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On some weighted fractional $p(\cdot,\cdot)-$Laplacian problems

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Abstract This paper is concerned with a class of fractional Kirchhoff $p(\cdot,\cdot)-$Laplacian problems with variable exponents and indefinite weights on a smooth bounded domain. Using the Mountain pass theorem combining with variational techniques, we prove the existence of a nontrivial solution.

1 Introduction

Let $\Omega \subset \mathbb{R}^N$ be a smooth bounded domain. In this paper, we focused on the existence of nontrivial solutions to the following fractional $p(x,y)$-Laplacian problem:

$$(\mathcal{P})\begin{cases} M\left(\displaystyle\int_{\Omega\times\Omega}\frac{|u(x)-u(y)|^{p(x,y)}}{p(x,y)|x-y|^{N+sp(x,y)}}dxdy\right)(-\Delta)^s_{p(x,y)}u(x)+|u|^{q_1(x)-2}u & \\ \quad =\lambda V_1(x)|u|^{q_2(x)-2}+\mu V_2(x)|u|^{r(x)-2} & \text{in } \Omega\,, \\ u=0 & \text{in } \partial\Omega\,, \end{cases}$$

where, M is a continuous function, λ, μ are positive parameters, and for $i=1,2$ V_i, are a weight functions. The variable exponents $p:\overline{\Omega}\times\overline{\Omega}\to(1,\infty)$, $q_i, r:\overline{\Omega}\to(1,\infty)$ are three continuous functions satisfying the following assumptions:

$$1<p^-=\min_{(x,y)\in\overline{\Omega}\times\overline{\Omega}}p(x,y)\leqslant p(x,y)\leqslant p^+=\max_{(x,y)\in\overline{\Omega}\times\overline{\Omega}}p(x,y)<+\infty,$$

$$1<q_i^-=\min_{x\in\bar{\Omega}}q_i(x)\le q_i(x)\le q_i^+=\max_{x\in\bar{\Omega}}q_i(x)<\infty,$$

and

$$1<r^-=\min_{x\in\bar{\Omega}}r(x)\le r(x)\le r^+=\max_{x\in\bar{\Omega}}r(x)<\infty.$$

We denote by $(-\Delta)^s_{p(x,y)}$ the fractional $p(\cdot,\cdot)-$Laplacian operator, which is defined by

$$(-\Delta)^s_{p(x,y)}u(x)=\mathcal{L}u(x)=P.V\int_\Omega\frac{|u(x)-u(y)|^{p(x,y)-2}(u(x)-u(y))}{|x-y|^{N+sp(x,y)}}dy,$$

where P.V is a commonly used abbreviation in the principal value sense.

In recent years, the literature has widely developed the study of fractional partial differential equations with variable growth. On the one hand, for their application in several scientific fields such as continuum mechanics, population dynamics, Covid-19, game theory, and others. We refer the reader to [3, 10, 13, 14, 19] for more details. On the other hand, for its mathematical importance in the theory of function spaces with variable exponents. For example, in [24], Kaufmann, Rossi, and Vidal were the first who introduced this kind of problem, they proved the existence and uniqueness of a solution to following nonlocal problem involving the fractional $p(x)-$Laplacian

$$\begin{cases} \mathcal{L}u(x) + |u(x)|^{q(x)-2}u(x) & = f(x) \quad \text{in} \quad \Omega, \\ u & = 0 \quad \text{in} \quad \partial\Omega, \end{cases} \tag{1.1}$$

where $f \in L^{a(x)}(\Omega)$ with $1 < a_- < a(x) < a_+$ for any $x \in \overline{\Omega}$. Bahrouni and Rădulescu in [7], studied the existence of solution to the following problem

$$\begin{cases} \mathcal{L}u(x) + |u(x)|^{q(x)-1}u(x) & = \lambda|u(x)|^{r(x)-1}u(x) \quad \text{in} \quad \Omega, \\ u & = 0 \quad \text{in} \quad \partial\Omega,, \end{cases} \tag{1.2}$$

where $\lambda > 0$, $1 < r(x) < p^- = \min_{x,y\in\Omega\times\Omega} p(x,y)$. for a deeper comprehension, we refer the reader to [1, 2, 6, 7, 8, 9, 15, 16, 20, 21, 22, 23, 30, 31, 32, 34] and the references therein. When, $p(\cdot,\cdot) = p =$ constant, we quote for example, the relevent work Di Nezza, Palatucci, and Valdinoci [14], see also [4, 8, 11, 26, 27, 28] and the references therein.
The main objective of this paper is to prove the existence of non-trivial solutions to the fractional Kirchhoff problem with variable exponents and indefinite weights $(\mathcal{P})$ in the framework of fractional spaces with variable exponents. When studying this kind of problem, the difficulty we encounter is that the operator associated with it is not monotone, which complicates the verification of the mountain pass hypothesis and Minty Browders' theorems.

The paper is structured as follows. Section 2 presents the relevant definitions and properties of Fractional Sobolev spaces with variable exponents. In Section 3, we introduce a suitable assumptions, and we show our main results.

2 Preliminaries and Notations

This section recalls the relevant definitions and properties of Sobolev spaces with variable exponents and Fractional Sobolev spaces with variable exponents, which help us in our analysis. We refer to [7, 16, 18, 20, 21, 24] and the references therein for more background.

