi) \right] - \frac{1}{4} \left[\Psi\sqrt{\Phi\Psi} - 4\Psi(\Phi + \Psi) \right] \\
 &= \frac{1}{4} \left[(\Phi - \Psi)\sqrt{\Phi\Psi} - 4(\Phi - \Psi)(\Phi + \Psi) \right] \\
 &= \frac{(\Phi - \Psi)}{4} \left[\sqrt{\Phi\Psi} - 4(\Phi + \Psi) \right] \\
 \Phi \frac{\partial f}{\partial \Phi} - \Psi \frac{\partial f}{\partial \Psi} &= \frac{(\Phi - \Psi)}{4} [G - 8A] \\
 (\ln \Phi - \ln \Psi) \left(\frac{\partial f}{\partial \Phi} - \frac{\partial f}{\partial \Psi} \right) &= (\ln \Phi - \ln \Psi) \frac{(\Phi - \Psi)}{4} [G - 8A] \geq 0
 \end{aligned}$$

This proves f is Schur geometrically convex.

Case-(3) Schur harmonically convex.

$$\begin{aligned}
\Phi^2 \frac{\partial f}{\partial \Phi} - \Psi^2 \frac{\partial f}{\partial \Psi} &= \frac{\Phi^2}{4} \left[\sqrt{\Phi\Psi} + \frac{\sqrt{\Psi}(\Phi + \Psi)}{2\sqrt{\Phi}} + 6\Psi - 4(\Phi + \Psi) \right] \\
&\quad - \frac{\Psi^2}{4} \left[\sqrt{\Phi\Psi} + \frac{\sqrt{\Phi}(\Phi + \Psi)}{2\sqrt{\Psi}} + 6\Phi - 4(\Phi + \Psi) \right] \\
&= \frac{1}{4} \left[\Phi^2 \sqrt{\Phi\Psi} + \Phi^2 \frac{\sqrt{\Psi}(\Phi + \Psi)}{2\sqrt{\Phi}} + 6\Phi^2\Psi - 4\Phi^2(\Phi + \Psi) \right] \\
&\quad - \frac{1}{4} \left[\Psi^2 \sqrt{\Phi\Psi} + \Psi^2 \frac{\sqrt{\Phi}(\Phi + \Psi)}{2\sqrt{\Psi}} + 6\Phi\Psi^2 - 4\Psi^2(\Phi + \Psi) \right] \\
&= \frac{1}{4} \left[(\Phi^2 - \Psi^2) \sqrt{\Phi\Psi} + \frac{(\Phi + \Psi)}{2} \left(\frac{\Phi^2 \sqrt{\Psi}}{\sqrt{\Phi}} - \frac{\Psi^2 \sqrt{\Phi}}{\sqrt{\Psi}} \right) \right. \\
&\quad \left. + 6\Phi^2\Psi - 6\Psi^2\Phi - 4\Phi^2(\Phi + \Psi) + 4\Psi^2(\Phi + \Psi) \right] \\
&= \frac{1}{4} \left[(\Phi - \Psi)(\Phi + \Psi) \sqrt{\Phi\Psi} + \frac{(\Phi + \Psi)}{2} \left(\frac{\Phi\Psi(\Phi - \Psi)}{\sqrt{\Phi\Psi}} \right) \right. \\
&\quad \left. + 6\Phi\Psi(\Phi - \Psi) - 4(\Phi + \Psi)^2(\Phi - \Psi) \right] \\
&= \frac{(\Phi - \Psi)}{4} \left[(\Phi + \Psi) \sqrt{\Phi\Psi} + \sqrt{\Phi\Psi} \frac{(\Phi + \Psi)}{2} + 6\Phi\Psi - 4(\Phi + \Psi)^2 \right] \\
&= \frac{(\Phi - \Psi)}{4} [2AG + AG + 6G^2 - 16A^2] \\
&= \frac{(\Phi - \Psi)}{4} [3AG + 6G^2 - 16A^2] \\
\Phi^2 \frac{\partial f}{\partial \Phi} - \Psi^2 \frac{\partial f}{\partial \Psi} &= \frac{(\Phi - \Psi)}{4} [3(AG + 2G^2) - 16A^2] \geq 0 \\
(\Phi - \Psi) \Phi^2 \frac{\partial f}{\partial \Phi} - \Psi^2 \frac{\partial f}{\partial \Psi} &= \frac{(\Phi - \Psi)^2}{4} [3(AG + 2G^2) - 16A^2] \geq 0
\end{aligned}$$

This proves f is Schur harmonically convex. \square

5 Conclusion

We conclude that the results listed in Main results are Schur convex for all positive values of $\Phi - \Psi$

References

- [1] J.S. Aujila and F.C Silva, Weak majorization inequalities and Convex functions, *Linear Algebra Appl.*,369(2003), 217-233.
- [2] P. S. Bullen, *Handbook of Means and Their Inequalities*, Kluwer Acad. Publ., Dordrecht, 2003.
- [3] Y.M. Chu and W.F. Xia, Solution of an open problem for Schur-Convexity or Schur-Concavity of gini means *Sci.China Ser A.*, 52(2009)10, 2099-2106.
- [4] Y. M. Chu and X. M. Zhang, The Schur geometrical Convexity of the extended mean values, *J. Convex Anal.*, 15(4)2008, 869-890.
- [5] Y.M. Chu and X.M. Zang, Necessary and sufficient conditions such that extended mean values are Schur-Convex or Schur-Concave, *Note on Schur-convex functions*, *Rocky Mountain J. Math.*, 48(2008)1,229-238.

