Palestine Journal of Mathematics

Vol. 12(4)(2023) , 67–74

Characterization of σ-prime rings involving derivation

Muzibur Rahman Mozumder, Md Arshad Madni and Wasim Ahmed

Communicated by Harikrishnan Panackal

MSC 2010 Classifications: Primary 16N60; Secondary 16W25.

Keywords and phrases: σ-prime ring, derivation, involution.

The authors would like to thank the reviewers and editor for their constructive comments and valuable suggestions that improved the quality of our paper.

Muzibur Rahman Mozumder was very thankful to the DST-SERB (project MATRICS, File No. MTR/2022/000153) for their necessary support and facility.

Abstract The primary objective of this paper is to investigate σ-centralizing mappings within σ-prime rings and assess the commutativity of σ-prime rings with involution that satisfy specific identities in derivations. Finally, we will furnish examples to demonstrate the necessity of these assumptions.

1 Introduction

Throughout this paper, we will use the symbol $\mathfrak{R}$ to denote an associative ring and $\mathfrak{Z}(\mathfrak{R})$ will represent the centre of the ring $\mathfrak{R}$. For any $x_1, x_2 \in \mathfrak{R}$, the notation $[x_1, x_2]$ illustrate the commutator $x_1x_2 - x_2x_1$, and $\mathfrak{R}$ is called 2-torsion free if $2x_1 = 0 \implies x_1 = 0$ for $x_1 \in \mathfrak{R}$. We use this basic identity $[x_1x_2, z] = x_1[x_2, z] + [x_1, z]x_2$, $[x_1, x_2z] = [x_1, x_2]z + x_2[x_1, z]$ for all $x_1, x_2, z \in \mathfrak{R}$ as and when required. Remember that an involution is defined as an anti-automorphism of order 2. A ring $\mathfrak{R}$ is designated as a σ-prime ring if the conditions $a\mathfrak{R}b = a\mathfrak{R}\sigma(b) = (0)$ or $\sigma(a)\mathfrak{R}b = a\mathfrak{R}b = (0)$ imply that either $a = 0$ or $b = 0$. It's important to note that while every prime ring with involution is a σ-prime ring, the converse may not hold true in all cases. For example: Consider the set S, defined as $S = \mathfrak{R} \times \mathfrak{R}^0$, where $\mathfrak{R}^0$ represents the opposite ring of $\mathfrak{R}$. We introduce a mapping σ on S defined as $\sigma(x, y) = (y, x)$. Consequently, S qualifies as a σ-prime ring; however, it does not meet the criteria for being a prime ring. We establish the terms hermitian for an element $x_1 \in \mathfrak{R}$ when $\sigma(x_1) = x_1$, and skew-hermitian when $\sigma(x_1) = -x_1$. The assemblage of hermitian elements and skew-hermitian elements in $\mathfrak{R}$ is denoted by $\mathfrak{J}_H$ and $\mathfrak{J}_S$, respectively. In the case where $\mathfrak{R}$ is 2-torsion free, each element $x_1 \in \mathfrak{R}$ can be uniquely expressed as $2x_1 = h + k$, where h belongs to $\mathfrak{J}_H$ (the set of hermitian elements), and k belongs to $\mathfrak{J}_S$ (the set of skew-hermitian elements). The involution σ is classified as first kind if the centre of $\mathfrak{R}$, denoted as $\mathfrak{Z}(\mathfrak{R})$, is contained within $\mathfrak{J}_H$. If $\mathfrak{Z}(\mathfrak{R})$ is not a subset of $\mathfrak{J}_H$, σ is referred to as second kind. It's important to note that when σ is of the second kind, it implies that $\mathfrak{J}_S \cap \mathfrak{Z}(\mathfrak{R})$. An element $x_1 \in \mathfrak{R}$ that satisfies the condition $x_1\sigma(x_1) = \sigma(x_1)x_1$ is termed a normal element. If all elements in R meet this condition, then $\mathfrak{R}$ is labeled as a normal ring. For an example of a normal ring, refer to [5].