2.1 Sobolev spaces with variable exponents

For $q \in C_+(\bar{\Omega}) = \left\{ q \in C(\bar{\Omega}) : 1 < \inf_{x\in\bar{\Omega}} q(x) \leqslant q(x) \leqslant q^+ < \infty \quad \text{for all } x \in \bar{\Omega} \right\}$. We define the variable exponent Lebesgue space $L^{q(.)}(\Omega)$ by

$$L^{q(.)}(\Omega) = \left\{ u : \Omega \to \mathbb{R} : u \text{ is measurable such that } \int_\Omega |u(x)|^{q(x)}dx < \infty \right\},$$

which is equipped with the following Luxemburg norm

$$|u|_{q(.)} = \inf \left\{ \mu > 0 : \int_\Omega \left| \frac{u(x)}{\mu} \right|^{q(x)} dx \le 1 \right\}.$$

Proposition 2.1. *The generalized Lebesgue space $\left(L^{q(.)}(\Omega); |\cdot|_{q(.)}\right)$ is a separable, uniformly convex, and Banach space.*

Proposition 2.2. *Let $q_1(x), q_2(x) \in C_+(\bar{\Omega})$ such that $q_1(x) \le q_2(x)$ for a.e $x \in \Omega$. Then we have the following continuous embedding*

$$L^{q_1(x)}(\Omega) \hookrightarrow L^{q_2(x)}(\Omega).$$

Proposition 2.3. *(Hölder's inequality) For $u \in L^{q(.)}(\Omega)$ and $v \in L^{q'(.)}(\Omega)$, we have*

$$\left| \int_\Omega uvdx \right| \leqslant \left(\frac{1}{q^-} + \frac{1}{q'^-} \right) \|u\|_{q(x)} \|v\|_{q'(x)} \leqslant 2\|u\|_{q(x)}\|v\|_{q'(x)},$$

where $\frac{1}{q'(x)} + \frac{1}{q(x)} = 1$.

The modular of the $L^{q(.)}(\Omega)$ space, which defined by

$$\rho_{q(x)}(u) = \int_\Omega |u(x)|^{q(x)}dx,$$

plays a significant role in manipulating generalized Lebesgue spaces with variable exponent. See [16, 33].

Proposition 2.4. *[18] Let $u \in L^{q(x)}(\Omega)$, $(u_k) \subset L^{q(x)}(\Omega)$.*

1) $||u||_{q(x)} < 1$ *(resp =1, > 1)* $\Leftrightarrow \rho_{q(x)}(u) < 1$ *(resp= 1 , >1).*

2) $||u||_{q(x)} \leq 1 \Rightarrow ||u||^{q^+}_{q(x)} \leq \rho_{q(x)}(u) \leq ||u||^{q^-}_{q(x)}$.

3) $||u||_{q(x)} \geq 1 \Rightarrow ||u||^{q^-}_{q(x)} \leq \rho_{q(x)}(u) \leq ||u||^{q^+}_{q(x)}$.

4) $\lim_{k\to+\infty} \|u_k - u\|_{L^{q(x)}(\Omega)} = 0 \quad \Leftrightarrow \lim_{k\to+\infty} \rho_{q(.)}(u_k - u) = 0$.

Lemma 2.5. *[18] Let $q, r \in C_+(\bar\Omega)$ such that $1 \leq r(x)q(x) < \infty$, for a.e $x \in \Omega$. Let $u \in L^{q(x)}(\Omega)$. Then*

1) $||u||_{r(x)q(x)} \leq 1 \Rightarrow ||u||^{r^+}_{q(x)r(x)} \leq |||u|^{r(x)}||_{q(x)} \leq ||u||^{r^-}_{q(x)r(x)}$.

2) $||u||_{r(x)q(x)} \geq 1 \Rightarrow ||u||^{r^-}_{q(x)r(x)} \geq |||u|^{r(x)}||_{q(x)} \geq ||u||^{r^+}_{q(x)r(x)}$.

In particular, $r(x) = r$ is constant, then

$$|||u|^r||_{q(x)} = ||u||^r_{q(x)r(x)}.$$

Now, we define the weighted variable exponent Lebesgue spaces $L^{q(x)}_V(\Omega)$ by

$$L^{q(x)}_V(\Omega) = \left\{u : \Omega \to \mathbb{R} : u \text{ is measurable such that } \int_\Omega V(x)|u(x)|^{q(x)}dx < +\infty\right\},$$

with the norm

$$\|u\|_{q,V} = \inf\left\{\mu > 0 : \int_\Omega V(x)\left|\frac{u(x)}{\mu}\right|^{q(x)} dx \leqslant 1\right\}.$$

Proposition 2.6. *The space $(L^{q(x)}_V(\Omega), \|u\|_{q,V})$ is a separable, uniformly convex, and Banach space.*

The weighted modular on $L^{q(x)}_V(\Omega)$ is defined as follows

$$\rho_{q,V} : L^{q(x)}_V(\Omega) \to \mathbb{R}$$
$$u \mapsto \int_\Omega V(x)|u(x)|^{q(x)}dx.$$

We refer the reader to [5, 15] for more background.

Proposition 2.7. *[32] Let $u, \{u_n\} \subset L^{q(x)}_V(\Omega)$, then we have*

1) $\|u\|_{q,V} < 1$ *(resp* $= 1, > 1) \Leftrightarrow \rho_{q,V}(u) < 1$ *(resp* $= 1, > 1)$.

2) $\|u\|_{q,V} < 1 \Rightarrow \|u\|^{q+}_{q,V} \leqslant \rho_{q,V}(u) \leqslant \|u\|^{q-}_{q,V}$.

3) $\|u\|_{q,V} > 1 \Rightarrow \|u\|^{q-}_{q,V} \leqslant \rho_{q,V}(u) \leqslant \|u\|^{q+}_{q,V}$.

4) $\lim_{n\to+\infty} \|u_n\|_{q,V} = 0 \Leftrightarrow \lim_{n\to+\infty} \rho_{q,V}(u_n) = 0$.

