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VALUE DISTRIBUTION OF MEROMORPHIC FUNCTION CONCERNING CERTAIN SMALL FUNCTION

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Abstract The paper uses the theory of quasinormal families to study the value distribution of derivatives of meromorphic functions. Let R be a nonzero rational function, and let f be a meromorphic function, whose zeros are multiple. We prove that if $\rho(f) > 2$, then $f'(z) = a(z)$ has infinitely many solutions in the complex plane, where $a(z) = R(z)\tan z$ or $a(z) = R(z)\mathrm{e}^z$.

1 Introduction

Let $\mathbb{C}$ be the complex plane and D be a domain on $\mathbb{C}$. For $z_0 \in \mathbb{C}$ and $r > 0$, we write $\overline{\Delta}(z_0, r) := \{z| \, |z-z_0| \le r\}$, $\Delta(z_0, r) := \{z| \, |z-z_0| < r\}$, $\Delta := \Delta(0, 1)$ and $\Delta'(z_0, r) := \{z| \, 0 < |z-z_0| < r\}$. We write $f_n \overset{\chi}{\Rightarrow} f$ in D to indicate that the sequence $\{f_n\}$ converges to f in the spherical metric uniformly on compact subsets of D and $f_n \Rightarrow f$ in D if the convergence is in the Euclidean metric.

For f meromorphic on $\mathbb{C}$ and D a domain on $\mathbb{C}$, set

$$f^{\#}(z) := \frac{|f'(z)|}{1 + |f(z)|^2} \text{ and } S(D, f) := \frac{1}{\pi} \iint_D [f^{\#}(z)]^2 \mathrm{d}x\mathrm{d}y.$$

Set $S(r, f) := S(\Delta(0, r), f)$. The Ahlfors–Shimizu characteristic is defined by $T_0(r, f) = \int_0^r \frac{S(t,f)}{t} \mathrm{d}t$. Let $T(r, f)$ denote the usual Nevanlinna characteristic function. Since $T(r, f) - T_0(r, f)$ is bounded as a function of r, we can replace $T_0(r, f)$ with $T(r, f)$ in the sequel.

Recall that a family $\mathcal{F}$ of functions meromorphic in D is said to be quasinormal in D if from each sequence $\{f_n\} \subset \mathcal{F}$ one can extract a subsequence $\{f_{n_k}\}$ which converges locally uniformly with respect to the spherical metric in $D \setminus E$, where the set E (which may depend on $\{f_{n_k}\}$) has no accumulation points in D. For further details, please see [1, pp. 131–132].

Our point of departure is the following classical result of Hayman in the value distribution theory of meromorphic functions.

Theorem 1.1. [2] *Let $f(z)$ be a transcendental meromorphic function. If $f(z) \neq 0$ for each z, then $f'(z) = 1$ has infinitely many solutions in the complex plane.*

A meromorphic function $a(z)$ is called a small function with respect to $f(z)$ provided that $T(r, a(z)) = o\{T(r, f(z))\}$ as $r \to \infty$ outside of a possible exceptional set of r of finite linear measure.

We wonder if Theorem 1.1 still holds provided that the constant 1 is replaced by a small function $a(z)$ with respect to $f(z)$.

In 2008, Pang et al. gave a generalized version of Theorem 1.1.

Theorem 1.2. [3] *Let $\alpha(z)$ be a nonzero rational function, and let $f(z)$ be a transcendental meromorphic function, whose zeros are multiple. Then $f'(z) = \alpha(z)$ has infinitely many solutions in the complex plane.*

In 2013, Yang and Nevo [4] proved the following result.

Theorem 1.3. *Let $f(z)$ be a meromorphic function in the complex plane, whose zeros are multiple, and let $\alpha(z)$ be a nonconstant elliptic function such that $T(r,\alpha)=o\{T(r,f)\}$ as $r\to\infty$. Then $f'(z)=\alpha(z)$ has infinitely many solutions in the complex plane.*

In 2017, Yang et al. obtained the following result.

Theorem 1.4. [5] *Let $f(z)$ be a meromorphic function in the complex plane, whose poles are multiple and whose zeros have multiplicity at least 3. Let $\alpha(z):=\beta(z)\exp(\gamma(z))$, where $\beta(z)$ is a nonconstant elliptic function and $\gamma(z)$ is an entire function. If $\sigma(f(z))>\sigma(\alpha(z))$, then $f'(z)=\alpha(z)$ has infinitely many solutions in the complex plane.*

In this paper, we continue to consider the value distribution of derivatives of meromorphic functions.

