Palestine Journal of Mathematics

Vol. 12(4)(2023) , 452–464

A NOTE ON (S, ω)-QUASI-ARMENDARIZ RINGS

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Communicated by Ayman Badawi

MSC 2010 Classifications: Primary 13B25, 16S36; Secondary 16W20.

Keywords and phrases: skew generalized power series ring; (S,ω)-quasi-Armendariz ring; left APP-ring; reflexive ring; right $p.q.$Baer ring.

Abstract. Let R be a ring, $(S, \leq)$ a strictly ordered monoid and $\omega : S \to$End(R) a monoid homomorphism. Properties of the ring $[[R^{S,\leq}, \omega]]$ of skew generalized power series with coefficients in R and exponents in S are considered. In this paper, we study some properties of (S, ω)-quasi-Armendariz under some suitable conditions. For example, we prove that, if I is a semiprime ring and R/I is $(S, \overline{\omega})$-quasi-Armendariz, when R is a completely S-compatible ring and $\overline{\omega} : S \to End(R/I)$ is the induced monoid homomorphism, then R is (S, ω)-quasi-Armendariz. If $[[R^{S,\leq}, \omega]]$ is right $p.q.$Baer, then R is right $p.q.$Baer and any S-indexed subset of $I(R)$ has a generalized join in $I(R)$. Also, we prove that, If R is S-compatible (S, ω)-quasi-Armendariz, then the ring $[[R^{S,\leq}, \omega]]$ is quasi-Baer (reflexive) if and only if R is quasi-Baer (reflexive, respectively). Moreover, some results of skew generalized power series $[[R^{S,\leq}, \omega]]$ are given.

1 Introduction

All rings considered here are associative with identity. We will write monoids multiplicatively unless otherwise indicated. If R is a ring and X is a nonempty subset of R, then the left (right) annihilator of X in R is denoted by $\ell_R(X)(r_R(X))$. We will denote by End(R) the monoid of ring endomorphisms of R, and by Aut(R) the group of ring automorphisms of R.

Let R be a ring, $(S, \leq)$ a strictly ordered monoid, and $\omega : S \to$End(R) a monoid homomorphism. For $s \in S$, let ω_s denote the image of s under ω, that is, $\omega_s = \omega(s)$. Let A be the set of all functions $f : S \to R$ such that the support supp$(f) = \{s \in S : f(s) \neq 0\}$ is artinian and narrow. Then for any $s \in S$ and $f, g \in A$ the set

$$X_s(f, g) = \{(u, v) \in supp(f) \times supp(g) : s = uv\}$$

is finite. Thus one can define the product $fg : S \to R$ of $f, g \in A$ as follows:

$$(fg)(s) = \sum_{(u,v)\in X_s(f,g)} f(u)\omega_u(g(v))$$

(by convention, a sum over the empty set is 0). With pointwise addition and multiplication as defined above, A becomes a ring, called the ring of skew generalized power series with coefficients in R and exponents in S, see [29] and denoted by $[[R^{S,\leq}, \omega]]$ (or by $R[[S, \omega]]$ when there is no ambiguity concerning the order $\leq$).

We will use the symbol 1 to denote the identity elements of the monoid S, the ring R, and the ring $[[R^{S,\leq}, \omega]]$ as well as the trivial monoid homomorphism $1 : S \to$End(R) that sends every element of S to the identity endomorphism. A subset $P \subseteq R$ will be called S-invariant if for every $s \in S$ it is ω_s-invariant (that is, $\omega_s(P) \subseteq P$). To each $r \in R$ and $s \in S$, we associate elements $c_r, e_s \in [[R^{S,\leq}, \omega]]$ defined by

$$c_r(x) = \begin{cases} r, & \text{if} \quad x = 1, \\ 0, & \text{if} \quad x \in S\backslash\{1\}, \end{cases} \qquad e_s(x) = \begin{cases} 1, & \text{if} \quad x = s, \\ 0, & \text{if} \quad x \in S\backslash\{s\}. \end{cases}$$

It is clear that $r \mapsto c_r$ is a ring embedding of R into $[[R^{S,\leq}, \omega]]$ and $s \mapsto e_s$, is a monoid embedding of S into the multiplicative monoid of the ring $[[R^{S,\leq}, \omega]]$, and $e_s c_r = c_{\omega_s(r)} e_s$.

Rege and Chhawchharia [23] introduced the notion of an Armendariz ring. They defined a ring R to be an Armendariz ring if whenever polynomials $f(x) = a_0 + a_1x + \cdots + a_nx^n, g(x) = b_0 + b_1x + \cdots + b_mx^m \in R[x]$ satisfy $f(x)g(x) = 0$, then $a_ib_j = 0$ for each i, j. (The converse is always true.) The name "Armendariz ring" was chosen because Armendariz [7, Lemma 1] had noted that a reduced ring satisfies this condition. Reduced rings (i.e., rings with no nonzero nilpotent elements). Some properties of Armendariz and reflexive rings have been studied in E. P. Armendariz [7], Anderson and Camillo [6], Kim and Lee [25], Ali [17], Huh, Lee and Smoktunowicz [30], and Lee and Wong [31].

Given a ring R and a ring endomorphism $\sigma : R \to R$, the skew polynomial ring $R[x; \sigma]$ consists of polynomials in the indeterminate x with coefficients from R, written on the left, where multiplication in $R[x; \sigma]$ is defined by

$$\left(\sum_i a_ix^i\right)\left(\sum_j b_jx^j\right) = \sum_{i,j} a_i\sigma^i(b_j)x^{i+j}.$$

Following Hong et al. [3], we say that a ring R with an endomorphism σ is σ-skew Armendariz if whenever polynomials $f(x) = a_0 + a_1x + \cdots + a_nx^n$ and $g(x) = b_0 + b_1x + \cdots + b_mx^m$ in $R[x; \sigma]$ satisfy $f(x)g(x) = 0$ then $a_i\sigma^i(b_j) = 0$ for all i, j. A stronger condition than Armendariz was studied by Kim et al. in [26]. A ring R is said to be power-serieswise Armendariz if whenever power series $f(x) = a_0 + a_1x + \cdots + a_nx^n$ and $g(x) = b_0 + b_1x + \cdots + b_mx^m$ in $R[[x]]$ satisfy $f(x)g(x) = 0$ then $a_ib_j = 0$ for all i, j. Armendariz rings were generalized to quasi-Armendariz rings by Hirano [35]. A ring R is called quasi-Armendariz provided that $a_iRb_j = 0$ for all i, j whenever $f(x) = a_0 + a_1x + \cdots + a_nx^n, g(x) = b_0 + b_1x + \cdots + b_mx^m \in R[x]$ satisfy $f(x)R[x]g(x) = 0$. In [24] Baser ana Kwak introduced the concept of σ-quasi-Armendariz ring. A ring R is called quasi-Armendariz ring with the endomorphism σ(or simply σ-quasi-Armendariz) if for $f(x) = a_0 + a_1x + \cdots + a_nx^n$ and $g(x) = b_0 + b_1x + \cdots + b_mx^m$ in $R[x; \sigma]$ satisfy $f(x)R[x; \sigma]g(x) = 0$ then $a_iR[x; \sigma]b_j = 0$ for all $0 \leq i \leq n, 0 \leq j \leq m$, or equivalently, $a_iR\sigma^tb_j = 0$ for any nonnegative integer t and all i, j [24]. Baser and Kwak [24] also showed that every σ-quasi-Armendariz ring is σ-skew quasi-Armendariz in case that σ is an epimorphism, but the converse does not hold, in general. The notion of σ-skew Armendariz rings is generalized as follows: Let σ be an endomorphism of a ring R. Then R is called a σ-skew quasi-Armendariz ring Definition 2.1 [4] if for $f(x) = a_0 + a_1x + \cdots + a_nx^n$ and $g(x) = b_0 + b_1x + \cdots + b_mx^m$ in $R[x; \sigma]$ satisfy $f(x)R[x; \sigma]g(x) = 0$ implies $a_iR\sigma^i(b_j) = 0$ for all $0 \leq i \leq n, 0 \leq j \leq m$, while Cortes Definition 3.11 [33] used the term quasi-skew Armendariz for what is called σ-skew quasi-Armendariz when σ is an automorphism. It is shown that the class of σ-skew quasi-Armendariz rings is Morita stable and that several extensions of a σ-skew quasi-Armendariz ring are also σ-skew quasi-Armendariz rings in [33] and [4]. Observe that every σ-skew Armendariz ring is σ-skew quasi-Armendariz when σ is an epimorphism, but the converse does not hold by Example 2.2(1) [4].