2.2 Fractional Sobolev space with variable exponent

In what follow, We assume that p is symmetric, that is $p(x,y)=p(y,x)$ for all $(x,y)\in\overline{\Omega}\times\overline{\Omega}$. For $s\in(0,1), p\in C_+(\bar{\Omega}\times\overline{\Omega})$ and $q\in C_+(\bar{\Omega})$ we define the fractional Sobolev space with variable exponent via the Gagliardo approach as follows:

$$W^{s,q(x),p(x,y)}(\Omega)=\left\{u\in L^{q(x)}(\Omega), \int_{\Omega\times\Omega}\frac{|u(x)-u(y)|^{p(x,y)}}{\mu^{p(x,y)}|x-y|^{N+sp(x,y)}}dxdy<\infty, \text{ for some } \mu>0\right\},$$

where $L^{q(x)}(\Omega)$ is the variable exponent Lebesgue space. Let

$$[u]_{s,p(x,y)}=\inf\left\{\mu>0, \int_{\Omega\times\Omega}\frac{|u(x)-u(y)|^{p(x,y)}}{\mu^{p(x,y)}|x-y|^{N+s\cdot p(x,y)}}dxdy<1\right\},$$

be the variable exponent Gagliardo seminorm and define

$$\|u\|_{W^{s,q(x),p(x,y)}}=[u]_{s,p(x,y)}+|u|_{q(x)}.$$

The space $W^{s,q(x),p(x,y)}(\Omega)$ is a separable reflexive and Banach space. We refer reader to [7, 21, 24] for more details. We define the modular $\rho_{p(x,y)}:W^{s,q(x),p(x,y)}(\Omega)\to\mathbb{R}$ defined as follows

$$\rho_{p(x,y)}(u)=\int_{\Omega\times\Omega}\frac{|u(x)-u(y)|^{p(x,y)}}{|x-y|^{N+sp(x,y)}}dxdy+\int_{\Omega}|u(x)|^{q(x)}dx,$$

and

$$\|u\|_{\rho_{p(x,y)}}=\inf\left\{\lambda>0:\rho_{p(x,y)}\left(\frac{u}{\lambda}\right)\leqslant 1\right\}.$$

Theorem 2.8. *(See [12]). Let $\Omega\subset\mathbb{R}^N$ be a smooth domain and $s\in(0,1)$. Let $p(x,y)\in C_+(\bar{\Omega}\times\overline{\Omega})$ be continuous variable exponent with $sp(x,y)<N$ for all $(x,y)\in\bar{\Omega}\times\overline{\Omega}$. Assume that $\beta:\Omega\to(1,\infty)$ is a continuous function such that*

$$p^*(x):=\frac{Np(x,x)}{N-sp(x,x)}>\beta(x)\geq\beta_->1,$$

for $x\in\bar{\Omega}$. Then, there exists a constant $C=C(N,s,p,q,\beta,\Omega)$ such that for every $u\in W^{s,q(x),p(x,y)}(\Omega)$, it holds that

$$\|u\|_{L^{\beta(x)}(\Omega)}\leq C\|u\|_{W^{s,q(x),p(x,y)}(\Omega)}.$$

Thus, the space $W^{s,q(x),p(x,y)}(\Omega)$ is continuously embedding in $L^{\beta(x)}(\Omega)$ for any $\beta\in(1,p^*(x))$. Moreover, this embedding is compact.

Remark 2.9. Suppose that $q(x)=p(x,x)$. Since

$$1<p(x,x)<\frac{Np(x,x)}{N-sp(x,x)}.$$

It follows that $[u]_{s,p(x,y)}$ is a norm on $W^{s,q(x),p(x,y)}(\Omega)$, which is equivalent to the norm $\|.\|$ on $W^{s,q(x),p(x,y)}(\Omega)$.

Proposition 2.10. *[7] For all $u,v\in W^{s,q(x),p(x,y)}(\Omega)$, we consider the following functional $I:W^{s,q(x),p(x,y)}(\Omega)\to(W^{s,q(x),p(x,y)}(\Omega))^*$ such that*

$$\langle I(u),v\rangle=\int_{\Omega\times\Omega}\frac{|u(x)-u(y)|^{p(x,y)-2}(u(x)-u(y))(v(x)-v(y))}{|x-y|^{N+s\cdot p(x,y)}}dxdy,$$

then:

1. *I is a bounded and strictly monotone operator.*

2. *I is a mapping of type (S_+), that is, if $u_n\rightharpoonup u\in W^{s,q(x),p(x,y)}(\Omega)$ and $\limsup_{n\to+\infty}I(u_n)(u_n-u)\leq 0$, then $u_n\to u\in W^{s,q(x),p(x,y)}(\Omega)$.*

3. *I is a homeomorphism.*

3 Assumptions And Main Result.

In this section, we introduce the concept of a weak solution for a problem $(\mathcal{P})$, and state the existence of such a solution.

3.1 Assumptions on data

(H1) $M : \mathbb{R} \to [0, \infty)$ is a continuous function, and there exist positive constants $0 < m_1 \leq m_2$ and $\alpha > 1$ such that

$$m_1 t^{\alpha-1} \leq M(t) \leq m_2 t^{\alpha-1},$$

for $t \in [0, \infty)$ and $\alpha p(x, x) < N$.

