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Average Distance and Average Restricted Detour Distance of a Straight Chain of k-Wheels Graph

Herish Omer Abdullah, Ivan Dler Ali and Rashad Rashid Haji

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A straight chain of β copies of k-wheels graph W_k^β is a sequence of β copies of k-wheels such that every two successive wheel have exactly one common edge. In this article, we investigate the average restricted detour distance, Hosoya polynomial, Wiener index, restricted detour polynomial and restricted detour index of the graph W_k^β for $k = 4, 5$ and $\beta \geq 2$. Moreover, we find the average distance and the average restricted detour distance of the graphs W_4^β and W_5^β for $\beta \geq 2$.

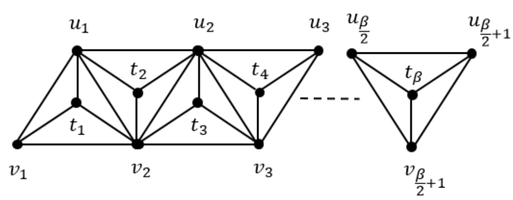
1 Introduction

Throughout this article, our graphs are finite, simple and connected. Let G be a graph with vertex set $V(G)$ and edge set $E(G)$. The distance $d(u, v)$ between the two vertices u and v of G is the number of edges in the shortest $u - v$ path in G[4]. An induced $u - v$ path of length $D^*(u, v)$ is called a restricted detour path. The restricted detour distance between two vertices u and v of G is the length of a longest $u - v$ path p for the induced condition $\langle V(P)\rangle = P$ and indicated $D^*(u, v)$[8, 10]. The restricted detour diameter δ^* of the graph G is defined to be the maximum restricted detour distance between any pair of vertices of G[7].
The Wiener index $W(G)$ of a graph G, is the sum of the distances between all pairs of vertices of G[2].
Let $C(G, k)$ denotes the number of pairs of vertices of G that are at distance k, for $k = 0, 1, \ldots, \delta$ where δ is the diameter of G. The Hosoya polynomial of G is $H(G; x) = \sum_{k=0}^{\delta} C(G, k)x^k$. The Wiener index and Hosoya polynomial have the relation $W(G) = \frac{d}{dx} H(G; x)\,|_{x=1}$. The average distance $\mu(G)$ is the expected distance between all the distinct unordered pairs of vertices of G, divided by $|V(G)|(|V(G)| - 1)$ where $|V(G)|$ is the order of G, that is, $\mu(G) = \frac{\sum_{\{u,v\}\subseteq V(G)} d(u,v)}{|V(G)|(|V(G)|-1)}$ [3, 6, 9]. Therefore $\mu(G) = \frac{\sum_{\{u,v\}\subseteq V(G)} d(u,v)}{\binom{|V(G)|}{2}} = \frac{2W(G)}{|V(G)|(|V(G)|-1)}$. The restricted detour polynomial depends on restricted detour distance, which is $D^*(G; x) = \sum_{\{u,v\}\subseteq V(G)} x^{D^*(u,v)}$, where the summation is taken over all unordered pairs of distinct vertices u and $v$of G. The

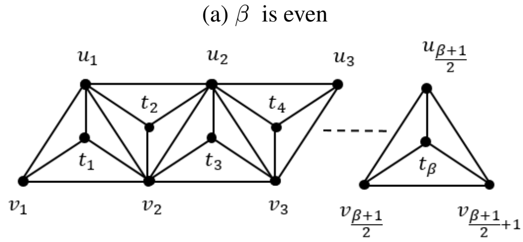
index $dd^*(x) = \sum_{\{u,v\}\subseteq V(G)} D^*(u,v)$ is also based on the restricted detour distance, where the summation is taken over all unordered pairs of distinct vertices u and $v$of $G[1]$.
Also $dd^*(x) = \frac{d}{dx} D^*(G;x)|_{x=1}$.
Finally, we define the average restricted detour distance as $\mu^*(G) = \frac{2dd^*(G)}{|V(G)|(|V(G)|-1)}$.
Restricted detour polynomials and restricted detour indices are computed in 2012 for a hexagonal chain and a ladder graph by Ali. A. Ali And G. A. Mohammed-Saleh [1], in 2017 for edge identification of two wheel graphs by Ivan Dler Ali and Herish Omer Abdullah [10], also computed in 2017 for some classes of thorn graphs by Gashaw A. Mohammed-Saleh, Herish O. Abdullah and Mohammed R. Ahmed [5], and in 2021 for a prism and some wheel related graphs by Herish O. Abdullah and Ivan Dler Ali [8]. The general form of average distance and average restricted detour distance of a straight chain of *k*-wheels graph is provided in this article.