If R is a ring and S is a strictly ordered monoid, then the ring R is called a generalized Armendariz ring if for each $f, g \in [[R^{S,\leq}]]$ such that $fg = 0$ implies that $f(u)g(v) = 0$ for each $u \in supp(f)$ and $v \in supp(g)$. In [36] called such ring S-Armendariz ring. If R is a ring, S be a torsion-free and cancellative monoid and $\leq$ a strict order on S, then the ring R is called a generalized quasi-Armendariz ring if for each $f, g \in [[R^{S,\leq}]]$ such that $f[[R^{S,\leq}]]g = 0$, then $f(u)Rg(v) = 0$ for each $u, v \in S$. Ali and Elshokry in [10], called such S-quasi-Armendariz ring and defined linearly S-quasi-Armendariz [34]. Marks et al. [12] a ring R is called (S, ω)-Armendariz, if whenever $f, g \in [[R^{S,\leq}, \omega]]$, $fg = 0$ implies $f(u)\omega_u(g(v)) = 0$ for all $u, v \in S$.

In this paper, we continue to study the concept of (S, ω)-quasi-Armendariz by [22], under some deferent conditions, which is unify the notions of (S, ω)-Armendariz and S-quasi-Armendariz ring. A ring R is called, (S, ω)-quasi-Armendariz, if whenever $f, g \in [[R^{S,\leq}, \omega]] \triangleq A$, $fAg = 0$ implies $f(u)R\omega_u(g(v)) = 0$ for all $u, v \in S$. We prove that, if I is a semiprime ring and R/I is $(S, \overline{\omega})$-quasi-Armendariz, where $\overline{\omega} : S \to End(R/I)$ is the induced monoid homomorphism, and R is a completely S-compatible ring, then R is (S, ω)-quasi-Armendariz (see Theorem 2.11), if the ring $[[R^{S,\leq}, \omega]]$ is $p.q.$Baer ring, then R is $p.q.$Baer and any S-indexed subset of $I(R)$ has a generalized join in $I(R)$ (see Theorem 3.3) and if $[[R^{S,\leq}, \omega]]$ is $p.q.$Baer ring, then R is (S,ω)-quasi-Armendariz (see Corollary 3.4). Under some additional conditions, the ring $[[R^{S,\leq},\omega]]$ is quasi-Baer (reflexive) if and only if R is quasi-Baer (reflexive, see Theorem 3.9 and Theorem 3.10, respectively). Also a necessary and sufficient condition is given for rings under which the ring $[[R^{S,\leq},\omega]]$ is (S,ω)-quasi-Armendariz.

2 (S,ω)-Quasi-Armendariz rings

In the following we discus some results for (S,ω)-Quasi-Armendariz rings which is an extend to the definition of S-Quasi-Armendariz rings. Clark defined quasi-Baer rings in [32]. A ring R is called quasi-Baer if the left annihilator of every left ideal of R is generated by an idempotent. Note that this definition is left-right symmetric. Some results of a quasi-Baer ring can be found in [27], [18] and [32] and used them to characterize when a finite dimensional algebra with unity over an algebraically closed field is isomorphic to a twisted matrix units semigroup algebra. As a generalization of quasi-Baer rings, Birkenmeier, Kim and Park in [19] introduced the concept of principally quasi-Baer rings. A ring R is called left principally quasi-Baer (or simply left $p.q.$-Baer) if the left annihilator of a principal left ideal of R is generated by an idempotent. Similarly, right $p.q.$-Baer rings can be defined. A ring is called $p.q.$-Baer if it is both right and left $p.q.$-Baer. Observe that biregular rings and quasi-Baer rings are $p.q.$-Baer. For more details and examples of left $p.q.$-Baer rings, see ([14]−[19]) and [38]. A ring R is called a right (resp., left) PP-ring if every principal right (resp., left) ideal is projective (equivalently, if the right (resp., left) annihilator of an element of R is generated (as a right (resp., left) ideal) by an idempotent of R). A ring R is called a PP-ring (also called a Rickart ring [5, p. 18]) if it is both right and left PP. We say a ring R is a left APP-ring if the left annihilator $l_R(Ra)$ is right s-unital as an ideal of R for any element $a \in R$. This concept is a common generalization of left $p.q.$-Baer rings and right PP-rings.

An ideal I of R is said to be right s-unital if, for each $a \in I$ there exists an element $e \in I$ such that $ae = a$. Note that if I and J are right s-unital ideals, then so is $I \cap J$ (if $a \in I \cap J$, then $a \in aIJ \subseteq a(I \cap J)$).

The following result follows from Tominaga Theorem 1 [21].

Lemma 2.1. *An ideal I of a ring R is left (resp. right) s-unital if and only if for any finitely many elements $a_1, a_2, \ldots, a_n \in I$, there exists an element $e \in I$ such that $a_i = ea_i (resp.\ a_i = a_i e)$ for each $i = 1, 2, \ldots, n$.*

Example 2.2. Here are some special cases of (S,ω)-quasi-Armendariz rings.
(1) Suppose R is quasi-Armendariz, as in [35]. This is the special case where $S = \mathbb{N} \cup \{0\}$ under addition, with the trivial order, and ω is trivial.
(2) Suppose R is σ-skew quasi-Armendariz for some $\sigma \in \mathrm{End}(R)$, as in [4]. This is the special case where $S = \mathbb{N} \cup \{0\}$ under addition, with the trivial order, and ω is determined by $\omega(1) = \sigma$.
(3) Suppose R is quasi-Armendariz relative to a monoid S, as in [42]. This is the special case where S is given the trivial order, and ω is trivial.
(4) Suppose R is S-quasi-Armendariz for some commutative, strictly ordered monoid $(S,\leq)$, as in [10]. This is the special case where ω is trivial (and S satisfies the extra conditions just described).

If $S = \{1\}$ then every ring is (S,ω)-quasi-Armendariz. In some of our results we will stipulate that $S \neq \{1\}$ to avoid trivialities.

Marks et al. in [12] they are studied compatibility of (S,ω)-Armendariz rings. To said that, when to suppose R is a ring and σ is an endomorphism of R. Then the skew power series ring $R[[x;\sigma]]$ is a skew generalized power series ring for $S = \mathbb{N} \cup \{0\}$ with natural order $\leq$ and $\omega(n) = \sigma^n$. Noted that for elements a and b of an (S,ω)-Armendariz ring R, if $ab = 0$, then $a\sigma(b) = 0$ (that is, 'half' of the definition of compatibility must hold). Indeed, define $f, g \in R[[x;\sigma]]$ as follows:

$$f = a - ax, \quad g = b + \sigma(b)x + \sigma^2(b)x + \cdots.$$

Then $fg = 0$, and invoking the (S,ω)-Armendariz condition for the constant coefficient of f and the x-coefficient of g yields $a\sigma(b) = 0$.

Definition 2.3. [12] Let R be a ring, $(S,\le)$ a strictly ordered monoid and $\omega : S \to \mathrm{End}(R)$ a monoid homomorphism. We say that R is S-compatible (S-rigid) if ω_s is compatible (rigid) for every $s \in S$; to indicate the homomorphism ω, we will sometimes say that R is (S,ω)-compatible ((S,ω)-rigid).