(H2) $p : \overline{\Omega} \times \overline{\Omega} \to (1, \infty), q_1, q_2, r : \overline{\Omega} \to (1, \infty)$, be continuous functions such that

$$1 < q_2^- \leqslant q_2(x) \leqslant q_2^+ < p^- \leqslant p(x,y) \leqslant p^+ < r^- \leqslant r(x) \leqslant r^+ < +\infty,$$

and

$$1 < q_1^- \leqslant q_1(x) \leqslant q_1^+ < p^- \leqslant p(x,y) \leqslant p^+ < r^- \leqslant r(x) \leqslant r^+ < +\infty,$$

where $q_2^- := \min_{x\in\overline{\Omega}} q_2(x)$, $q_2^+ := \sup_{x\in\overline{\Omega}} q_2(x)$, $q_1^- := \min_{x\in\overline{\Omega}} q_1(x)$, $q_1^+ := \sup_{x\in\overline{\Omega}} q_1(x)$, $r^- := \min_{x\in\overline{\Omega}} r(x)$, $r^+ := \sup_{x\in\overline{\Omega}} r(x)$, $p^- := \min_{(x,y)\in\overline{\Omega}\times\overline{\Omega}} p(x,y)$ and $p^+ := \sup_{(x,y)\in\overline{\Omega}\times\overline{\Omega}} p(x,y)$.

(H3) $V_1 \in C(\bar{\Omega})$ is non-negative weighted function satisfy the following assumptions:

$$V_1 \in L^{l_1(.)}(\Omega),$$

with $l_1'(.)q_2(.) < p^*(.)$.

(H4) $V_2 \in C(\bar{\Omega})$ is non-negative weighted function satisfy the following assumptions:

$$V_2 \in L^{l_2(.)}(\Omega),$$

with $l_2'(.)r(.) < p^*(.)$.

3.2 Main Result

Definition 3.1. A function $u \in W^{s,q(x),p(x,y)}(\Omega)$ is called a weak solution to a problem $(\mathcal{P})$ if and only if

$$M\left(\int_{\Omega\times\Omega} \frac{|u(x)-u(y)|^{p(x,y)}}{p(x,y)|x-y|^{N+sp(x,y)}}dxdy\right)\int_{\Omega\times\Omega} \frac{|u(x)-u(y)|^{p(x,y)-2}(u(x)-u(y))(v(x)-v(y))}{|x-y|^{N+sp(x,y)}}dxdy$$
$$+\int_\Omega |u(x)|^{q_1(x)-2}uvdx = \lambda\int_\Omega V_1(x)|u(x)|^{q_2(x)-2}uvdx + \mu\int_\Omega V_2(x)|u(x)|^{r(x)-2}uvdx,$$

for all $v \in W^{s,q(x),p(x,y)}(\Omega)$.

Theorem 3.2. *Let $p \in C_+(\bar{\Omega} \times \overline{\Omega})$ be a continuous variable exponent and let $s \in (0, 1)$, with $sp(x, y) < N$ for all $(x, y) \in \overline{\Omega} \times \overline{\Omega}$. If hypotheses $(H1), (H2)$ and $(H3)$ hold. Then there exists $\lambda^* > 0$, such that for any $\beta \in (0, \lambda^*)$, problem $(\mathcal{P})$ has at least one nontrivial weak solution. Where $\beta = min(\lambda, \mu)$*

In order to formulate the variational approach of problem $(\mathcal{P})$, we introduce the functional $\psi_{\lambda,\mu} : W^{s,q(x),p(x,y)}(\Omega) \to \mathbb{R}$ defined by

$$\psi_{\lambda,\mu}(u) = \hat{M}\Big(\int_{\Omega\times\Omega} \frac{|u(x)-u(y)|^{p(x,y)}}{p(x,y)|x-y|^{N+sp(x,y)}}dxdy\Big) + \int_\Omega \frac{|u(x)|^{q_1(x)}}{q_1(x)}dx - \lambda\int_\Omega \frac{V_1(x)|u(x)|^{q_2(x)}}{q_2(x)}dx$$
$$-\mu\int_\Omega \frac{V_2(x)|u(x)|^{r(x)}}{r(x)}dx,$$

where $\hat{M}(t)=\displaystyle\int_0^t M(s)ds$. By standard arguments, we can show that $\psi_{\lambda,\mu}\in C^1(W^{s,q(.),p(.,.)},\mathbb{R})$ and

$$\begin{aligned}
&\langle\psi'_{\lambda\mu}(u),v\rangle\\
&=M\left(\int_{\Omega\times\Omega}\frac{|u(x)-u(y)|^{p(x,y)}}{p(x,y)|x-y|^{N+sp(x,y)}}dxdy\right)\int_{\Omega\times\Omega}\frac{|u(x)-u(y)|^{p(x,y)-2}(u(x)-u(y))(v(x)-v(y))}{|x-y|^{N+sp(x,y)}}dxdy\\
&+\int_\Omega|u(x)|^{q_1(x)-2}uvdx-\lambda\int_\Omega V_1(x)|u(x)|^{q_2(x)-2}uvdx-\mu\int_\Omega V_2(x)|u(x)|^{r(x)-2}uvdx.
\end{aligned}$$

Lemma 3.3. *The functional $\psi_{\lambda,\mu}$ satisfies the Palais-Smale condition in $W^{s,q(.),p(.,.)}(\Omega)$.*