2 Straight 4-Wheel Chains

A straight 4-wheel chain W_4^β for $\beta \geq 2$ is a graph consisting of a chain of β copies of 4-wheels so that every two successive wheel have exactly one common edge. This forming a chain as shown in 1.



(a) β is even



(b) β is odd

Figure 1: The graph W_4^β

By direct computations, we see that the order of the graph W_4^β is $p(W_4^\beta) = 2\beta + 2$, and the size of the graph W_4^β is $q(W_4^\beta) = 5\beta + 1$. Next proposition computes the diameter and restricted detour diameter of the graph W_4^β.

Proposition 2.1. *For $\beta \geq 2$, the diameter $\delta(W_4^\beta)$ of the graph W_4^β is equal to $\beta - \left\lfloor \frac{\beta-1}{2} \right\rfloor$ and the restricted detour diameter $\delta^*(W_4^\beta)$ of the graph W_4^β is equal to $\beta - \left\lfloor \frac{\beta-1}{3} \right\rfloor$.*

Proof: It clear that from the Figure 1, and by mathematical induction on the number of copies β of the graph W_4^β, the diameter $\delta(W_4^\beta)$ is equal to $\beta - \left\lfloor \frac{\beta-1}{2} \right\rfloor$ and the restricted detour diameter

$\delta^*(W_4^{\beta})$ is equal to $\beta - \left\lfloor \frac{\beta-1}{3} \right\rfloor$.

Obviously, the Hosoya polynomial and the restricted detour polynomial of the 2-copies of 4-wheel graphs W_4^2 are given by

$$H\left(W_4^2;x\right) = D^*\left(W_4^2;x\right) = 6 + 11x + 4x^2.$$

□

The Hosoya polynomial of W_4^{β} for $\beta \geq 2$ is given in the next result.

Theorem 2.2. *If $\beta \geq 2$, then the Hosoya polynomial of the graph W_4^{β} is given by*

$$H(W_4^{\beta};x) = (2\beta+2) + (5\beta+1)\,x + (8\beta-12)\,x^2 + 4\sum_{m=4}^{\beta}(\beta-m+1)x^{m-\left\lfloor \frac{m-1}{2}\right\rfloor}. \quad (2.1)$$

Proof: We use mathematical induction on the number of copies β in the graph W_4^{β}.

The result is obvious for $\beta = 2$.

For $\beta = 3$, then from Figure 1a and by directed calculations we obtain $H\left(W_4^3;x\right) = 8 + 16x + 12x^2$, which satisfies the relation (2.1).

Assume that the theorem is true for $\beta = k > 3$, that is

$$H(W_4^{k};x) = (2k+2) + (5k+1)\,x + (8k-12)\,x^2 + 4\sum_{m=4}^{k}(k-m+1)x^{m-\left\lfloor \frac{m-1}{2}\right\rfloor}.$$

Now, to prove the theorem is true for $\beta = k+1$.

First, we assume k is an even integer and let w_1 and w_2 be any two vertices of $V(W_4^{k+1})$, and consider the following cases:

(i) If w_1 and w_2 are belong to the first k copies of W_4^{k+1}, that is, if $w_1, w_2 \in V(W_4^k)$, then we get the polynomial

$$H(W_4^{k+1};x) = H(W_4^{k};x) = (2k+2)+(5k+1)\,x+(8k-12)\,x^2+4\sum_{m=4}^{k}(k-m+1)x^{m-\left\lfloor \frac{m-1}{2}\right\rfloor}.$$

(ii) If $w_1 = t_{k+1}$ and $w_2 = v_{\frac{k}{2}+2}$, then we get the polynomial $G(x) = x$.

(iii) If w_1 belongs to the first k copies of W_4^{k+1} and $w_2 \in \{t_{k+1},\ v_{\frac{k}{2}+2}\}$, then we get the polynomials.