The following result appeared in [12].

Lemma 2.4. *Let R be a ring, $(S,\le)$ a strictly ordered monoid and $\omega : S \to End(R)$ a monoid homomorphism. Then $[[R^{S,\le},\omega]]$ is reduced if and only if R is reduced.*

Proposition 2.5. *Let $(S,\le)$ be a strictly totally ordered monoid, $\omega : S \to End(R)$ a monoid homomorphism and R is S-compatible. Then, R is (S,ω)-quasi-Armendariz if and only if, for any $f,g \in [[R^{S,\le},\omega]]$, $f[[R^{S,\le},\omega]]g = 0$ implies $f(u)Rg(v) = 0$ for all $u,v \in S$.*

Proof. It follows from the definition of (S,ω)-quasi-Armendariz. □

For every nonempty subset X of R, we denote $[[X^{S,\le},\omega]] = \{f \in [[R^{S,\le},\omega]] \mid f(s) \in X \cup \in \{0\}$ for every $s \in S\}$.

Proposition 2.6. *Let $(S,\le)$ be a strictly totally ordered monoid, $\omega : S \to End(R)$ a monoid homomorphism and R is S-compatible. Then the following conditions are equivalent:*
(1) *R is (S,ω)-quasi-Armendariz.*
(2) *For any $f \in [[R^{S,\le},\omega]]$, $r_{[[R^{S,\le},\omega]]}(f[[R^{S,\le},\omega]]) = [[r_R(I)^{S,\le},\omega]]$, where I be the right ideal of R generated by $\{f(u) \mid u \in S\}$.*

Proof. (1) $\Rightarrow$ (2) Assume that $g \in r_{[[R^{S,\le},\omega]]}(f[[R^{S,\le},\omega]])$. By (1), $f(u)Rg(v) = 0$ for all $u,v \in S$. So $g(v) \in r_R(f(u)R)$ for every $u,v \in S$. So $g \in [[r_R(I)^{S,\le},\omega]]$. Conversely, suppose that $g \in [[r_R(I)^{S,\le},\omega]]$. Then $g(v) \in r_R(I)$ for each $v \in S$. So $f(u)Rg(v) = 0$ for all $u,v \in S$. Since R is S-compatible, we have $f(u)R\omega_t(g(v)) = 0$ for any $u,v,t \in S$. So by compatibility again $f(u)\omega_u(R\omega_t(g(v))) = 0$, and hence $f(u)\omega_u(h(t)\omega_t(g(v))) = 0$, for any $u,v,t \in S$ and any $h \in [[R^{S,\le},\omega]]$. Thus for any $h \in [[R^{S,\le},\omega]]$ and any $s \in S$,

$$(fhg)(s) = \sum_{(u,t,v)\in X_s(f,h,g)} f(u)\omega_u(h(t)\omega_t(g(v))) = 0.$$

So, $g \in r_{[[R^{S,\le},\omega]]}(f[[R^{S,\le},\omega]])$.

(2) $\Rightarrow$ (1). Assume that $f[[R^{S,\le},\omega]]g = 0$ for elements $f,g \in [[R^{S,\le},\omega]]$. So

$$g \in r_{[[R^{S,\le},\omega]]}(f[[R^{S,\le},\omega]]).$$

By (2) $g \in [[r_R(I)^{S,\le},\omega]]$, where I be the right ideal of R generated by $\{f(u) \mid u \in S\}$. Therefore $f(u)Rg(v) = 0$ for all $u,v \in S$. □

Lemma 2.7. *Let $(S,\le)$ be a strictly totally ordered monoid, $\omega : S \to End(R)$ a monoid homomorphism and R is S-compatible. Then for any $a \in R$,*

$$[[r_R(aR)^{S,\le},\omega]] = r_{[[R^{S,\le},\omega]]}(c_a[[R^{S,\le},\omega]]).$$

Proof. Let $g \in r_{[[R^{S,\le},\omega]]}(c_a[[R^{S,\le},\omega]])$. Then for every $r \in R$,

$$0 = (c_a c_r g)(s) = \sum_{(u,v)\in X_s(c_{ar},g)} c_{ar}(u)\omega_u(g(v)) = arg(s).$$

Thus $aRg(s) = 0$, for every $s \in S$. Hence $g \in [[r_R(aR)^{S,\le},\omega]]$.

Conversely let $g \in [[r_R(aR)^{S,\le},\omega]]$. So $aRg(v) = 0$, for every $v \in S$. Thus by S-compatible of R, for every $f \in [[R^{S,\le},\omega]]$, $af(u)R\omega_u(g(v)) = 0$ for every $u,v \in S$. For any $s \in S$,

$$(c_a fg)(s) = \sum_{(u,t,v)\in X_s(c_a,f,g)} c_a(u)\omega_u(f(t)\omega_t(g(v)))$$

$$= \sum_{(t,v)\in X_s(f,g)} af(t)\omega_t(g(v)) = 0.$$

Therefore, $g \in r_{[[R^{S,\le},\omega]]}(c_a[[R^{S,\le},\omega]])$ and the result follows. □

Lemma 2.8. *Let $(S,\leq)$ be a strictly totally ordered monoid, $\omega : S \to End(R)$ a monoid homomorphism and R is S-compatible. If $[[R^{S,\leq},\omega]]$ is a right APP-ring, then R is a right APP-ring.*

Proof. Let $a,b \in R$ and $bRa = 0$. Then

$$[[r_R(bR)^{S,\leq},\omega]] = r_{[[R^{S,\leq},\omega]]}(c_b[[R^{S,\leq},\omega]]),$$

by Lemma 2.7. Since $[[R^{S,\leq},\omega]]$ is right APP, $r_{[[R^{S,\leq},\omega]]}(c_b[[R^{S,\leq},\omega]])$ is left s-unital. Since $a \in r_R(bR)$, we have $c_b[[R^{S,\leq},\omega]]c_a = 0$. So there exists, $f \in r_{[[R^{S,\leq},\omega]]}(c_b[[R^{S,\leq},\omega]])$ such that $c_a = fc_a$. Then $a = c_a(0) = (fc_a)(0) = f(0)a$. Since $c_b[[R^{S,\leq},\omega]]f = 0$ and $[[r_R(bR)^{S,\leq},\omega]] = r_{[[R^{S,\leq},\omega]]}(c_b[[R^{S,\leq},\omega]])$, we conclude that $f(0) \in r_R(bR)$. Therefore $r_R(bR)$ is left s-unital. This means that R is a right APP-ring. □

Theorem 2.9. *Let $(S,\leq)$ be a strictly totally ordered monoid and $\omega : S \to End(R)$ a monoid homomorphism. Assume that R is S-compatible and $[[R^{S,\leq},\omega]]$ is a right APP-ring, then R is (S,ω)-quasi-Armendariz.*

Proof. Apply Lemma 2.8 and Proposition 2.9 [10], respectively. □

Definition 2.10. [11, Definition 2.24] Let R be a ring, $(S,\leq)$ a strictly ordered monoid and $\omega : S \to \mathrm{End}(R)$ a monoid homomorphism. We say that a ring R is completely S-compatible if, for any ideal I of R, R/I is S-compatible, to indicate the homomorphism ω, we will sometimes say that R is completely (S,ω)-compatible.

Clearly, every completely S-compatible ring is S-compatible. Another description of complete S-compatibility of R that we shall often use is that for all $I \subseteq R$, $a,b \in R$, we have $ab \in I \Leftrightarrow a\omega(b) \in I$.