2 Lemmas

Lemma 2.1. *Let $\mathcal{F}$ be a family of meromorphic functions in D, and suppose that there exists $M\geq 1$ such that $|f'(z)|\leq M$ whenever $f(z)=0$. Then if $\mathcal{F}$ is not normal at z_0, there exist, for each $0\leq\alpha\leq 1$,*

(a) *points z_n, $z_n\to z_0$,*

(b) *functions $f_n\in\mathcal{F}$, and*

(c) *positive numbers $\rho_n\to 0$*

such that $\rho_n^{-\alpha}f_n(z_n+\rho_n\zeta)=g_n(\zeta)\overset{\chi}{\Rightarrow}g(\zeta)$ on $\mathbb{C}$, where g is a nonconstant meromorphic function on $\mathbb{C}$ such that $g^{\#}(\zeta)\leq g^{\#}(0)=M+1$. In particular, g has order at most 2.

This is the local version of [6, Lemma 2] (cf. [7, Lemma 1]; [8, pp. 216-217]). The proof consists of a simple change of variable in the result cited from [6]; cf. [9, pp. 299-300]. For a thorough discussion of related issues, see [10].

Lemma 2.2. [4, Lemma 3.9] *Let $\{f_n\}$ be a family of meromorphic functions in D, all of whose zeros are multiple, and let $\{\psi_n\}$ be a sequence of meromorphic functions in D such that $\psi_n\overset{\chi}{\Rightarrow}\psi$ in D, where $\psi(z)\not\equiv 0,\infty$ in D. If for each $n\in\mathbb{N}$, $f_n'(z)\neq\psi_n(z)$ for all $z\in D$, then $\{f_n\}$ is quasinormal in D.*

Lemma 2.3. [11, Lemma 17] *Let $\{f_n\}$ be a family of meromorphic functions in D, whose zeros are multiple. Let $\{\psi_n\}$ be a sequence of meromorphic functions in D such that $\psi_n(z)\overset{\chi}{\Rightarrow}\psi(z)$ in D, where ψ is a nonzero holomorphic function in D. Let $E\subset D$ be a set which has no accumulation points in D. Assume that*

(a) *ψ and ψ_n have the same zeros with the same multiplicity;*

(b) *for each $n\in\mathbb{N}$ and each $z\in D$, $f_n'(z)\neq\psi_n(z)$;*

(c) *for each $a^*\in E$, no subsequence of $\{f_n\}$ is normal at a^*;*

(d) *$f_n(z)\overset{\chi}{\Rightarrow}f(z)$ in $D\setminus E$.*

Then

(e) *for each $a^*\in E$, there exist $r_{a^*}>0$ and $N_a>0$ such that for sufficiently large n, $n(\Delta(a^*,r_{a^*}),\frac{1}{f_n})<N_{a^*}$, where r_{a^*} and N_{a^*} only depend on a^*;*

(f) *for each $a^*\in E$, $f(z)=\int_{a^*}^{z}\psi(\zeta)\mathrm{d}\zeta$ in $D\setminus E$.*

Lemma 2.4. [12, Lemma 2.5] *Let $\{f_n\}$ be a family of meromorphic functions in $\Delta(z_0,r)$. Suppose that*

(a) *there exists $M_0>0$ such that $n\big(\Delta(z_0,r),\frac{1}{f_n}\big)\leq M_0$ for sufficiently large n, and*

(b) *$f_n\overset{\chi}{\Longrightarrow}f$ in $\Delta'(z_0,r)$, where $f(\not\equiv 0)$ may be ∞ identically.*

Then there exists $M > 0$ such that $S\big(\Delta(z_0, r/4),\ f_n\big) < M$ for sufficiently large n.

Lemma 2.5. *Let $f(z)$ be a meromorphic function of order $\rho(f) > 2$ in $\mathbb{C}$. Then there exist $\alpha_n \to \infty$ and $\delta_n \to 0$ such that $f^{\#}(\alpha_n) \to \infty$ and $S(\Delta(\alpha_n, \delta_n), f) \to \infty$ as $n \to \infty$.*

Remark 2.6. From the process of proof of Lemma 10 in [13] (see [13, p.12]), it is easy to see that Lemma 2.5 holds. A full and complete proof of Lemma 2.5 is given in [14, p.1278].