$H(v_{\frac{k}{2}+2}, W_4^{k+1};x) = 1 + 3x + 4x^2 + 2\sum_{m=4}^{k+1} x^{m-\left\lfloor \frac{m-1}{2}\right\rfloor}$, and

$$H\left(t_{k+1}, W_4^{k+1};x\right) = 1 + 2x + 4x^2 + 2\sum_{m=4}^{k+1} x^{m-\left\lfloor \frac{m-1}{2}\right\rfloor}.$$

Now, by combining the results obtained from the cases (i),(ii) and (iii) we note that the pair $\{t_{k+1},\ v_{\frac{k}{2}+2}\}$ of distance one is counted twice, so we obtain

$$H(W_4^{k+1};x) = H(W_4^{k};x) + G(x) + H(v_{\frac{k}{2}+2}, W_4^{k+1};x) + H(t_{k+1}, W_4^{k+1};x) - x.$$

Hence

$$H(W_4^{k+1};x) = (2k+2) + (5k+1)\,x + (8k-12)\,x^2 + 4\sum_{m=4}^{k}(k-m+1)\,x^{m-\left\lfloor \frac{m-1}{2}\right\rfloor} + x$$
$$+ 2 + 5x + 8x^2 + 4\sum_{m=4}^{k+1} x^{m-\left\lfloor \frac{m-1}{2}\right\rfloor} - x.$$

Or,

$$H(W_4^{k+1};x) = [2\,(k+1)+2] + [5\,(k+1)+1]\,x + [8\,(k+12)-1]\,x^2$$
$$+ 4\sum_{m=4}^{k+1}[(k+1)-m+1]x^{m-\left\lfloor \frac{m-1}{2}\right\rfloor}.$$

So, (2.1) is true for even k. Similarly, we can show that (2.1) is true for odd k. Hence the relation (2.1) is true for all $\beta = k+1$. This completes the proof. □

Now, taking the derivative of $H(W_4^{\beta};x)$ given from Theorem 2.2 with respect to x at $x=1$; we get the following interesting result.

Corollary 2.3. *For $\beta \geq 2$, the Wiener index of the graph W_4^{β} is given by*

$$W(W_4^{\beta}) = \begin{cases} \frac{1}{3}\beta^3 + \frac{5}{2}\beta^2 + \frac{8}{3}\beta + 1, & \text{if } \beta \text{ is even,} \\ \frac{1}{3}\beta^3 + \frac{5}{2}\beta^2 + \frac{8}{3}\beta + \frac{1}{2}, & \text{if } \beta \text{ is odd.} \end{cases}$$

Proof: Taking the derivative of $H(W_4^{\beta};x)$ given from Theorem 2.2 with respect to x then putting $x=1$; we get

$$W(W_4^{\beta}) = 21\beta - 23 + 4(\beta+1)\sum_{m=4}^{\beta} m - 4\sum_{m=4}^{\beta} m^2 - 4(\beta+1)\sum_{m=4}^{\beta}\left\lfloor \frac{m-1}{2}\right\rfloor + 4\sum_{m=4}^{\beta} m\left\lfloor \frac{m-1}{2}\right\rfloor. \tag{2.2}$$

One can easily see that

$$\sum_{m=4}^{\beta}\left\lfloor \frac{m-1}{2}\right\rfloor = \begin{cases} \frac{1}{4}\beta\,(\beta-2) - 1, & \text{if } \beta \text{ is even,} \\ \frac{1}{4}\,(\beta+1)\,(\beta-3)\,, & \text{if } \beta \text{ is odd.} \end{cases} \tag{2.3}$$

and

$$\sum_{m=4}^{\beta} m\left\lfloor \frac{m-1}{2}\right\rfloor = \begin{cases} \frac{1}{24}\beta\,(\beta-2)\,(4\beta+5) - 3, & \text{if } \beta \text{ is even,} \\ \frac{1}{24}\,(\beta+1)\,(\beta-1)\,(4\beta+3) - 3, & \text{if } \beta \text{ is odd.} \end{cases} \tag{2.4}$$

Now, substituting (2.3) and (2.4) in (2.2) and then simplifying we get the required result. □