Proof. Let $\{u_n\}_{n\in\mathbb{N}}\subset W^{s,q(x),p(x,y)}(\Omega)$ such that $\psi_{\lambda,\mu}(u_n)\longrightarrow c$ in $W^{s,q(x),p(x,y)}(\Omega)$ and $d\psi_{\lambda,\mu}(u_n)\longrightarrow 0$ in $(W^{s,q(x),p(x,y)}(\Omega))^*$ as $n\longrightarrow+\infty$, where $(W^{s,q(x),p(x,y)}(\Omega))^*$ is the dual space of $W^{s,q(x),p(x,y)}(\Omega)$. Now, we will show that $\{u_n\}$ is bounded in $W^{s,q(x),p(x,y)}(\Omega)$. We assume by contradiction for a subsequence, still denoted by $\{u_n\}$, we get $\|u_n\|_{W^{s,q(x),p(x,y)}}\longrightarrow+\infty$ as $n\longrightarrow+\infty$. For n large enough and $\lambda\in(0,\lambda^*)$, by hypothesis $(H1),(H2)$ and $(H3)$, we get

$$\begin{aligned}
1+c+\|u_n\|_{W^{s,q(x),p(x,y)}}&\geq\psi_{\lambda,\mu}(u_n)-\frac{1}{r^-}\psi'_{\lambda,\mu}(u_n)(u_n)\\
&=\hat{M}\left(\int_{\Omega\times\Omega}\frac{|u_n(x)-u_n(y)|^{p(x,y)}}{p(x,y)|x-y|^{N+sp(x,y)}}dxdy\right)+\int_\Omega\frac{|u_n(x)|^{q_1(x)}}{q_1(x)}dx\\
&-\lambda\int_\Omega\frac{V_1(x)|u_n(x)|^{q_2(x)}(x)}{q_2(x)}dx-\mu\int_\Omega\frac{V_2(x)|u_n(x)|^{r(x)}(x)}{r(x)}dx\\
&-\frac{1}{r^-}M\left(\int_{\Omega\times\Omega}\frac{|u_n(x)-u_n(y)|^{p(x,y)}}{p(x,y)|x-y|^{N+sp(x,y)}}dxdy\right)\int_{\Omega\times\Omega}\frac{|u_n(x)-u_n(y)|^{p(x,y)}}{|x-y|^{N+sp(x,y)}}dxdy\\
&-\frac{1}{r^-}\int_\Omega|u_n(x)|^{q_1(x)}dx+\frac{\lambda}{r^-}\int_\Omega V_1(x)|u_n(x)|^{q_2(x)}dx+\frac{\mu}{r^-}\int_\Omega V_2(x)|u_n(x)|^{r(x)}dx\\
&\geq\left(\frac{m_1}{\alpha(p^+)^\alpha}-\frac{m_2}{r^-(p^-)^{\alpha-1}}\right)\int_{\Omega\times\Omega}\frac{|u_n(x)-u_n(y)|^{p(x,y)}}{|x-y|^{N+sp(x,y)}}dxdy\\
&+\left(\frac{1}{q_1^+}-\frac{1}{r^-}\right)\int_\Omega|u_n(x)|^{q_1(x)}dx-\lambda\int_\Omega\frac{V_1(x)|u_n(x)|^{q_2(x)}}{q_2(x)}dx\\
&-\lambda\frac{1}{r^-}\int_\Omega V_1(x)|u_n(x)|^{q_2(x)}dx+\frac{1}{r^-}\int_\Omega V_2(x)|u_n(x)|^{r(x)}dx\\
&-\int_\Omega\frac{1}{r(x)}V_2(x)|u_n(x)|^{r(x)}dx\\
&\geq\left(\frac{m_1}{\alpha(p^+)^\alpha}-\frac{m_2}{r(p^-)^{\alpha-1}}\right)||u_n||^{\alpha p^-}-\lambda\left(\frac{1}{q_2^-}-\frac{1}{r^-}\right)|V_1|_{\alpha_1(x)}|u_n|^{q_2}_{\frac{\alpha_1(x)q_2(x)}{\alpha_1(x)-1}}\\
&\geq\left(\frac{m_1}{\alpha(p^+)^\alpha}-\frac{m_2}{r(p^-)^{\alpha-1}}\right)||u_n||^{\alpha p^-}-\lambda^*\left(\frac{1}{q_2^-}-\frac{1}{r^-}\right)|V_1|_{\alpha_1(x)}C_1\\
&\times\left(||u_n||^{q_2^+}_{W^{s,q(x),p(x,y)}}+||u_n||^{q_2^-}_{W^{s,q(x),p(x,y)}}\right).
\end{aligned}$$

Dividing the above inequality by $||u_n||^{p^-}_{W^{s,q(x),p(x,y)}}$, and passing to the limit as $n\longrightarrow+\infty$, we obtain a contradiction. So $\{u_n\}_{n\in\mathbb{N}}$ is bounded in $W^{s,q(x),p(x,y)}(\Omega)$. Since $W^{s,q(x),p(x,y)}(\Omega)$ is a reflexive Banach space. Then, there exists u in $W^{s,q(x),p(x,y)}$ such that, the subsequence $\{u_n\}$ converge weakly in $W^{s,q(x),p(x,y)}(\Omega)$. And, since $q_1(x), \alpha(x) = \dfrac{l_1(x)q_2(x)}{l_1(x)-q_2(x)}$, $\beta = \dfrac{l_2(x)r(x)}{l_2(x)-r(x)} < p^*(x)$, we deduce that there exists a compact embedding $W^{s,q(x),p(x,y)}(\Omega) \hookrightarrow L^{q_1(x)}(\Omega), W^{s,q(x),p(x,y)}(\Omega) \hookrightarrow L^{\beta(x)}(\Omega)$ and $W^{s,q(x),p(x,y)}(\Omega) \hookrightarrow L^{\alpha(x)}(\Omega)$. So

$$\{u_n\} \to u \text{ stongly in } L^{q_1(x)}(\Omega) \text{ as } n \to \infty,$$

$$\{u_n\} \to u \text{ stongly in } L^{\alpha(x)}(\Omega) \text{ as } n \to \infty,$$

and

$$\{u_n\} \to u \text{ stongly in } L^{\beta(x)}(\Omega) \text{ as } n \to \infty.$$