Lemma 2.7. *Let d be an integer, and let f be a transcendental meromorphic function, all but finite many of whose zeros are multiple. Set $g(z) := \frac{f(z)}{z^d}$. If $\rho(f) > 2$, then there exist $\alpha_n \to \infty$ and $\delta_n \to 0$ such that*

$$\frac{f(\alpha_n)}{\alpha_n^d} \to 0\,, \quad \frac{f'(\alpha_n)}{\alpha_n^d} \to \infty \ \text{ and } \ S(\Delta(\alpha_n, \delta_n), g) \to \infty \ \text{ as } \ n \to \infty.$$

Proof. Clearly, $\rho(g) > 2$. By Lemma 2.5, there exist $\beta_n \to \infty$ and $\varepsilon_n \to 0$ such that $g^{\#}(\beta_n) \to \infty$ and $S(\Delta(\beta_n, \varepsilon_n), g) \to \infty$ as $n \to \infty$.

Set $g_n(z) := g(z + \beta_n)$ for $z \in \Delta$. Clearly, $g_n^{\#}(0) = g^{\#}(\beta_n) \to \infty$ and hence $\{g_n\}$ is not normal at 0. It is also clear that all zeros of $g_n(z)$ are multiple for sufficiently large n in Δ. Using Lemma 2.1 for $\alpha = 1/2$, there exist points $z_n \to 0$, positive numbers $\rho_n \to 0$ and a subsequence of $\{g_n\}$ (still denoted by $\{g_n\}$) such that

$$G_n(\zeta) = \frac{g_n(z_n + \rho_n\zeta)}{\rho_n^{1/2}} \overset{\chi}{\Longrightarrow} G(\zeta) \ \text{ in } \ \mathbb{C},$$

where G is a nonconstant meromorphic function in $\mathbb{C}$, all of whose zeros are multiple.

Set $\alpha_n := \beta_n + z_n + \rho_n\zeta_0$, where ζ_0 is not a zero or pole of $G'(\zeta)$ (In fact, $G'(\zeta)$ is not a constant function. Otherwise, either G is a constant function, or G has a simple zero). Noting that

$$\frac{g(\alpha_n)}{\rho_n^{1/2}} = \frac{g_n(z_n + \rho_n\zeta_0)}{\rho_n^{1/2}} = G_n(\zeta_0) \to G(\zeta_0),$$

$$\rho_n^{1/2} g'(\alpha_n) = \rho_n^{1/2} g_n'(z_n + \rho_n\zeta_0) = G_n'(\zeta_0) \to G'(\zeta_0),$$

we have

$$\alpha_n \to \infty, \quad g(\alpha_n) \to 0 \ \text{ and } \ g'(\alpha_n) \to \infty.$$

A simple calculation shows that

$$\frac{f(\alpha_n)}{\alpha_n^d} = g(\alpha_n) \to 0,$$

$$\frac{f'(\alpha_n)}{\alpha_n^d} = \frac{\big(z^d g(z)\big)'}{\alpha_n^d}\Bigg|_{z=\alpha_n} = \frac{d}{\alpha_n} \cdot g(\alpha_n) + g'(\alpha_n) \to \infty.$$

Set $\delta_n := \varepsilon_n + |\alpha_n - \beta_n| = \varepsilon_n + |z_n + \rho_n\zeta_0|$. Obviously, $\delta_n \to 0$ and $\Delta(\beta_n, \varepsilon_n) \subset \Delta(\alpha_n, \delta_n)$, and hence $S(\Delta(\alpha_n, \delta_n), g) \to \infty$ as $n \to \infty$. □

Lemma 2.8. *Let d be an integer, and let f be a transcendental meromorphic function, all but finite many of whose zeros are multiple. Set $g(z) := \frac{f(z)}{z^d e^z}$. If $\rho(f) > 2$, then there exist $\alpha_n \to \infty$ and $\delta_n \to 0$ such that*

$$\frac{f(\alpha_n)}{\alpha_n^d e^{\alpha_n}} \to 0\,, \quad \frac{f'(\alpha_n)}{\alpha_n^d e^{\alpha_n}} \to \infty \ \text{ and } \ S(\Delta(\alpha_n, \delta_n), g) \to \infty \ \text{ as } \ n \to \infty.$$

Proof. Since $\rho(z^d e^z) = 1$, we have $\rho(g) > 2$. Set $g_n(z) := g(z + \beta_n)$ for $z \in \Delta$.