Corollary 2.4. *For $\beta \geq 2$, the average distance of the graph W_4^β is given by*

$$\mu\left(W_4^\beta\right) = \begin{cases} \frac{2\beta^3+15\beta^2+16\beta+6}{6(\beta+1)(2\beta+1)}, & \text{if } \beta \text{ is even,} \\ \frac{2\beta^3+15\beta^2+16\beta+3}{6(\beta+1)(2\beta+1)}, & \text{if } \beta \text{ is odd.} \end{cases}$$

Proof: It follows from Corollary 2.3 and the relation $\mu\left(W_4^\beta\right) = \frac{2W(W_4^\beta)}{(2\beta+2)(2\beta+1)}$. □

The restricted detour polynomial of the graph W_4^β for $\beta \geq 2$ is obtained in the next theorem.

Theorem 2.5. *For $\beta \geq 2$ we have*

$$D^*(W_4^\beta;x) = (2\beta+2) + (5\beta+1)\,x + 4\,(\beta-1)\,x^2 + 4\sum_{m=3}^{\beta}(\beta-m+1)x^{m-\left\lfloor\frac{m-1}{3}\right\rfloor} \quad (2.5)$$

Proof: We use mathematical induction on the number of β copies of the graph W_4^β.

The result is obvious for $\beta = 2$.

For $\beta = 3$, then from Figure 1a and by directed calculations we obtain $D^*\left(W_4^3;x\right) = 8 + 16x + 8x^2 + 4x^3$, which satisfies the relation (2.5).

Assume that the theorem is true for $\beta = k > 3$, that is

$$D^*(W_4^k;x) = (2k+2) + (5k+1)\,x + 4\,(k-1)\,x^2 + 4\sum_{m=3}^{k}(k-m+1)x^{m-\left\lfloor\frac{m-1}{3}\right\rfloor}.$$

Now, to prove the theorem is true for $\beta = k+1$.

First, we assume k is an even integer and let $w_1$and w_2 be any two vertices of $V(W_4^{k+1})$, and consider the following cases:

(i) If w_1 and w_2 are belong to the first k copies of W_4^{k+1}, that is $w_1, w_2 \in V(W_4^k)$, then we get the polynomial

$$D^*(W_4^{k+1};x) = D^*(W_4^k;x) = (2k+2)+(5k+1)\,x+4\,(k-1)\,x^2+4\sum_{m=3}^{k}(k-m+1)x^{m-\left\lfloor\frac{m-1}{3}\right\rfloor}.$$

(ii) If $w_1 = t_{k+1}$and $w_2 = v_{\frac{k}{2}+2}$then we get the polynomial $G(x) = x$.

(iii) If w_1 belongs to the first k copies of W_4^{k+1} and $w_2 \in \{t_{k+1}, v_{\frac{k}{2}+2}\}$, then we get the polynomials $D^*(v_{\frac{k}{2}+2}, W_4^{k+1};x) = 1 + 3x + 2x^2 + 2\sum_{m=3}^{k+1} x^{m-\left\lfloor\frac{m-1}{3}\right\rfloor}$, and

$$D^*\left(t_{k+1}, W_4^{k+1};x\right) = 1 + 2x + 2x^2 + 2\sum_{m=3}^{k+1} x^{m-\left\lfloor\frac{m-1}{3}\right\rfloor}.$$

Now, combining the results given from the cases (i),(ii) and (iii) we note that the distance of the pair $\{t_{k+1},\ v_{\frac{k}{2}+2}\}$ of distance one is counted twice, so we obtain

$$D^*(W_4^{k+1};x) = D^*(W_4^k;x) + G(x) + D^*(v_{\frac{k}{2}+2}, W_4^{k+1};x) + D^*\left(t_{k+1}, W_4^{k+1};x\right) - x.$$

This leads to the following polynomial

$$D^*(W_4^{k+1};x) = 2(k+1) + (5k+1)x + 4(k-1)x^2 + 4\sum_{m=3}^{k}(k-m+1)x^{m-\left\lfloor\frac{m-1}{3}\right\rfloor} + x$$
$$+ 2 + 5x + 4x^2 + \sum_{m=3}^{k+1} x^{m-\left\lfloor\frac{m-1}{3}\right\rfloor} - x \tag{2.6}$$