Theorem 2.11. *Let $(S,\leq)$ be a strictly totally ordered monoid, $\omega : S \to End(R)$ a monoid homomorphism and I an ideal of R with $\omega_s(I) \subseteq I$ for all $s \in S$. Assume that R is a completely S-compatible ring. If I is a semiprime ring and R/I is $(S,\overline{\omega})$-quasi-Armendariz, where $\overline{\omega} : S \to End(R/I)$ is the induced monoid homomorphism, then R is (S,ω)-quasi-Armendariz.*

Proof. Let $0 \neq f,g \in [[R^{S,\leq},\omega]]$ be such that $f[[R^{S,\leq},\omega]]g = 0$. Assume that $\pi(f) = u_0$, $\pi(g) = v_0$. Then for any $(u,v) \in X_{u_0+v_0}(f,g)$, $u_0 \leq u, v_0 \leq v$. If $u_0 < u$, since $\leq$ is a strict order, $u_0 + v_0 < u + v_0 \leq u + v = u_0 + v_0$, a contradiction. Thus $u = u_0$. Similarly, $v = v_0$. Note that for $\overline{f},\overline{g}$ the corresponding skew generalized power series of f and g in $[[(R/I)^{S,\leq},\overline{\omega}]]$, $\overline{f}[[(R/I)^{S,\leq},\overline{\omega}]]\overline{g} = \overline{0}$. Thus, we have $f(u)Rg(v) \subseteq I$, for each $u,v \in S$, since R/I is $(S,\overline{\omega})$-quasi-Armendariz and by the definition 2.10. For any $r \in R$,

$$0 = (fc_rg)(u_0+v_0) = \sum_{(u,v)\in X_{u_0+v_0}(f,c_rg)} f(u)\omega_u(rg(v)) = f(u_0)\omega_{u_0}(rg(v_0)).$$

Then $f(u_0)Rg(v_0) = 0$ by the compatibility of ω.

Now, let $\lambda \in S$ with $u_0 + v_0 \leq \lambda$ and assume that for any $u \in supp(f)$ and any $v \in supp(g)$, if $u+v < \lambda$, then $f(u)R\omega_u(g(v)) = 0$. We claim that $f(u)R\omega_u(g(v)) = 0$, for each $u \in supp(f)$ and each $v \in supp(g)$ with $u + v = \lambda$. For convenience, we write

$$X_\lambda(f,g) = \{(u,v) \mid u+v = \lambda, u \in supp(f), v \in supp(g)\}$$

as $\{(u_i,v_i) \mid i = 1,2,\ldots,n\}$ such that $u_1 < u_2 < \cdots < u_n$, where n is a positive integer (Note that if $u_1 = u_2$, then from $u_1 + v_1 = u_2 + v_2$ we have $v_1 = v_2$, and then $(u_1,v_1) = (u_2,v_2)$). Then for any $r \in R$, we have;

$$0 = (fc_rg)(\lambda) = \sum_{(u,v)\in X_\lambda(f,c_rg)} f(u)\omega_u(rg(v)) = \sum_{i=1}^{n} f(u_i)\omega_{u_i}(rg(v_i)). \tag{2.1}$$

Let p be an element of R. Multiplying Eq. (2.4) by $f(u_1)p$, from the left side, we can get $\sum_{i=1}^{n} f(u_1)pf(u_i)\omega_{u_i}(rg(v_i)) = 0$. Note that any $i \geq 2$, $u_1 < u_i$, then $u_1 + v_i < u_i + v_i = \lambda$ for any $i \geq 2$. Thus, $f(u_1)R\omega_{u_1}(g(v_i)) = 0$ for any $i \geq 2$ by the induction hypothesis. Hence

$f(u_1)Rg(v_i) = 0$ and so $f(u_1)R\omega_{u_1}(g(v_i)) = 0$ by the compatibility of ω. So we can get $f(u_1)Rf(u_1)Rg(v_1) = 0$. Hence, $(Rf(u_1)Rf(u_1)Rg(v_1))^2 = 0$. Since $Rf(u_1)Rg(v_1)R \subseteq I$ and I is semiprime ring, $f(u_1)Rg(v_1) = 0$. Now Eq. (2.4) becomes

$$\sum_{i=2}^{n} f(u_i)\omega_{u_i}(rg(v_i)) = 0. \tag{2.2}$$

Multiplying Eq. (2.5) by $f(u_2)p$ from the left side, we obtain $\sum_{i=2}^{n} f(u_2)pf(u_i)\omega_{u_i}(rg(v_i)) = 0$. Thus, $f(u_2)R\omega_{u_2}(g(v_i)) = 0$ for any $i \geq 3$ by the induction hypothesis. Hence $f(u_2)Rg(v_i) = 0$ and so $f(u_2)R\omega_{u_2}(g(v_i)) = 0$ by the compatibility of ω. So we can get $f(u_2)Rf(u_2)Rg(v_2) = 0$. Hence, $(Rf(u_2)Rf(u_2)Rg(v_2))^2 = 0$. Since $Rf(u_2)Rg(v_2)R \subseteq I$ and I is semiprime ring, $f(u_2)Rg(v_2) = 0$ in the same way as above. Continuing this process, we can prove $f(u_i)Rg(v_j) = 0$ for any i, j. Thus $f(u)Rg(v) = 0$ for any $u \in supp(f)$ and $v \in supp(g)$ with $u+v = \lambda$. Therefore, by transfinite induction, $f(u)Rg(v) = 0$ for any $u \in supp(f)$ and $v \in supp(g)$. Thus, R is (S,ω)-quasi-Armendariz. □

Since any reduced ring is a semiprime. Here we have.

Corollary 2.12. *Let R be a ring, $(S,\leq)$ a strictly totally ordered monoid, $\omega : S \to End(R)$ a monoid homomorphism and I an ideal of R and R is a completely S-compatible ring. If I is reduced and R/I is $(S,\overline{\omega})$-quasi-Armendariz, where $\overline{\omega} : S \to End(R/I)$ is the induced monoid homomorphism, then R is (S,ω)-quasi-Armendariz.*

Corollary 2.13. *Let R be a ring, $(S,\leq)$ a strictly totally ordered monoid, $\omega : S \to End(R)$ a compatible monoid homomorphism. If R is reduced, then R is (S,ω)-quasi-Armendariz.*

Corollary 2.14. *Let $(S,\leq)$ be a strictly totally ordered monoid, $\omega : S \to End(R)$ a monoid homomorphism and I an ideal of R. Assume that R is a completely S-compatible ring. If I is reduced and R/I is $(S,\overline{\omega})$-Armendariz, where $\overline{\omega} : S \to End(R/I)$ is the induced monoid homomorphism, then R is (S,ω)-Armendariz.*

Corollary 2.15. *Let S be a commutative, cancellative and torsion-free monoid, $\omega : S \to End(R)$ a monoid homomorphism. Assume that R is a completely S-compatible ring. If one of the following conditions holds, then R is (S,ω)-quasi-Armendariz.*
(1) R is reduced.
(2) R/I is $(S,\overline{\omega})$-quasi-Armendariz for some ideal I of R and I is reduced, where $\overline{\omega} : S \to End(R/I)$ is the induced monoid homomorphism.

Proof. If S is commutative, cancellative and torsion-free, then by Ribenboim [28] there exists a compatible strict total ordered $\leq$ on S. Now the results follows from Corollaries 2.12 and 2.13. □

Corollary 2.16. *[9, Proposition 1.10] Let M be a strictly totally ordered monoid and I an ideal of R. If I is a semiprime ring and R/I is quasi-Armendariz relative to a monoid, then R is quasi-Armendariz relative to a monoid.*

Let I be an index set and R_i be a ring for each $i \in I$. Let $(S,\leq)$ be a strictly ordered monoid and $\omega^i : S \to End(R_i)$ a monoid homomorphism. Then the mapping $\omega : S \to End(\prod_{i\in I} R_i)$ is a monoid homomorphism given by $\omega_s(\{r_i\}_{i\in I}) = \{(\omega^i)_s(r_i)\}_{i\in I}\}$ for all $s \in S$.