A mapping ψ applied to $\mathfrak{R}$ is referred to as a derivation on $\mathfrak{R}$ if it satisfies the conditions $\psi(x_1+x_2) = \psi(x_1)+\psi(x_2)$ and $\psi(x_1x_2) = \psi(x_1)x_2+x_1\psi(x_2)$ for all $x_1, x_2 \in \mathfrak{R}$. A map f that operates from $\mathfrak{R}$ into itself is termed a centralizing map on $\mathfrak{R}$ if the condition $[f(x_1), x_1] \in \mathfrak{Z}(\mathfrak{R})$ holds for all $x_1 \in \mathfrak{R}$. Specifically, if $[f(x_1), x_1] = 0$ holds for all $x_1 \in \mathfrak{R}$, it is referred to as a commuting map. Inspired by the concept of a centralizing map, a map f that operates from $\mathfrak{R}$ into itself is denoted as σ-centralizing if it adheres to the condition $[f(x_1), \sigma(x_1)] \in \mathfrak{Z}(\mathfrak{R})$ for all $x_1 \in \mathfrak{R}$. Similarly, it is termed σ-commuting if $[f(x_1), \sigma(x_1)] = 0$ for all $x_1 \in \mathfrak{R}$. Therefore, it is reasonable to investigate the aforementioned mappings in the context of prime rings, σ-prime rings and semi-prime rings with involution.

Posner previously demonstrated that in the presence of a nonzero centralizing derivation within a prime ring, the prime ring must necessarily be commutative. In recent years, several algebraists have established the commutativity theorem for prime and semi-prime rings by incorporating automorphisms and derivations into their investigations for detail see [1, 3, 4, 7, 8]. In 2014, S. Ali and colleagues, as documented in [2], initiated an investigation into the σ-version of Posner's theorem. They established that if $\mathfrak{R}$ is a prime ring with involution σ, and the characteristic of $\mathfrak{R}$ is not equal to 2, and if ψ is a non-zero derivation of $\mathfrak{R}$ satisfying the condition$[\psi(x), \sigma(x)] \in \mathfrak{Z}(\mathfrak{R})$ and $\psi(\mathfrak{J}_S \cap \mathfrak{Z}(\mathfrak{R})) \neq 0$ for all $x \in \mathfrak{R}$, then $\mathfrak{R}$ is proven to be commutative. The primary objective of this paper is to investigate the σ-version of Posner's theorem as it pertains to σ-prime rings. Furthermore, we have also established the commutativity of σ-prime rings by considering differential identities.

2 MAIN RESULTS

Lemma 2.1. *Consider $\mathfrak{R}$ as a σ-prime ring with the involution σ, and let ψ be a derivation on $\mathfrak{R}$ that commutes with σ. For any element a in $\mathfrak{R}$, if $a\psi(x_1) = 0$ for all $x_1 \in \mathfrak{R}$, then it follows that a must be 0, or ψ is the zero derivation.*

Proof. We have give that $a\psi(x_1) = 0$ for all $x_1 \in \mathfrak{R}$, on replacing x_1 by x_1x_2, we obtain $ax_1\psi(x_2) = 0$ for all $x_1, x_2 \in \mathfrak{R}$. On changing x_2 by $\sigma(x_2)$, we get $ax_1\psi(\sigma(x_2)) = 0$ for all $x_1, x_2 \in \mathfrak{R}$, then we have $a\mathfrak{R}\psi(x_2) = a\mathfrak{R}\psi(\sigma(x_2))$=(0), by the definition of σ-prime rings we have either $a = 0$ or $\psi(x_2) = 0$ for all $x_2 \in \mathfrak{R}$, implies $\psi = 0$. □

Lemma 2.2. *Let $\mathfrak{R}$ be a σ-prime rings, with involution σ and $I \neq (0)$ be a right ideal in $\mathfrak{R}$. If ψ be a derivation on $\mathfrak{R}$ which is zero on I and commutes with σ, then ψ is zero on $\mathfrak{R}$.*

Proof. As we have that, $\psi(I) = 0$ implies $0 = \psi(I\mathfrak{R}) = \psi(I)\mathfrak{R} + I\psi(\mathfrak{R}) = I\psi(\mathfrak{R})$, so by Lemma 2.1, we have $\psi(\mathfrak{R}) = (0)$ implies $\psi = 0$. □