Or,

$$\begin{aligned} D^*\left(W_4^{k+1};x\right) &= [2(k+1)+2] + [5(k+1)+1]x + 4[(k+1)-1]x^2 \\ &+ 4\sum_{m=3}^{k+1}[(k+1)-m+ 1]x^{m-\left\lfloor\frac{m-1}{3}\right\rfloor} \end{aligned} \tag{2.7}$$

So, (2.5) is true for even k. Similarly, (2.5) is true for odd k. This completes the proof. □

Now, taking the derivative of $D^*(W_4^\beta;x)$ given from Theorem 2.5 with respect to x at $x=1$; and using the facts that

$$\sum_{m=3}^{\beta}\left\lfloor\frac{m-1}{3}\right\rfloor = \begin{cases} \frac{1}{6}\beta(\beta-3), & \text{if } \beta\equiv 0 \ mod\ 3, \\ \frac{1}{6}(\beta-1)(\beta-2), & \text{otherwise.} \end{cases}$$

and

$$\sum_{m=3}^{\beta} m\left\lfloor\frac{m-1}{3}\right\rfloor = \begin{cases} \frac{1}{18}\beta(\beta-3)(2\beta+3), & \text{if } \beta\equiv 0 \ mod\ 3, \\ \frac{1}{18}(\beta-1)\left(2\beta^2-\beta-4\right), & \text{if } \beta\equiv 1 \ mod\ 3, \\ \frac{1}{18}(\beta-2)(\beta+1)(2\beta-1), & \text{if } \beta\equiv 2 \ mod\ 3. \end{cases}$$

we get the following result.

Corollary 2.6. *For $\beta\geq 2$, the restricted detour index of the graph W_4^β is given by*

$$dd^*(W_4^\beta) = \begin{cases} \frac{4}{9}\beta^3+\frac{8}{3}\beta^2+\frac{7}{3}\beta+1, & \text{if } \beta\equiv 0 \ mod\ 3, \\ \frac{4}{9}\beta^3+\frac{8}{3}\beta^2+\frac{7}{3}\beta+\frac{5}{9}, & \text{if } \beta\equiv 1 \ mod\ 3, \\ \frac{4}{9}\beta^3+\frac{8}{3}\beta^2+\frac{7}{3}\beta+\frac{1}{9}, & \text{if } \beta\equiv 2 \ mod\ 3. \end{cases}$$

Proof: Note that $dd^*\left(W_4^\beta\right) = \frac{d}{dx}D^*(W_4^\beta;x)\Big|_{x=1}$.From Theorem 2.5, the result follows. □

Corollary 2.7. *For $\beta\geq 2$, the average restricted detour distance of the graph W_4^β is given by*

$$\mu^*(W_4^\beta) = \begin{cases} \frac{4\beta^3+24\beta^2+21\beta+9}{9(\beta+1)(2\beta+1)}, & \text{if } (\beta \ mod\ 3)\equiv 0, \\ \frac{4\beta^3+24\beta^2+21\beta+5}{9(\beta+1)(2\beta+1)}, & \text{if } (\beta \ mod\ 3)\equiv 1, \\ \frac{4\beta^3+24\beta^2+21\beta+1}{9(\beta+1)(2\beta+1)}, & \text{if } (\beta \ mod\ 3)\equiv 2. \end{cases}$$

Proof: Obvious. □

3 Straight 5-Wheel Chains

A straight 5–wheel chain W_5^β for, $\beta \geq 2$ is a graph consisting of a chain of β copies of 5-wheel so that every two successive wheels have exactly one common edge. This forming a chain as depicted in the following Figure.

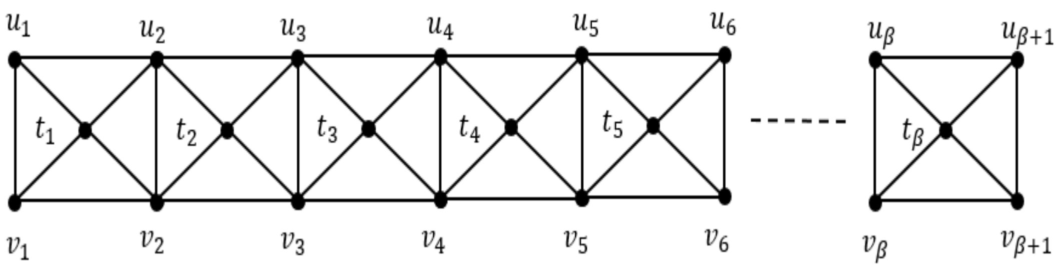


Figure 2: A straight 5-wheel chain W_5^β

In this section, we will obtain Hosoya polynomial, Wiener index, restricted detour polynomial, restricted detour index, average distance and average restricted detour distance of the graph W_5^β for $\beta \geq 2$.