Using the same argument as in Lemma 2.7, we can show that

$$\alpha_n \to \infty, \quad g(\alpha_n) \to 0 \ \text{ and } \ g'(\alpha_n) \to \infty,$$

where α_n has the same definition as in Lemma 2.7. A simple calculation shows that

$$\frac{f(\alpha_n)}{\alpha_n^d e^{\alpha_n}} = g(\alpha_n) \to 0,$$

$$\frac{f'(\alpha_n)}{\alpha_n^d \, e^{\alpha_n}} = \frac{\left(z^d \, e^z g(z)\right)'}{\alpha_n^d e^{\alpha_n}}\Bigg|_{z=\alpha_n} = \left(1 + \frac{d}{\alpha_n}\right) g(\alpha_n) + g'(\alpha_n) \to \infty.$$

Set $\delta_n := \varepsilon_n + |\alpha_n - \beta_n|$. Then we also have $\delta_n \to 0$ and $S(\Delta(\alpha_n, \delta_n), g) \to \infty$. □

3 Main results

Theorem 3.1. *Let R be a nonzero rational function, and let f be a meromorphic function, all but finite many of whose zeros are multiple. If $\rho(f) > 2$, then $f'(z) = R(z)\tan z$ has infinitely many solutions in the complex plane (including the possibility of infinitely many common poles of f and* $\tan z$*).*

Remark 3.2. Theorem 3.1 still holds provided that $\tan z$ is replaced by $\cot z$.

Proof. We present the proof of Theorem 3.1 using reduction to absurdity.

Now let us assume that $f'(z) = R(z)\tan z$ has finitely many solutions in the complex plane. Let $R(z) \sim cz^d$ as $z \to \infty$, where c is a finite nonzero complex number and d is an integer.

Set $g(z) := \frac{f(z)}{z^d}$. Clearly, $\rho(g) > 2$. By Lemma 2.7, there exist $\alpha_n \to \infty$ and $\delta_n \to 0$ such that

$$S(\Delta(\alpha_n, \delta_n), g) \to \infty \ \text{ as } \ n \to \infty, \tag{3.1}$$

$$\frac{f(\alpha_n)}{\alpha_n^d} \to 0 \ \text{ and } \ \frac{f'(\alpha_n)}{\alpha_n^d} \to \infty \ \text{ as } \ n \to \infty. \tag{3.2}$$

Write $\alpha_n := x_n + iy_n$. Taking a subsequence and renumbering if necessary, we may assume that $y_n \to y^*$ as $z \to \infty$.

We consider the following two cases.

Case 1 $y^* \neq \pm\infty$.

There exist integers j_n and points $\widehat{x}_n \in (-\frac{\pi}{2}, \frac{\pi}{2}]$ such that $\widehat{x}_n = x_n - \pi j_n$. Taking a suitable subsequence and renumbering if necessary, we may assume that $\widehat{x}_n \to x^*$. Clearly, $x^* \in [-\frac{\pi}{2}, \frac{\pi}{2}]$.

Set $\beta^* = x^* + iy^*$, $\beta_n := \beta^* + \pi j_n$ and $\sigma_n := |\alpha_n - \beta_n| + \delta_n$. Then we have

$$\beta_n \to \infty, \quad \sigma_n \to 0 \ \text{ and } \ S\big(\Delta(\beta_n, \sigma_n), g\big) \to \infty. \tag{3.3}$$

In fact, a simple calculation shows that

$$|\alpha_n - \beta_n| = |(\widehat{x}_n - x^*) + i(y_n - y^*)| \leq |\widehat{x}_n - x^*| + |y_n - y^*| \to 0, \tag{3.4}$$

$$\sigma_n = |\alpha_n - \beta_n| + \delta_n \to 0.$$

It is easy to see that $\Delta(\alpha_n, \delta_n) \subset \Delta(\beta_n, \sigma_n)$ and hence $S\big(\Delta(\beta_n, \sigma_n), g\big) \to \infty$.