Proposition 2.17. *Let R_i be a ring, $(S,\leq)$ a strictly totally ordered monoid, $\omega^i : S \to End(R_i)$ a compatible monoid homomorphism, for each i in a finite index set I. If R_i is (S,ω^i)-quasi-Armendariz for each i, then $R = \prod_{i\in I} R_i$ is (S,ω)-quasi-Armendariz, where $\omega = \prod_{i\in I}\omega^i$.*

Proof. Let $R = \prod\limits_{i\in I} R_i$ be the direct product of rings $(R_i)_{i\in I}$ and R_i is (S,ω^i)-quasi-Armendariz for each $i \in I$. Denote the projection $R \to R_i$ as Π_i. Suppose that $f, g \in [[R^{S,\leq},\omega]]$ are such that $f[[R^{S,\leq},\omega]]g = 0$, where $\omega = \prod\limits_{i\in I}\omega^i$. Set $f_i = \prod_i f$, $g_i = \prod_i g$ and $h_i = \prod_i h$. Then

$f_i, g_i \in [[R_i^{S,\leq}, \omega^i]]$. For any $u, v \in S$, assume $f(u) = (a_i^u)_{i\in I}$, $g(v) = (b_i^v)_{i\in I}$. Now, for any $h \in [[R^{S,\leq}, \omega]]$, any $r \in R$ and any $s \in S$,

$$\begin{aligned}(fc_rg)(s) &= \sum_{(u,v)\in X_s(f,c_rg)} f(u)\omega_u(rg(v))\\ &= \sum_{(u,v)\in X_s(f,c_rg)} (a_i^u)_{i\in I}(\prod_{i\in I}\omega^i)_u(r_ib_i^v)_{i\in I}\\ &= \sum_{(u,v)\in X_s(f,c_rg)} (a_i^u)_{i\in I}(\prod_{i\in I}\omega_u^i)(r_ib_i^v)_{i\in I}\\ &= \sum_{(u,v)\in X_s(f,c_rg)} (a_i^u\omega_u^i(r_ib_i^v))_{i\in I}\\ &= \sum_{(u,v)\in X_s(f,c_rg)} (f_i(u)\omega_u^i(r_ig_i(v)))_{i\in I}\\ &= \Big(\sum_{(u,v)\in X_s(f,c_rg)} f_i(u)\omega_u^i(r_ig_i(v)\Big)_{i\in I}\\ &= \Big(\sum_{(u,v)\in X_s(f_i,c_{r_i}g_i)} f_i(u)\omega_u^i(r_ig_i(v))\Big)_{i\in I}\\ &= ((f_ih_ig_i)(s))_{i\in I}.\end{aligned}$$

Since $(fc_rg)(s) = 0$ we have

$$(f_ic_{r_i}g_i)(s) = 0.$$

Thus, $f_ih_ig_i = 0$. Now it follows $f_i(u)\omega_u^i(r_ig_i(v)) = 0$ for any $r \in R$, any $u, v \in S$ and any $i \in I$, since R_i is (S, ω^i)-quasi-Armendariz. Hence, for any $u, v \in S$,

$$f(u)\omega_u(rg(v)) = (f_i(u)\omega_u^i(r_ig_i(v)))_{i\in I} = 0$$

since I is finite. Thus, $f(u)Rg(v) = 0$ by the compatibility of ω. This means that R is (S, ω)-quasi-Armendariz. □

3 Some results on ring extensions of skew generalized power series quasi-Armendariz

Recall that an idempotent $e \in R$ is left (resp. right) semicentral in R if $ere = re$ (resp. $ere = er$) for all $r \in R$ (see [20]). Equivalently, $e^2 = e \in R$ is left (resp. right) semicentral if eR (resp. Re) is an ideal of R. Since the right annihilator of a right ideal is an ideal, we see that the right annihilator of a principal right ideal is generated by a left semicentral idempotent in a right $p.q.$Baer ring. The set of all left semicentral idempotents of R is denoted by $S_l(R)$.

The following result appeared in Lemma 3 [39].

Lemma 3.1. *Let R be a ring and $(S, \leq)$ a strictly totally ordered monoid satisfying that $0 \leq s$ for all $s \in S$. If $\phi \in [[R^{S,\leq}]]$ is a left semicemtral idempotent, then $\phi(0) \in R$ is a left semicentral idempotent and $\phi[[R^{S,\leq}]] = c_{\phi(0)}[[R^{S,\leq}]]$.*

Let $I(R)$ be the set of all idempotents of R. G be a subset of $I(R)$. We say that G is S-indexed if there exists an artinian and narrow subset I of S such that G is indexed by I (see [41]).

Definition 3.2. [40] Let G be an S-indexed subset of $I(R)$. We say that G has a generalized join in $I(R)$ if there exists an idempotent $e \in I(R)$ such that
(1) $gR(1-e) = 0$ for any $g \in G$, and
(2) If $f \in I(R)$ is such that $gR(1-f) = 0$ for any $g \in G$, then $eR(1-f) = 0$.

Theorem 3.3. *Let R be a ring, $(S, \leq)$ a strictly totally ordered monoid, $\omega : S \to End(R)$ a monoid homomorphism and R is S-compatible satisfying the condition that $0 \leq s$ for all $s \in S$. If $[[R^{S,\leq}, \omega]]$ is right $p.q.$Baer, then R is right $p.q.$Baer and any S-indexed subset of $I(R)$ has a generalized join in $I(R)$.*

Proof. Let a be an element of R. Then, by Lemma 2.7, $r_{[[R^{S,\leq},\omega]]}(c_a[[R^{S,\leq}, \omega]]) = [[r_R(aR)^{S,\leq}, \omega]]$. On the other hand, since $[[R^{S,\leq}, \omega]]$ is right $p.q.$Baer, there exists a left semicentral idempotent $f \in [[R^{S,\leq}, \omega]]$ such that $r_{[[R^{S,\leq},\omega]]}(c_a[[R^{S,\leq}, \omega]]) = f[[R^{S,\leq}, \omega]]$. We will show that

$r_R(aR) = f(0)R$ with $f(0)^2 = f(0)$, which will imply that R is a $p.q.$Baer. By Lemma 3.1, $f(0)$ is an idempotent of R and

$$f[[R^{S,\leq},\omega]] = c_{f(0)}[[R^{S,\leq},\omega]].$$

Thus, by compatibility, for any $r \in R$,

$$c_a c_r c_{f(0)} = 0,$$

which implies that $arf(0) = 0$. Hence $f(0) \in r_R(aR)$. Conversely, assume that $b \in r_R(aR)$. Then for any $g \in [[R^{S,\leq},\omega]]$ and any $v \in S$,

$$(c_a g c_b)(v) = ag(v)b = 0,$$

Thus, $c_a g c_b = 0$. This means that $c_b \in r_{[[R^{S,\leq},\omega]]}(c_a[[R^{S,\leq},\omega]])$. So $c_b = c_{f(0)}h$ for some $h \in [[R^{S,\leq},\omega]]$, which implies that $b \in f(0)R$. Thus, $r_R(aR) = f(0)R$. This means that R is right $p.q.$Baer ring.