By simple calculations, we see that the order of the graph W_5^β is $p(W_5^\beta) = 3\beta + 2$, and the size of the graph W_5^β is $q(W_5^\beta) = 7\beta + 1$.

Next proposition computes the diameter and restricted detour diameter of the graph W_5^β

Proposition 3.1. *For $\beta \geq 2$, the diameter $\delta(W_5^\beta)$ of the graph W_5^β is equal to $\beta + 1$ and the restricted detour diameter $\delta^*(W_5^\beta)$ of the graph W_5^β is equal to 2β.*

Proof: It clear that from the Figure 2, and by mathematical induction on the number of copies β of the graph W_5^β, the diameter $\delta(W_5^\beta)$ is equal to $\beta+1$ and the restricted detour diameter $\delta^*(W_5^\beta)$ is equal to 2β.

By direct calculations, we see that the Hosoya polynomial of the 2-copies of 5-wheels graph W_5^2 is $H\left(W_5^2; x\right) = 8 + 15x + 11x^2 + 2x^3$, and the restricted detour polynomial of the 2-copies of 5-wheels graph W_5^2 is $D^*\left(W_5^2; x\right) = 8 + 15x + 5x^2 + 6x^3 + 2x^4$. □

The Hosoya polynomial of the graph W_5^β, for $\beta \geq 2$ is given in the next result.

Theorem 3.2. *If $\beta \geq 2$, then the Hosoya polynomial of the graph W_5^β is given by*

$$H(W_5^\beta; x) = (3\beta + 2) + (7\beta + 1)\,x + (9\beta - 7)\,x^2 + 2\,(\beta - 1)\,x^3 + \sum_{m=3}^{\beta} (\beta - m + 1)(7 + 2x)x^m \tag{3.1}$$

Proof: We use mathematical induction on the number of copies β in the graphW_5^β.

The result is obvious for $\beta = 2$.

For $\beta = 3$, then from Figure 3.1 and by directed calculations we obtain
$H(W_5^3; x) = 11 + 22x + 20x^2 + 11x^3 + 2x^4$, which satisfies the relation (3.1).
Assume that the theorem is true for $\beta = k > 3$, that is,

$$H(W_5^k; x) = (3k+2) + (7k+1)\,x + (9k-7)\,x^2 + 2\,(k-1)\,x^3 + \sum_{m=3}^{k}(k-m+1)(7+2x)x^m.$$

Now, to prove the theorem is true for $\beta = k+1$.

Let w_1 and w_2 be any two vertices of $V(W_5^{k+1})$, and consider the following cases:

(i) If w_1 and w_2 are belong to the first k copies of W_5^{k+1}, that is, if $w_1, w_2 \in V(W_5^k)$, then we get the polynomial

$$H\left(W_5^{k+1}; x\right) = H(W_5^k; x) = (3k+2) + (7k+1)\,x + (9k-7)\,x^2 + 2\,(k-1)\,x^3 + \sum_{m=3}^{k}(k-m+1)(7+2x$$

(ii) If $w_1, w_2 \in \{u_{k+2}, v_{k+2}, t_{k+1}\}$, then we get the polynomial $G(x) = 3x$.