Now, we will prove that $\{u_n\} \to u$ stongly in $W^{s,q(x),p(x,y)}(\Omega)$ as $n \to \infty$. We first show that the following equality holds

$$\lim_{n\to\infty} \int_\Omega |u_n|^{q_1(x)-2} u_n (u_n - u)\, dx = 0.$$

Using Hölder 's inequality, we have

$$\int_\Omega |u_n|^{q_1(x)-2} u_n (u_n - u)\, dx \le ||u_n|^{q_1(x)-2} u_n|_{\frac{q_1(x)}{q_1(x)-1}} |u_n - u|_{q_1(x)}.$$

Now, if $||u_n|^{q_1(x)-2}u_n|_{\frac{q_1(x)}{q_1(x)-1}} > 1$, by proposition we get

$$||u_n|^{q_1(x)-2}u_n|_{\frac{q_1(x)}{q_1(x)-1}} \le |u_n|_{q_1(x)}^{q_1^+}.$$

So, by the compact embedding $W^{s,q(x),p(x,y)}(\Omega) \hookrightarrow L^{q_1(x)}(\Omega)$ and the boundedness of $\{u_n\}$ in $W^{s,q(x),p(x,y)}(\Omega)$ we get

$$\lim_{n\to\infty} \int_\Omega |u_n|^{q_1(x)-2} u_n (u_n - u)\, dx = 0.$$

We show that

$$\lim_{n\to\infty} \int_\Omega V_1(x) |u_n|^{q_2(x)-2} u_n (u_n - u)\, dx = 0.$$

Combining with hypotheses $(H3)$ and Hölder 's inequality, we have

$$\begin{aligned}\int_\Omega V_1(x) |u_n|^{q_2(x)-2} u_n (u_n - u)\, dx &\le |V_1|_{l_1(x)} \left\| u_n|^{q_2(x)-2} u_n (u_n - u)\right|_{l_1'(x)} \\ &\le |V_1|_{l_1(x)} ||u_n|^{q_1(x)-2}u_n|_{\frac{q_1(x)}{q_1(x)-1}} |u_n - u|_{\alpha(x)}.\end{aligned}$$

If $||u_n|^{q_1(x)-2}u_n|_{\frac{q_1(x)}{q_1(x)-1}} > 1$, we have

$$||u_n|^{q_1(x)-2}u_n|_{\frac{q_1(x)}{q_1(x)-1}} \le |u_n|_{q_2(x)^+}^{q_2^+}.$$

Thanks to the compact embedding $W^{s,q(x),p(x,y)}(\Omega) \hookrightarrow L^{\alpha(x)}(\Omega)$ and boundedness of $\{u_n\}$ in $W^{s,q(x),p(x,y)}(\Omega)$ we get,

$$\lim_{n\to\infty} \int_\Omega V_1(x) |u_n|^{q_2(x)-2} u_n (u_n - u)\, dx = 0.$$

Similarly, we obtain

$$\lim_{n\to\infty} \int_\Omega V_2(x) |u_n|^{r(x)-2} u_n (u_n - u)\, dx = 0.$$

Since $\psi'_{\lambda,\mu}(u_n) \to 0$ and $\{u_n\}$ is bounded in $W^{s,q(x),p(x,y)}(\Omega)$ we have

$$\begin{aligned}\left|\left\langle \psi'_{\lambda,\mu}(u_n), u_n - u\right\rangle\right| &\leq \left|\left\langle \psi'_{\lambda,\mu}(u_n), u_n\right\rangle\right| + \left|\left\langle \psi'_{\lambda,\mu}(u_n), w\right\rangle\right| \\ &\leq \left\|\psi'_{\lambda,\mu}(u_n)\right\|_{W^{s,q(x),p(x,y)}(\Omega)^*} \|u_n\|_{W^{s,q(x),p(x,y)}(\Omega)} \\ &+ \left\|\psi'_{\lambda,\mu}(u_n)\right\|_{W^{s,q(x),p(x,y)}(\Omega)^*} \|u\|_{W^{s,q(x),p(x,y)}(\Omega)}.\end{aligned}$$

Therefore,

$$\lim_{n\to\infty} \left\langle \psi'_{\lambda,\mu}(u_n), u_n - u\right\rangle = 0.$$

So,

$$\begin{aligned}&M\left(\int_{\Omega\times\Omega} \frac{|u_n(x)-u_n(y)|^{p(x,y)}}{p(x,y)|x-y|^{N+p(x,y)s}}dxdy\right) \\ &\times \int_{\Omega\times\Omega} \frac{|u_n(x)-u_n(y)|^{p(x,y)-2}(u_n(x)-u_n(y))\left((u_n-u)(x)-(u_n-u)(y)\right)}{|x-y|^{N+p(x,y)s}}dxdy \to 0,\end{aligned}$$

as $n \to +\infty$.

If

$$\int_{\Omega\times\Omega} \frac{|u_n(x)-u_n(y)|^{\mu(x,y)}}{p(x,y)|x-y|^{N+p(x,y)s}}\,dxdy \to 0 \text{ as } n\to\infty,$$

then

$$u_n \to u \text{ strongly in } W^{s,q(x),p(x,y)}(\Omega).$$

If

$$\int_{\Omega\times\Omega} \frac{|u_n(x)-u_n(y)|^{p(x,y)}}{p(x,y)|x-y|^{N+p(x,y)s}}dxdy \to z \text{ as } n\to\infty,$$

we have

$$M\left(\int_{\Omega\times\Omega} \frac{|u_n(x)-u_n(y)|^{p(x,y)}}{p(x,y)|x-y|^{N+p(x,y)s}}dxdy\right) \to M(z) > m_1 z^{\alpha-1} > 0.$$

So it follows that

$$\lim_{n\to\infty} \langle I(u_n), u_n - u\rangle = 0.$$

Combining this with proposition 2.6, we deduce that $\{u_n\} \to u$ stongly in $W^{s,q(x),p(x,y)}(\Omega)$ as $n \to \infty$.