Suppose that G is an S-indexed subset of $I(R)$. Then there exists an artinian and narrow subset I of S such that $G = \{e_s \in I(R) \mid s \in I\}$. Define $\phi \in [[R^{S,\leq},\omega]]$ via

$$\phi(s) = \begin{cases} e_s, & s \in I; \\ 0, & s \notin I. \end{cases}$$

Since $[[R^{S,\leq},\omega]]$ is right $p.q.$Baer, there exists a left semicentral idempotent $f \in [[R^{S,\leq},\omega]]$ such that

$$r_{[[R^{S,\leq},\omega]]}(\phi[[R^{S,\leq},\omega]]) = f[[R^{S,\leq},\omega]].$$

By Lemma 3.1, $f(0)$ is an idempotent of R and

$$c_{f(1)}[[R^{S,\leq},\omega]] = f[[R^{S,\leq},\omega]].$$

Thus,

$$r_{[[R^{S,\leq},\omega]]}(\phi[[R^{S,\leq},\omega]]) = c_{f(0)}[[R^{S,\leq},\omega]].$$

Now for any $r \in R$, $0 = (\phi c_r c_{f(0)})(s) = \phi(s)rf(0)$. Thus, $e_s rf(0) = 0$, for all $s \in I$. Let $g = 1 - f(0)$. Then $e_s r(1-g) = 0$, for all $r \in R$. Thus, $e_s R(1-g) = 0$. Suppose that e is an idempotent of R such that $e_s R(1-e) = 0$. Then $e_s re = e_s r$, for all $r \in R$. Thus, for any $a \in R$ and for any $\psi \in [[R^{S,\leq},\omega]]$, any $t \in S$,

$$(\phi\psi c_a c_{1-e})(t) = \sum_{(u,v)\in X_t(\phi,\psi)} \phi(u)\omega_u(\psi(v)a(1-e)) = 0.$$

This means that

$$c_a c_{1-e} \in r_{[[R^{S,\leq},\omega]]}(\phi[[R^{S,\leq},\omega]]),$$

for all $a \in R$. Thus, $c_a c_{1-e} = c_{f(0)} c_a c_{1-e}$, which implies that $ga(1-e) = 0$, for all $a \in R$. Thus, $gR(1-e) = 0$. Hence g is a generalized join of the S-indexed subset G. □

Corollary 3.4. *Let R be a ring, $(S,\leq)$ a strictly totally ordered monoid, $\omega : S \to End(R)$ a monoid homomorphism and R is S-compatible satisfying the condition that $0 \leq s$ for all $s \in S$. If $[[R^{S,\leq},\omega]]$ is right $p.q.$Baer, then R is (S,ω)-quasi-Armendariz.*

Ali and Elshokry [10], observed that, the relations between the right (left) annihilators in the ring R and the right (left) annihilators in generalized power series $[[R^{S,\leq}]]$, when R is an S-quasi-Armendariz ring.

In this note we investigate the relations between the right (left) annihilators in the ring R and the right (left) annihilators in the skew generalized power series $[[R^{S,\leq},\omega]]$. In this case R is (S,ω)-quasi-Armendariz.

For a subset U of R, we define the following:
$r_R(U) = \{f \in [[R^{S,\leq},\omega]] | c_r f = 0 \text{ for all } r \in U\}$,
$[[r_R(U)^{S,\leq},\omega]] = \{f \in [[R^{S,\leq},\omega]] | f(s) \in r_R(U) \text{ for all } s \in supp(f)\}$.

Lemma 3.5. *Let R be a ring, $(S,\leq)$ a strictly totally ordered monoid and $\omega : S \to End(R)$ a compatible monoid homomorphism. If $U \subseteq R$, then*

$$([[R^{S,\leq},\omega]])\ell_R(U) = \ell_{[[R^{S,\leq},\omega]]}(U), (r_R(U)[[R^{S,\leq},\omega]] = r_{[[R^{S,\leq},\omega]]}(U)).$$

Proof. Let $f \in r_R(U)[[R^{S,\leq},\omega]]$. Then, for each $u \in supp(f)$ we have $f(u) \in r_R(U)$. Thus, for each $v \in U$ we have $0 = (c_v f)(u) = c_v(0)\omega_0(f(u)) = vf(u)$. Consequently, $f(u) \in r_R(U)$ for each $u \in supp(f)$. Hence, $f \in r_{[[R^{S,\leq},\omega]]}(U)$ and it follows that $r_R(U)[[R^{S,\leq},\omega]] \subseteq r_{[[R^{S,\leq},\omega]]}(U)$.

Conversely, let $f \in r_{[[R^{S,\leq},\omega]]}(U)$. Then for each $f(u) \in r_R(U)$ for each $u \in supp(f)$. So, for each $v \in U$ and $u \in supp(f)$, we have, $0 = vf(u) = v\omega_o(f(u)) = c_v(0)\omega_0(f(u)) = (c_v f)(u)$. Consequently, $f \in r_R(U)[[R^{S,\leq},\omega]]$ and it follows that $r_{[[R^{S,\leq},\omega]]}(U) \subseteq r_R(U)[[R^{S,\leq},\omega]]$. So, $r_R(U)[[R^{S,\leq},\omega]] = r_{[[R^{S,\leq},\omega]]}(U)$. The proof of the left is similar. □

By Lemma 3.5, we have two maps $\phi : rAnn_R(id(R)) \to rAnn_{[[R^{S,\leq},\omega]]}(id([[R^{S,\leq},\omega]]))$ and $\psi : \ell Ann_R(id(R)) \to \ell Ann_{[[R^{S,\leq},\omega]]}(id([[R^{S,\leq},\omega]]))$ defined by $\phi(I) = I[[R^{S,\leq},\omega]]$ and $\psi(J) = [[R^{S,\leq},\omega]]J$ for every $I \in rAnn_R(id(R)) = \{r_R(U)|U$ is an ideal of $R\}$ and $J \in \ell Ann_R(id(R)) = \{\ell_R(U)|U$ is an ideal of $R\}$, respectively. Obviously, ϕ is injective.

In the following Theorem we show that ϕ and ψ are bijective maps if and only if R is (S,ω)-quasi-Armendariz. This Theorem is a generalization of a result of Hashemi and Moussavi Proposition 2.5 [8] that generalizes a result of Hirano Proposition 3.4 [35]).

Theorem 3.6. *Let R be a ring, $(S,\leq)$ a strictly totally ordered monoid and $\omega : S \to End(R)$ a compatible monoid homomorphism. If $[[R^{S,\leq},\omega]]$ the skew generalized power series, then the following are equivalent:*
(1) R is (S,ω)-quasi-Armendariz.
(2) ϕ is a bijective.
(3) ψ is a bijective.

Proof. (1)⇒(2) Let $Y \subseteq [[R^{S,\leq},\omega]]$ and $\gamma = \cup_{f\in Y} C(f)$. From Lemma 3.5 it is sufficient to show that $r_{[[R^{S,\leq},\omega]]}(f) = r_R C(f)[[R^{S,\leq},\omega]]$ for all $f \in Y$. In fact, let $g \in r_{[[R^{S,\leq},\omega]]}(f)$ and for any $h \in [[R^{S,\leq},\omega]]$. Then $fhg = 0$ and by assumption $f(u_i)tg(v_j) = 0$ for each $u_i \in supp(f), t \in R$ and each $v_j \in supp(g)$ since R is an (S,ω)-quasi-Armendariz and compatibility of ω. Then for a fixed $u_i \in supp(f), t \in R$ and each $v_j \in supp(g), 0 = f(u_i)tg(v_j) = (c_{f(u_i)}c_t g)(v_j)$ and it follows that $g \in r_R \cup_{u_i \in supp(f)} c_{f(u_i)}c_t[[R^{S,\leq},\omega]] = r_R C(f)[[R^{S,\leq},\omega]]$. So $r_{[[R^{S,\leq},\omega]]}(f) \subseteq r_R C(f)[[R^{S,\leq},\omega]]$.