(iii) If w_1 belongs to the first k copies of W_5^{k+1} and $w_2 \in \{u_{k+2}, v_{k+2}, t_{k+1}\}$, then we get the polynomials

$$H(u_{k+2}, W_5^{k+1}; x) = 1 + 3x + 3x^2 + x^3 + \sum_{m=3}^{k+1}(2+x)\,x^m,$$

$$H(v_{k+2}, W_5^{k+1}; x) = 1 + 2x + 3x^2 + x^3 + \sum_{m=3}^{k+1}(2+x)\,x^m$$

and

$$H(t_{k+1}, W_5^{k+1}; x) = 1 + 2x + 3x^2 + \sum_{m=3}^{k+1}(3x^m).$$

Now, by combining the results obtained from the cases (i),(ii) and (iii) we note that the distances of each of the pairs of vertices $\{u_{k+2}, t_{k+1}\}$, $\{u_{k+2}, v_{k+2}\}$ and $\{v_{k+2}, t_{k+1}\}$ each of distance one are counted twice, we get following polynomial

$$H\left(W_5^{k+1}; x\right) = H\left(W_5^k; x\right) + G(x) + H\left(u_{k+2}, W_5^{k+1}; x\right) + H\left(v_{k+2}, W_5^{k+1}; x\right) + H\left(t_{k+1}, W_5^{k+1}; x\right) - 3x.$$

Hence

$$H(W_5^{k+1}; x) = (3k+2) + (7k+1)\,x + (9k-7)\,x^2 + 2\,(k-1)\,x^3 + \sum_{m=3}^{k}(k-m+1)\,(7+2x)\,x^m$$
$$+\,3x + 3 + 7x + 9x^2 + 2x^3 + \sum_{m=3}^{k+1}(7+2x)x^m - 3x$$

Or,

$$H\left(W_5^{k+1}; x\right) = [3\,(k+1)+2] + [7\,(k+1)+1]\,x + [9\,(k+1)-7]\,x^2 + 2[(k+1)-1]x^3$$
$$+ \sum_{m=3}^{k+1}[(k+1)-m+1](7+2x)x^m.$$

Hence the result holds for all $\beta = k+1$. This completes the proof. $\square$

The following results are direct consequences of Theorem 3.2.

Corollary 3.3. *For $\beta \geq 2$, the Wiener index of the graph W_5^β is given by*

$$W(W_5^\beta) = \frac{3}{2}\beta^3 + \frac{11}{2}\beta^2 + 4\beta + 1.$$

Proof: Obvious. □

Corollary 3.4. *For $\beta \geq 2$, the average distance of the graph W_5^β is given by*

$$\mu\left(W_5^\beta\right) = \frac{3\beta^3 + 11\beta^2 + 8\beta + 2}{(3\beta+2)(3\beta+1)}.$$

Proof: It follows from the corollary 3.3 and the relation $\left(W_5^\beta\right) = \frac{2W(W_4^\beta)}{(3\beta+2)(3\beta+1)}$. □

We obtain the restricted detour polynomial of the chain W_5^β, for $\beta \geq 2$ in the next theorem.

Theorem 3.5. *For $\beta \geq 2$, we have*

$$D^*(W_5^\beta;x) = (3\beta+2) + (7\beta+1)\,x + (3\beta-1)\,x^2 + 6\,(\beta-1)\,x^3 + 2\,(\beta-1)\,x^4 + \sum_{m=3}^{\beta}(\beta-m+1)(1+6x+2x^2)x^{2m-2}. \tag{3.2}$$

Proof: We use mathematical induction on the number of copies β of the graph W_5^β.
The result is obvious for $\beta = 2$.
For $\beta = 3$, then from Figure 2 and by directed calculations we get $D^*(W_5^3;x) = 11 + 22x + 8x^2 + 12x^3 + 5x^4 + 6x^5 + 2x^6$, which satisfies the relation (3.2).
Assume that the theorem is true for $\beta = k > 3$, that is

$$D^*(W_5^k;x) = (3k+2) + (7k+1)\,x + (3k-1)\,x^2 + 6\,(k-1)\,x^3 + 2\,(k-1)\,x^4 + \sum_{m=3}^{k}(k-m+1)(1+6x+2x^2)x^{2m-2}.$$

Now, we have to prove that the result is true for $\beta = k+1$. Let $w_1$and w_2 be any two vertices of $V(W_5^{k+1})$, and consider the following cases:

(i) If w_1 and w_2 are belong to the first k copies of W_5^{k+1}, that is $w_1, w_2 \in V(W_5^k)$, then we get the polynomial

$$D^*(W_5^{k+1};x) = D^*(W_5^k;x) = (3k+2) + (7k+1)\,x + (3k-1)\,x^2 + 6\,(k-1)\,x^3 + 2\,(k-1)\,x^4 + \sum_{m=3}^{k}(k-m+1)(1+6x+2x^2)x^{2m-2}.$$

(ii) If $w_1, w_2 \in \{u_{k+2}, v_{k+2}, t_{k+1}\}$, then we get the polynomial $G(x) = 3x$.