Lemma 3.4. *Assume that the hypotheses $(H1), (H2), (H3)$ and $(H4)$ are fulfilled. Then for all $\rho \in (0,1)$ there exist $\beta^* > 0$ and a constant $a > 0$ such that, for any $\beta \in (0,\beta^*)$, we have $\psi_{\lambda,\mu}(u) \geq a > 0$ for any $u \in W^{s,q(x),p(x,y)}$ with $\|u\|_{W^{s,q(x),p(x,y)}} = \rho$, where $\beta = \min(\lambda,\mu)$.*

Proof. Since the embedding $W^{s,q(x),p(x,y)}(\Omega) \hookrightarrow L^{l_1'(.)q_2(.)}(\Omega)$ and $W^{s,q(x),p(x,y)}(\Omega) \hookrightarrow L^{l_2'(.)r(.)}(\Omega)$ are continuous, we have

$$|u|_{l_1'(.)q_2(.)} \leq c_1\|u\|_{W^{s,q(x),p(x,y)}}$$

and

$$|u|_{l_2'(.)r(.)} \leq c_2\|u\|_{W^{s,q(x),p(x,y)}}.$$

We fix $\rho \in (0,1)$ such that $\rho < \dfrac{1}{c_1}$ and $\rho < \dfrac{1}{c_2}$. So $|u|_{l_1'(.)q_2(.)} < 1$ and $|u|_{l_2'(.)r(.)} < 1$, for all $u \in W^{s,q(x),p(x,y)}$. Moreover, by assumption $(H1)$, we have $\hat{M}(t) \geq \dfrac{m_1}{\alpha}t^\alpha$ for all $t \in [0,+\infty)$.

Consequently, by combining Hölder 's inequality with lemma 2.4, we obtain that for all $u \in W^{s,q(x),p(x,y)}(\Omega)$ with $\|u\|_{W^{s,q(x),p(x,y)}} = \rho$.

$$\begin{aligned}
\psi_{\lambda,\mu}(u) &= \hat{M}\left(\int_{\Omega\times\Omega}\frac{|u(x)-u(y)|^{p(x,y)}}{p(x,y)|x-y|^{N+sp(x,y)}}dxdy\right) + \int_\Omega \frac{|u(x)|^{q_1(x)}}{q_1(x)}dx - \lambda\int_\Omega \frac{V_1(x)|u(x)|^{q_2(x)}}{q_2(x)}dx \\
&- \mu\int_\Omega \frac{V_2(x)|u(x)|^{r(x)}}{r(x)}dx \\
&\geq \hat{M}\left(\frac{1}{p^+}\int_{\Omega\times\Omega}\frac{|u(x)-u(y)|^{p(x,y)}}{|x-y|^{N+sp(x,y)}}dxdy\right) - \frac{\lambda}{q_2^-}\int_\Omega V_1(x)|u(x)|^{q_2(x)}dx - \frac{\mu}{r^-}\int_\Omega V_2(x)|u(x)|^{r(x)}dx \\
&\geq \frac{m_1}{\alpha(p^+)^\alpha}\|u\|^{\alpha p^+}_{W^{s,q(x),p(x,y)}} - \frac{\lambda}{q_2^-}|V_1|_{l_1(x)}||u|^{q(x)}|_{l_1'(x)} - \frac{\mu}{r^-}|V_2|_{l_2(x)}||u|^{r(x)}|_{l_2'(x)} \\
&\geq \frac{m_1}{\alpha(p^+)^\alpha}\|u\|^{\alpha p^+}_{W^{s,q(x),p(x,y)}} - \frac{\lambda}{q_2^-}|V_1|_{l_1(x)}c_1^{q_2^-}\|u\|^{q_2^-}_{W^{s,q(x),p(x,y)}} - \frac{\mu}{r^-}|V_2|_{l_2(x)}c_2^{r^-}\|u\|^{r^-}_{W^{s,q(x),p(x,y)}} \\
&\geq \frac{m_1}{\alpha(p^+)^\alpha}\|u\|^{\alpha p^+}_{W^{s,q(x),p(x,y)}} - \beta c\left(c_2^{r^-}, c_1^{q_2^-}\right)\frac{1}{q_2^-}\left(|V_1|_{l_1(x)}\rho^{q_2^-} + |V_2|_{l_2(x)}\rho^{r^-}\right) \\
&= \rho^{r^-}\left(\frac{m_1}{\alpha(p^+)^\alpha}\rho^{\alpha p^+ - r^-} - \beta c\left(c_2^{r^-}, c_1^{q_2^-}\right)\frac{1}{q_2^-}\left(|V_1|_{l_1(x)}\rho^{q_2^- - r^-} + |V_2|_{l_2(x)}\right)\right).
\end{aligned}$$

By the above inequality, we remark that if we define

$$\beta^* = \frac{m_1\rho^{\alpha p^+ - r^-}q_2^-}{\alpha(p^+)^\alpha c\left(c_2^{r^-}, c_1^{q_2^-}\right)\left(|V_1|_{l_1(x)}\rho^{q_2^- - r^-} + |V_2|_{l_2(x)}\right)},$$

then for any $\beta \in (0, \beta^*)$ and $u \in W^{s,q(x),p(x,y)}(\Omega)$ with $\|u\|_{W^{s,q(x),p(x,y)}} = \rho$ there exists $a > 0$ such that $\psi_{\lambda,\mu}(u) \geq a > 0$.