Set

$$g_n(z) := g(z + \beta_n) \ \text{and} \ f_n(z) := \frac{f(z + \beta_n)}{\beta_n^d} \ \text{for} \ z \in \Delta.$$

Then we see that

$$S\big(\Delta(0, \sigma_n), g_n\big) = S\big(\Delta(\beta_n, \sigma_n), g\big) \to \infty, \tag{3.5}$$

$$g_n(z) = \frac{\beta_n^d}{(z + \beta_n)^d} \cdot f_n(z) \ \text{ and } \ \frac{\beta_n^d}{(z + \beta_n)^d} \Rightarrow 1 \ \text{ in } \ \Delta. \tag{3.6}$$

By (3.2), we have

$$f_n(\alpha_n - \beta_n) = \frac{f(\alpha_n)}{\alpha_n^d} \cdot \frac{\alpha_n^d}{\beta_n^d} \to 0 \ \text{ and } \ f_n'(\alpha_n - \beta_n) = \frac{f'(\alpha_n)}{\alpha_n^d} \cdot \frac{\alpha_n^d}{\beta_n^d} \to \infty. \tag{3.7}$$

Then, (3.4) and (3.7) imply that no subsequence of $\{f_n\}$ is normal at $z = 0$.

Now, we have, for sufficiently large n, (a1) all zeros of f_n are multiple in Δ,

(a2) $f_n'(z) \neq \psi_n$ in Δ, where $\psi_n = \frac{R(z+\beta_n)}{\beta_n^d}\tan(z+\beta_n) \overset{\chi}{\Rightarrow} c\tan(z+\beta^*)$ in Δ.

By Lemma 2.2, $\{f_n\}$ is quasinormal in Δ. Thus there exist a subsequence of $\{f_n\}$ (still denoted by $\{f_n\}$) and $\delta \in (0,1)$ such that

(b1) no subsequence of $\{f_n\}$ is normal at 0,

(b2) $f_n \overset{\chi}{\Rightarrow} f^*$ in $\Delta'(0,\delta)$, where f^* is meromorphic or identically infinite there.

By Lemma 2.3, we have

(c1) there exist $r_0 \in (0,\delta)$ and $N_0 > 0$ such that $n\big(\Delta(0,r_0), 1/f_n\big) < N_0$ for sufficiently large n,

(c2) $f^*(z) = \int_0^z c\tan(\zeta+\beta^*)\,\mathrm{d}\zeta$ in $\Delta'(0,\delta)$.

Since $f^*(z)$ is a single-valued meromorphic function, $\int_0^z c\tan(\zeta+\beta^*)\,\mathrm{d}\zeta$ must also be a single-valued meromorphic function, and hence $\tan(\zeta+\beta^*)$ must be holomorphic in $\Delta'(0,\delta)$. It follows from (3.6), (c1) and (c2) that

(d1) $n\big(\Delta(0,r_0), 1/g_n\big) < N_0$ for sufficiently large n,

(d2) $g_n(z) \Rightarrow \int_0^z c\tan(\zeta+\beta^*)\,\mathrm{d}\zeta$ in $\Delta'(0,r_0)$.

By Lemma 2.4, there exists $M > 0$ such that $S\big(\Delta(0,r_0/4),\, g_n\big) < M$ for sufficiently large n. This contradicts (3.5).

Case 2 $y^* = +\infty$ or $y^* = -\infty$.

Without loss of generality, we may assume that $y^* = +\infty$.

Set

$$g_n(z) := g(z+\alpha_n) \text{ and } f_n(z) := \frac{f(z+\alpha_n)}{\alpha_n^d} \text{ for } z \in \Delta.$$

Then we see that

$$S\big(\Delta(0,\delta_n), g_n\big) = S\big(\Delta(\alpha_n,\delta_n), g\big) \to \infty, \tag{3.8}$$

$$g_n(z) = \frac{\alpha_n^d}{(z+\alpha_n)^d}\cdot f_n(z) \ \text{ and } \ \frac{\alpha_n^d}{(z+\alpha_n)^d} \Rightarrow 1 \ \text{ in } \ \Delta. \tag{3.9}$$

It follows from (3.2) that

$$f_n(0) = \frac{f(\alpha_n)}{\alpha_n^d} \to 0 \ \text{ and } \ f_n'(0) = \frac{f'(\alpha_n)}{\alpha_n^d} \to \infty.$$

Then, no subsequence of $\{f_n\}$ is normal at $z=0$.