- [6] N. Elezovic and J. Pecaric, Note on Schur-convex functions, *Rocky Mountain J. Math.*, 29(1998), 853-856.
- [7] K.Z. Guan. Some properties of class of symmetric functions, *Math. Inequal. Appl.*, 9(4)(2006), 567-576.
- [8] K.Z. Guan. Schur convexity of complete symmetric function, *Math. Inequal. Appl.*, 3(1)(2007), 70-80.
- [9] K.Z. Guan. A class of symmetric functions for multiplicatively convex function, *Math. Inequal. Appl.*, 10(4)(2007), 745-753.
- [10] G. H. Hardy, J. E. Littlewood and G. Pölya, *Inequalities*, 2nd edition, Cambridge University Press, Cambridge, 1959.
- [11] Huan Nan Shi, Jain Zang, and Da-mao Li, Schur-Geometric Convexity for Difference of Means, *Applied Mathematics E-Notes*, 10(2010), 274-284.
- [12] F.K.Hun, wang, U.G. Rothblum and L. Shepp, Manotone optimal multipartition using Schur convexity with respect to partial orders, *SIAM J. Discrete math* 6(1993)4,533-547.
- [13] V. Lokesha, K. M. Nagaraja, B. Naveen Kumar and Y.-D. Wu, Shur convexity of Gnan mean for positive arguments, *Notes on Number Theory and Discrete Mathematics*, 17(4) (2011), 37-41.
- [14] A. M. Marshall and I. Olkin, *Inequalities: Theory of Majorization and Its Application*, New York : Academics Press, 1979.
- [15] K. M. Nagaraja, V. Lokesha and S. Padmanabhan, A simple proof on strengthening and extension of inequalities, *Advn. Stud. Contemp. Math.*, 17(1) (2008), 97-103.
- [16] Merkle, Milan. "Conditions for convexity of a derivative and applications to the gamma and digamma function." *Facta Universitatis. Series: Mathematics and Informatics* 16 (2001): 13-20.
- [17] K Murali, KM Nagaraja, Schur convexity of Stolarsky's extended mean values, *Journal of Mathematical Inequalities* 10 (3), 725-35
- [18] K. M. Nagaraja, Murali K, V. Lokesha, Schur Convexity and Concavity of Gnan Mean, *Proceedings of the Jangjeon Mathematical Society* Vol.17 No.3 (2014) 355 - 367
- [19] P.S.K. Reddy, K.M. Nagaraja, R.S. Kumar, α - Centroidal Means and its Applications to Graph Labeling, *Palestine Journal of Mathematics*, Vol. 10(2), (2021) 704-709
- [20] K. M. Nagaraja, P Reddy, A Note on D' Ocagne Identity on Generalized Fibonacci and Lucas numbers, *Palestine Journal of Mathematics*, Vol. 10(2), (2021) 751-755
- [21] Rooin, Jamal & Hassani, Mehdi. Some New Inequalities Between Important Means and Applications to Ky Fan- type Inequalities. *Mathematical Inequalities & Applications*. 10. 10.7153/mia, (2007) 10-48.
- [22] H. N. Shi, Schur-Convex Functions relate to Hadamard-type inequalities, *J. Math. Inequal.*, 1(1)(2007), 127-136.
- [23] H. N. Shi, S. H. Wu and F. Qi, An alternative note on the Schur-Convexity of the extended mean values, *Math. Inequal. Appl.*, 9(2)(2006), 219-224.
- [24] Ying Wu, Feng Qi and Huan-Nan Shi, Schur-Harmonic Convexity for differences of some special means in two variables, *J. Math. Inequ.* 8(2)(2014),321-330.
- [25] X. M. Zang, The Schur geometrical Convexity of integral arithmetic mean, *Inte. J. Pure Appl. Math.*, 41(7)(2007), 919-925.

Author information

Sandeep Kumar, Department of Mathematics, Presidency University, Bangalore-560 064, India.
E-mail: skumars1504@gmail.com

Sangeetha B. K., Department of Mathematics, Presidency University, Bangalore-560 064, India.
E-mail: vinaysangeetha3@gmail.com

Received: 2023-04-23

Accepted: 2023-05-17