(iii) If w_1 belongs to the first k copies of W_5^{k+1}, and $w_2 \in \{u_{k+2}, v_{k+2}, t_{k+1}\}$, then we get the polynomials

$$D^*(u_{k+2}, W_5^{k+1}; x) = 1 + 3x + x^2 + 2x^3 + x^4 + \sum_{m=3}^{k+1} (2x + x^2)x^{2m-2},$$

$$D^*(v_{k+2}, W_5^{k+1}; x) = 1 + 2x + x^2 + 2x^3 + x^4 + \sum_{m=3}^{k+1} (2x + x^2)x^{2m-2},$$

and

$$D^*(t_{k+1}, W_5^{k+1}; x) = 1 + 2x + x^2 + 2x^3 + \sum_{m=3}^{k+1} (1 + 2x)x^{2m-2}.$$

Now, by combining the results given from the cases (i),(ii) and (iii) and notice that the distances of each of the pairs of vertices $\{u_{k+2}, t_{k+1}\}$, $\{u_{k+2}, v_{k+2}\}$ and $\{v_{k+2}, t_{k+1}\}$ each of distance one are counted twice, we get following polynomial

$$D^*(W_5^{k+1}; x) = D^*(W_5^k; x) + G(x) + D^*(u_{k+2}, W_5^{k+1}; x) + D^*(v_{k+2}, W_5^{k+1}; x) + D^*(t_{k+1}, W_5^{k+1}; x) - 3x.$$

Hence

$$\begin{aligned} D^*(W_5^{k+1}; x) &= (3k + 2) + (7k + 1)\, x + (3k - 1)\, x^2 + 6\,(k - 1)\, x^3 + 2\,(k - 1)\, x^4 \\ &+ \sum_{m=3}^{k} (k - m + 1)\,(1 + 6x + 2)\, x^{2m-2} + 3x + 3 + 7x + 3x^2 + 6x^3 + 2x^4 \\ &+ \sum_{m=3}^{k+1} (1 + 6x + 2)x^{2m-2} - 3x. \\ &= [3\,(k + 1) + 2] + [7\,(k + 1) + 1]\, x + [3(k + 1) - 1]x^2 + 6[(k + 1) - 1]x^3 \\ &+ 2[(k + 1) - 1]x^4 + \sum_{m=3}^{k} [(k + 1) - m + 1]\,(1 + 6x + 2)\, x^{2m-2}. \end{aligned}$$

Thus, the relation (3.2) is true for all $\beta = k + 1$. This completes the proof. □

Now, taking the derivative of $D^*(W_5^\beta; x)$ given from Theorem 3.5 with respect to x at $x = 1$; we get the following interesting result.

Corollary 3.6. *For $\beta \geq 2$, the restricted detour index of the graph W_5^β is given by*

$$dd^*(W_5^\beta) = 3\beta^3 + 5\beta^2 + 3\beta + 1.$$

Proof: Note that $dd^*(W_5^\beta) = \frac{d}{dx} D^*(W_5^\beta; x)\Big|_{x=1}$.From Theorem 3.5, the result follows. □

Corollary 3.7. *For $\beta \geq 2$, the average restricted detour distance of the graph W_5^β is given by*

$$\mu^*(W_5^\beta) = \frac{6\beta^3 + 10\beta^2 + 6\beta + 2}{(3\beta + 2)(3\beta + 1)}.$$

Proof: Obvious. □

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Author information

Herish Omer Abdullah, Salahaddin University-Erbil, Kurdistan Region,, Iraq..
E-mail: herish.abdullah@su.edu.krd

Ivan Dler Ali, Salahaddin University-Erbil, Kurdistan Region,, Iraq..
E-mail: evan.ali1@su.edu.krd

Rashad Rashid Haji, Salahaddin University-Erbil, Kurdistan Region,, Iraq..
E-mail: rashad.haji@su.edu.krd

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