Now, we have, for sufficiently large n,

(A1) all zeros of f_n are multiple in Δ,

(A2) $f_n'(z) \neq \psi_n$ in Δ, where $\psi_n = \frac{R(z+\alpha_n)}{\alpha_n^d}\tan(z+\alpha_n) \Rightarrow c\,i$ in Δ.
(A note: $\tan z = \frac{1}{i}\frac{e^{iz}-e^{-iz}}{e^{iz}+e^{-iz}} \to i$ as $\mathrm{Im}(z) \to \infty$.)

By Lemma 2.2, $\{f_n\}$ is quasinormal in Δ. Thus there exist a subsequence of $\{f_n\}$ (still denoted by $\{f_n\}$) and $\delta \in (0,1)$ such that

(B1) no subsequence of $\{f_n\}$ is normal at 0,

(B2) $f_n \overset{\chi}{\Rightarrow} f^*$ in $\Delta'(0,\delta)$, where f^* is meromorphic or identically infinite there.

By Lemma 2.3, we have

(C1) there exist $r_0 \in (0,\delta)$ and $N_0 > 0$ such that $n\big(\Delta(0,r_0), 1/f_n\big) < N_0$ for sufficiently large n,

(C2) $f^*(z) = \int_0^z c\,i\,\mathrm{d}\zeta = c\,iz$ in $\Delta'(0,\delta)$.

It follows from (3.9), (C1) and (C2) that

(D1) $n\big(\Delta(0,r_0), 1/g_n\big) < N_0$ for sufficiently large n,

(D2) $g_n(z) \Rightarrow c\,iz$ in $\Delta'(0,r_0)$.

By Lemma 2.4, there exists $M > 0$ such that $S\big(\Delta(0, r_0/4),\, g_n\big) < M$ for sufficiently large n. This contradicts (3.8). □

Theorem 3.3. *Let R be a nonzero rational function, and let f be a meromorphic function, all but finite many of whose zeros are multiple. If $\rho(f) > 2$, then $f'(z) = R(z)\mathrm{e}^z$ has infinitely many solutions in the complex plane.*

Proof. We assume that $f'(z) = R(z)\mathrm{e}^z$ has at most finitely many zeros and derive a contradiction. Let $R(z) \sim cz^d$ as $z \to \infty$, where c is a finite nonzero complex number and d is an integer.

Set $g(z) := \frac{f(z)}{z^d \mathrm{e}^z}$. By Lemma 2.8, there exists a sequence $\alpha_n \to \infty$ and $\delta_n \to 0$ such that

$$S(\Delta(\alpha_n, \delta_n), g) \to \infty \quad \text{as} \quad n \to \infty,$$

$$\frac{f(\alpha_n)}{\alpha_n^d \mathrm{e}^{\alpha_n}} \to 0 \quad \text{and} \quad \frac{f'(\alpha_n)}{\alpha_n^d \mathrm{e}^{\alpha_n}} \to \infty \quad \text{as} \quad n \to \infty. \tag{3.10}$$

Set

$$g_n(z) := g(z + \alpha_n) \text{ and } f_n(z) := \frac{f(z+\alpha_n)}{\alpha_n^d \mathrm{e}^{\alpha_n}} \text{ for } z \in \Delta.$$

Then we see that

$$S\big(\Delta(0, \delta_n), g_n\big) = S\big(\Delta(\alpha_n, \delta_n), g\big) \to \infty,$$

$$g_n(z) = \frac{\alpha_n^d \mathrm{e}^{\alpha_n}}{(z+\alpha_n)^d\, \mathrm{e}^{z+\alpha_n}} \cdot f_n(z) \ \text{ and } \ \frac{\alpha_n^d \mathrm{e}^{\alpha_n}}{(z+\alpha_n)^d\, \mathrm{e}^{z+\alpha_n}} \Rightarrow \mathrm{e}^{-z} \ \text{ in } \ \Delta.$$

It follows from (3.10) that

$$f_n(0) = \frac{f(\alpha_n)}{\alpha_n^d \mathrm{e}^{\alpha_n}} \to 0 \ \text{ and } \ f_n'(0) = \frac{f'(\alpha_n)}{\alpha_n^d \mathrm{e}^{\alpha_n}} \to \infty.$$

Then, no subsequence of $\{f_n\}$ is normal at $z = 0$.

The rest of the proof is similar to the proof of Theorem 3.1 in **case 2** and we omit the details here. □

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