Conversely, let $g \in r_R C(f)[[R^{S,\leq},\omega]]$, then $c_{f(u_i)}c_t g = 0$ for each $u_i \in supp(f), t \in R$. Hence, $0 = (c_{f(u)}c_t g)(v) = f(u)tg(v)$ for each $u \in supp(f), t \in R$ and $v \in supp(g)$. Since R is an S-compatible, $f(u)\omega_u(tg(v)) = 0$ for each $u \in supp(f), t \in R$ and $v \in supp(g)$. Thus,

$$(fhg)(s) = \sum_{(u,v)\in X_s(f,c_t g)} f(u)\omega_u(tg(v)) = 0$$

and it follows that $g \in r_{[[R^{S,\leq},\omega]]}(f)$. Hence $r_R C(f)[[R^{S,\leq},\omega]] \subseteq r_{[[R^{S,\leq},\omega]]}(f)$ and it follows that $r_R C(f)[[R^{S,\leq},\omega]] = r_{[[R^{S,\leq},\omega]]}(f)$. So

$$r_{[[R^{S,\leq},\omega]]}(Y) = \cap_{f\in Y} r_{[[R^{S,\leq},\omega]]}(f) = \cap_{f\in Y} r_R C(f)[[R^{S,\leq},\omega]] = r_R(\gamma)[[R^{S,\leq},\omega]].$$

(2)⇒(1) Suppose that $f, g \in [[R^{S,\leq},\omega]]$ be such that $f[[R^{S,\leq},\omega]]g = 0$. Then $g \in r_{[[R^{S,\leq},\omega]]}(f)$ and by assumption $r_{[[R^{S,\leq},\omega]]}(f) = \gamma[[R^{S,\leq},\omega]]$ for some right ideal γ of R. Consequently, $0 = fc_t c_{g(v)}$ and for any $u \in supp(f), 0 = (fc_t c_{g(v)})(u) = f(u)\omega_u(tg(v))$ for each $u \in supp(f), t \in R$ and $v \in supp(g)$. Hence, R is (S,ω)-quasi-Armendariz. The proof of (1)⇔(3) is similar to the proof of (1)⇔(2). □

A submodule N of a left R-module M is called a pure submodule if $L \otimes_R N \to L \otimes_R M$ is a monomorphism for every right R-module L. By Proposition 11.3.13 [2], an ideal I is right s-unital if and only if R/I is flat as a left R-module if and only if I is pure as a left ideal of R.

Theorem 3.7. *Let R be a ring, $(S,\leq)$ a strictly totally ordered monoid, $\omega : S \to End(R)$ a monoid homomorphism and R is S-compatible. Then the following statements are equivalent:*
(1) $r_R(aR)$ is pure as a right ideal in R for any element $a \in R$;
(2) $r_{[[R^{S,\leq},\omega]]}(f[[R^{S,\leq},\omega]])$ is pure as a right ideal in $[[R^{S,\leq},\omega]]$ for any element $f \in [[R^{S,\leq},\omega]]$.
In this case R is an (S,ω)-quasi-Armendariz ring.

Proof. Assume that the condition (1) holds. Firstly, by using the same method of the proof of Proposition 2.9 [10] we can proved that R is an (S,ω)-quasi-Armendariz. Finally, by using Lemma 2.1 we can see that the condition (2) holds.

Conversely, suppose that the condition (2) holds. Let a be an element of R.
Then $r_{[[R^{S,\leq},\omega]]}(C_a[[R^{S,\leq},\omega]])$ is left s-unital. Hence, for any $b \in r_R(aR)$, there exists an element $f \in [[R^{S,\leq},\omega]]$ such that $bf = b$. Let $f(0)$ be the constant term of f. Then $f(0) \in r_R(aR)$ and $f(0)b = b$. This implies that $r_R(aR)$ is left s-unital. Therefore condition (1) holds. □

Corollary 3.8. *Let R be a commutative ring, $(S,\leq)$ a strictly totally ordered monoid, $\omega : S \to End(R)$ a monoid homomorphism and R is S-compatible. Then each principal ideal of R is flat if and only if each principal ideal of $[[R^{S,\leq},\omega]]$ is flat.*
In this case R is an (S,ω)-Armendariz ring.

Proof. For each $a \in R$, $R/r_R(a) \cong aR$ holds. Hence the result follows from Theorem 3.7. □

It was proved in Theorem 1.8 [20] that, a ring R is quasi-Baer if and only if $R[x]$ is quasi-Baer if and only if $R[[x]]$ is quasi-Baer.

Theorem 3.9. *Let R be a ring, $(S,\leq)$ a strictly totally ordered monoid and $\omega : S \to End(R)$ a monoid homomorphism. If R is S-compatible (S,ω)-quasi-Armendariz, and for any $\phi^2 = \phi$ there exists $e^2 = e$ such that $\phi = C_e$. Then, R is quasi-Baer ring if and only if $[[R^{S,\leq},\omega]]$ is quasi-Baer ring.*

Proof. $(\Rightarrow)$ Let $U \in [[R^{S,\leq},\omega]]$ is a subset, since R is quasi-Baer, there exist $e^2 = e \in R$ such that $r_R(C_U) = eR$, where $C_U R$ denotes generated by C_U subset of R. We want to show that $r_{[[R^{S,\leq},\omega]]}(U) = C_e[[R^{S,\leq},\omega]]$. For any $f,h \in [[R^{S,\leq},\omega]], g \in U, s \in S$

$$(ghC_ef)(s) \sum_{(u,v,t)\in X_s(g,h,f)} g(u)\omega_u(h(v)\omega_v(ef(t))).$$

Because $g(u)h(v) \in C_U R$ and $ef(t) \in eR$, so $(ghC_ef)(s) = 0$ therefore $ghC_ef = 0$. This means that $r_{[[R^{S,\leq},\omega]]}(U) \supseteq C_e[[R^{S,\leq},\omega]]$. Conversely, let $f \in r_{[[R^{S,\leq},\omega]]}(U), g \in U$, then $g[[R^{S,\leq},w]]f = 0$. Because R is S-compatible (S,ω)-quasi-Armendariz, for any $u \in S$ and any $v \in S$, we have $g(u)Rf(v) = 0$. This means for any $s \in S, f(s) \in r_R(C_U R)$. Therefore, there exist $r_s \in R$ such that $f(s) = er_s$. We have map $h : S \to R$ as follows

$$h(s) = \begin{cases} r_s, & s \in supp(f); \\ 0, & s \in S - supp(f), \end{cases}$$

so because $supp(h) = supp(f)$ we have $h \in [[R^{S,\leq},\omega]]$. Easy to show $f = C_eh \in C_e[[R^{S,\leq},\omega]]$, So $r_{[[R^{S,\leq},\omega]]}(U) \subseteq C_e[[R^{S,\leq},\omega]]$. Thus, $r_{[[R^{S,\leq},\omega]]}(U) = C_e[[R^{S,\leq},\omega]]$. Therefore, $[[R^{S,\leq},\omega]]$ is quasi-Baer ring.

$(\Leftarrow)$ For any subset $Q \in R$, let

$$V = \{f \in [[R^{S,\leq},\omega]] \mid f(s) \in Q, s \in S\}$$

and let $V[[R^{S,\leq},\omega]]$ denotes the subsets of $[[R^{S,\leq},\omega]]$, which is generated by V. Therefore there exist $e^2 = e \in R$ such that

$$r_{[[R^{S,\leq},\omega]]}(V[[R^{S,\leq},\omega]]) = C_e[[R^{S,\leq},\omega]].$$

We can show that $r_R(Q) = eR$. For any $q \in Q, r \in R$, we have $(C_qC_eC_r)(0) = qer = 0$. So $eR \subseteq r_R(Q)$, and let $a \in r_R(Q)$, because $C_a \in r_{[[R^{S,\leq},\omega]]}(V[[R^{S,\leq},\omega]])$, there exist $h \in [[R^{S,\leq},\omega]]$ such that $C_a = C_eh$. So $a = C_eh(0) = eh(0) \in eR$, this means that $r_R(Q) \subseteq eR$, so $r_R(Q) = eR$. Therefore, R is quasi-Baer ring. □

According to [13], a right ideal I is reflexive if $xRy \in I$ implies $yRx \in I$ for $x, y \in R$. Hence we shall call a ring R a reflexive ring if 0 is a reflexive ideal (i.e., $aRb = 0$ implies $bRa = 0$ for $a, b \in R$). Moreover, a right ideal I is called completely reflexive if $xy \in I$ implies $yx \in I$. A ring R is completely reflexive if (0) has the corresponding property. It is clear that every completely reflexive ring is reflexive. Here we have some results of reflexive ring, under the condition that (S, ω)-quasi-Armendariz.