Lemma 3.5. *Assume that the conditions $(H1), (H2), (H3)$, and $(H4)$ hold. Then for any $\lambda > 0$ and $\mu > 0$, there exists $v \in W^{s,q(x),p(x,y)}(\Omega)$ such that $v \geq 0$, $v \neq 0$ and $\psi_{\lambda,\mu}(tv) < 0$ for all $t > 0$ small enough.*

Proof. We denote $r_0^- := \inf_{x\in\Omega_0} r(x)$, $p_0^- := \inf_{(x,y)\in\Omega_0\times\Omega_0} p(x,y)$, $p_0^+ := \sup_{(x,y)\in\Omega_0\times\Omega_0} p(x,y)$ $q_{1,0}^- := \inf_{x\in\Omega_0} q_1(x)$, $q_{1,0}^+ := \sup_{x\in\Omega_0} q_1(x)$, $q_{2,0}^- := \inf_{x\in\Omega_0} q_2(x)$ and $q_{2,0}^+ := \sup_{x\in\Omega_0} q_2(x)$, where $\Omega_0 \subset\subset \Omega$.

Since $q_2 \in C(\Omega_0)$, there exists an open set $\Omega_1 \subset \Omega_0$ such that $|q_2(x) - q_{2,0}^-| < \varepsilon$, for all $x \in \Omega_1$ and $\varepsilon > 0$. Thus $q_2(x) < q_{2,0}^- + \varepsilon$, for all $x \in \Omega_1$. We will use the same idea $r \in C(\Omega_0)$, there exists an open set $\Omega_2 \subset \Omega_0$ such that $|r(x) - r^-| < \varepsilon$, for all $x \in \Omega_2$ and $\varepsilon > 0$. Thus $r(x) < r^- + \varepsilon$, for all $x \in \Omega_2$.

Let $v \in C_0^\infty(\Omega)$ be such that $supp(v) \subset \Omega_1 \subset \Omega_0$, $v = 1$ in a subset $\Omega_1^{`} \subset supp(v)$, and

$0 \leqslant v \leqslant 1$ in Ω_1. It follows that

$$\begin{aligned}
\psi_{\lambda,\mu}(tv) &= \hat{M}\left(\int_{\Omega\times\Omega} \frac{|tv(x)-tv_(y)|^{p(x,y)}}{p(x,y)|x-y|^{N+sp(x,y)}}dxdy\right) + \int_{\Omega} \frac{|tv(x)|^{q_1(x)}}{q_1(x)}dx \\
&- \lambda\int_{\Omega} \frac{V_1(x)|tv(x)|^{q_2(x)}}{q_2(x)}dx - \mu\int_{\Omega} \frac{V_2(x)|tv(x)|^{r(x)}(x)}{r(x)}dx \\
&\leq \frac{m_2 t^{\alpha p_0^-}}{\alpha(p_0^-)^\alpha}(\int_{\Omega_0\times\Omega_0} \frac{|v(x)-v(y)|^{p(x,y)}}{|x-y|^{N+sp(x,y)}}dxdy) + \frac{t^{q_{1,0}^-}}{q_{1,0}^-}\int_{\Omega_0}|v(x)|^{q_1(x)}dx \\
-\lambda&\frac{t^{q_{2,0}^-}}{q_{2,0}^+}\int_{\Omega_1} V_1(x)|v(x)|^{q_2}dx - \mu\frac{t^{r_0^+}}{r_0^+}\int_{\Omega_2} V_2(x)|v(x)|^{r(x)}dx \\
&\leq t^\beta\left(\frac{m_2}{\alpha(p_0^-)^\alpha}\int_{\Omega_0\times\Omega_0} \frac{|v(x)-v(y)|^{p(x,y)}}{|x-y|^{N+sp(x,y)}}dxdy + \frac{1}{q_{1,0}^-}\int_{\Omega_0}|v(x)|^{q_1(x)}dx\right) \\
&- \frac{min(\lambda,\mu)}{r^-}t^{r^-+\varepsilon}\left(\int_{\Omega_1} V_1(x)|v(x)|^{q_2(x)}dx + \int_{\Omega_2} V_2(x)|v(x)|^{r(x)}dx\right).
\end{aligned}$$

Therefore,

$$\psi_{\lambda,\mu}(tv) < 0,$$

for all $0 < t < \delta^{\frac{1}{\beta-r^--\varepsilon}}$, where $0 < \delta < \delta_0$ and δ_0 is given by

$$\delta_0 = \frac{min(\lambda,\mu)\left(\int_{\Omega_1} V_1(x)|v(x)|^{q_2(x)}dx + \int_{\Omega_2} V_2(x)|v(x)|^{r(x)}dx\right)}{r^-\left(m_2\int_{\Omega_0\times\Omega_0} \frac{|v(x)-v(y)|^{p(x,y)}}{|x-y|^{N+sp(x,y)}}dxdy) + \frac{1}{q_{1,0}^-}\int_{\Omega_0}|v(x)|^{q_1(x)}dx\right)}$$

Since $v = 1$ in Ω_1, then $\|u\|_{W^{s,q(x),p(x,y)}} > 0$, which complete the proof of this Lemma.
Proof. The proof of Theorem 3.2. By the Mountain pass Theorem, we conclude that the functional $\psi_{\lambda,\mu}(u)$ has a least one nontrivial critical point u in $W^{s,q(x),p(x,y)}$, which is a weak solution of our problem $(\mathcal{P})$, this achieves the proof.

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