Theorem 3.10. *Let R be a ring, $(S, \leq)$ a strictly totally ordered monoid and $\omega : S \to End(R)$ a monoid homomorphism. Assume that R is S-compatible (S, ω)-quasi-Armendariz. Then R is reflexive ring if and only if $[[R^{S,\leq}, \omega]]$ is reflexive.*

Proof. $(\Rightarrow)$ Let R be reflexive ring. Suppose that $f, g \in [[R^{S,\leq}, \omega]]$ are such that $f[[R^{S,\leq}, \omega]]g = 0$. Since R is (S, ω)-quasi-Armendariz, we have $f(u)R\omega_u(g(v)) = 0$ for any $u \in supp(f)$ and $v \in supp(g)$, so $f(u)Rg(v) = 0$ by compatibility. Since R is reflexive, we have $g(v)Rf(u) = 0$ for all $u, v \in S$. Now for any $h \in [[R^{S,\leq}, \omega]]$, and any $t, s \in S$,

$$(ghf)(s) = \sum_{(v,t,u)\in X_s(g,h,f)} g(v)h(t)f(u) = 0.$$

Thus $ghf = 0$. This show that $g[[R^{S,\leq}, \omega]]f = 0$. This means that $[[R^{S,\leq}, \omega]]$ is reflexive.
$(\Leftarrow)$ Let $a, b \in R$ be such that $aRb = 0$. Then $C_a[[R^{S,\leq}, \omega]]C_b = 0$. Hence $C_b[[R^{S,\leq}, \omega]]C_a$ $= 0$ by reflexive. So $bRa = 0$. Therefore R is reflexive. □

Proposition 3.11. *Let $(S, \leq)$ be a strictly totally ordered monoid and $\omega : S \to End(R)$ a monoid homomorphism. Assume that R is S-compatible left APP-ring. Then R is reflexive ring if and only if $[[R^{S,\leq}, \omega]]$ is reflexive.*

Proof. By Proposition 2.9 [10], if R is S-compatible left APP-ring, then R is (S, ω)-quasi-Armendariz. Thus, the result follows from Theorem 3.10. □

Proposition 3.12. *Let $(S, \leq)$ be a strictly totally ordered monoid and $\omega : S \to End(R)$ a compatible monoid homomorphism. Assume that R is (S, ω)-Armendariz and semicommutative. Then R is reflexive ring if and only if $[[R^{S,\leq}, \omega]]$ is reflexive.*

Corollary 3.13. *Let $(S, \leq)$ be a strictly totally ordered monoid and $\omega : S \to End(R)$ a compatible monoid homomorphism. Assume that R is a semiprime. Then R is reflexive ring if and only if $[[R^{S,\leq}, \omega]]$ is reflexive.*

Corollary 3.14. *Let $(S, \leq)$ be a strictly totally ordered monoid, $\omega : S \to End(R)$ a compatible monoid homomorphism and R a reduced ring. Then R is reflexive ring if and only if $[[R^{S,\leq}, \omega]]$ is reflexive.*

Proof. It follows from Lemma 2.4 and Proposition 3.12. □

Proposition 3.15. *Let $(S, \leq)$ be a strictly ordered monoid, $\omega : S \to End(R)$ a monoid homomorphism and e be a central idempotent of a ring R with $\omega_s(e) = e$. Then, R is (S, ω)-quasi-Armendariz if and only if eR and $(1 - e)R$ are (S, ω)-quasi-Armendariz.*

Proof. $(\Rightarrow)$. Suppose that R is (S, ω)-quasi-Armendariz. Let $f, g \in [[(eR)^{S,\leq}, \omega]]$ such that $f[[(eR)^{S,\leq}, \omega]]g = 0$. Note that $fe = f$ and $eg = g$. For any $r \in R$, $fc_rg = f(c_{er})g = 0$, and so $f[[R^{S,\leq}, \omega]]g = 0$. Since R is (S, ω)-quasi-Armendariz, $f(u)R\omega_u(g(v)) = 0$. Since e is central $f(u)(eR)\omega_u(g(v)) = 0$. Therefore, eR is (S, ω)-quasi-Armendariz. Similarly, we can show that $(1 - e)R$ is (S, ω)-quasi-Armendariz.

$(\Leftarrow)$. Assume that both eR and $(1 - e)R$ are (S, ω)-quasi-Armendariz. Let $f, g \in [[R^{S,\leq}, \omega]]$ be such that $f[[R^{S,\leq}, \omega]]g = 0$. We will show that $f(u)R\omega_u(g(v)) = 0$. For any $r \in R$, $c_e f(c_{er})c_e g = c_e(fc_r g) = 0$ and $c_{1-e}f(c_{(1-e)r})c_{1-e}g = c_{1-e}(f(c_r)g) = 0$, and so

$$c_e f[[(eR)^{S,\leq}, \omega]]c_e g = 0, c_{1-e}f[[((1 - e)R)^{S,\leq}, \omega]]c_{1-e}g = 0.$$

Since eR and $(1 - e)R$ are (S, ω)-quasi-Armendariz, we have $e(f(u)R\omega_u(g(v))) = 0$ and $(1 - e)(f(u)R\omega_u(g(v))) = 0$. Thus,

$$f(u)R\omega_u(g(v)) = e(f(u)R\omega_u(g(v))) + (1 - e)f(u)R\omega_u(g(v))) = 0.$$

Therefore, R is (S, ω)-quasi-Armendariz. □

Hirano, [35] showed that semiprime rings are quasi-Armendariz rings but not conversely. Moreover, he proved that the class of quasi-Armendariz rings is Morita stable Theorem 3.12 and Proposition 3.13 [35], extending the class of quasi-Armendariz rings, through several extensions. Most of these properties are not satisfied over Armendariz rings Examples 1 and 3 [25]. We will now prove a proposition that unifies some results in [1] and [35] within the context of skew generalized power series rings. To prove that, the class of (S,ω)-quasi-Armendariz rings is Morita stable, we need the following.

Proposition 3.16. *Let R be a ring, $(S,\leq)$ a strictly ordered monoid and $\omega : S \to End(R)$ a monoid homomorphism. If R is (S,ω)-quasi-Armendariz ring, then, for any nonzero idempotent $e \in R$, with $\omega_s(e) = e$ for all $s \in S$, eRe is (S,ω)-quasi-Armendariz ring.*

Proof. Let $f, g \in [[(eRe)^{S,\leq},\omega]]$ be an elements satisfying $f[[(eRe)^{S,\leq},\omega]]g = 0$. Since $fc_e = f$ and $c_e g = g$, we obtain $f[[R^{S,\leq},\omega]]g = 0$, and hence $f(u)R\omega_u(g(v)) = 0$. Since e is an idempotent we have $f(u)eRe\omega_u(g(v)) = 0$. Thus, eRe is (S,ω)-quasi-Armendariz. □

Corollary 3.17. *Let R be a ring, $(S,\leq)$ a strictly ordered monoid and $\omega : S \to End(R)$ a monoid homomorphism. If R is (S,ω)-quasi-Armendariz ring and if R is Morita equivalent to a ring T, then T is (S,ω)-quasi-Armendariz.*

Lemma 3.18. *[37, Proposition 3.7] Let $e \in R$ be an idempotent. If R is a left APP-ring, then eRe is a left APP-ring.*

Corollary 3.19. *Let $(S,\leq)$ be a strictly totally ordered monoid and $\omega : S \to$End(R) a monoid homomorphism. Assume that e be an idempotent. If R is left APP-ring. Then eRe is (S,ω)-quasi-Armendariz.*

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Received: January 13, 2020.
Accepted: December 3, 2021.