Lemma 2.3. *Let $\mathfrak{R}$ is σ-prime rings with involution σ contains a commutative non-zero right ideal I and σ commutes with derivation on $\mathfrak{R}$, then $\mathfrak{R}$ is commutative.*

Proof. For $x_1 \in I$, we have $[x_1, I] = (0) = I_{x_1}(I)$ so by Lemma 2.1, we have $I_{x_1} = (0)$ on $\mathfrak{R}$ gives us, $x_1 \in \mathfrak{Z}(\mathfrak{R})$ implies $[x_1, \mathfrak{R}] = (0)$ for all $x_1 \in I$, $[a, I] = (0) = I_a(I)$ for all $a \in \mathfrak{R}$, using Lemma 2.1, we have $I_a = 0$ for all $a \in \mathfrak{R}$ implies $a \in \mathfrak{Z}(\mathfrak{R})$ for all $a \in \mathfrak{R}$, yields the desired result. □

Lemma 2.4. *Let b and ab, is in the centre of σ-prime ring $\mathfrak{R}$ and σ commutes with ψ, if $b \neq 0$, then a must be in $\mathfrak{Z}(\mathfrak{R})$.*

Proof. Since b and ab is in $\mathfrak{Z}(\mathfrak{R})$, then $0 = [ab, r] = [a, r]b$ for all $a \in \mathfrak{R}$, further implies $I_a(r)b = 0$, applying Lemma 2.1, we get either $b = 0$ or $I_a = 0$, since $b \neq 0$ then later case implies that $a \in \mathfrak{Z}(\mathfrak{R})$. □

Lemma 2.5. *Let $\mathfrak{R}$ be a σ-prime ring of characteristics different from 2, then $\mathfrak{R}$ is 2-torsion free.*

Proof. Assume $x_1 \in \mathfrak{R}$ and $2x_1 = 0$ implies, $2x_1rs = 0$ for all $r, s \in \mathfrak{R}$ and $x_1\mathfrak{R}(2s) = (0)$ for all $s \in \mathfrak{R}$. Since characteristics of $\mathfrak{R}$ is different from 2 and $\mathfrak{R} \neq (0)$, this gives us $s \neq 0 \in \mathfrak{R}$ satisfying $2s \neq 0$, gives us $(0) = x_1\mathfrak{R}(2s) = x_1\mathfrak{R}\sigma(2s)$, by the definition of σ-prime rings we obtain $x_1 = 0$, hence $\mathfrak{R}$ is 2-torsion free. □

Lemma 2.6. *Let $\mathfrak{R}$ be a 2-torsion free semi prime ring. If $b \in \mathfrak{R}$ commutes with all of its commutator $[b, x_1]$ for all $x_1 \in \mathfrak{R}$, then $b \in \mathfrak{Z}(\mathfrak{R})$.*

Lemma 2.7. *For σ-prime ring $\mathfrak{J}_H \cap \mathfrak{Z}(\mathfrak{R})$ and $\mathfrak{J}_S \cap \mathfrak{Z}(\mathfrak{R})$, are free from zero-divisor.*

Proof. Let $a, b \in \mathfrak{Z}(\mathfrak{R}) \cap \mathfrak{J}_H$ and $ab = 0$, implies $abr = 0$ for all $r \in \mathfrak{R}$ gives us $(0) = a\mathfrak{R}b = a\mathfrak{R}\sigma(b)$, by definition of σ-prime ring, we have $a = 0$ or $b = 0$, which completes the proof of Lemma. □

Proposition 2.8. *([2], Proposition 2.2) Let $(\mathfrak{R}, \sigma)$ be a 2-torsion free semiprime ring. If $f\colon \mathfrak{R} \to \mathfrak{R}$ is an additive mapping and satisfying $[f(x_1), \sigma(x_1)] \in \mathfrak{Z}(\mathfrak{R})$ for all $x_1 \in \mathfrak{R}$, then $[f(x_1), \sigma(x_1)] = 0$ for all $x_1 \in \mathfrak{R}$.*

Lemma 2.9. *Let $\mathfrak{R}$ be a σ-prime ring of characteristics different from 2, where involution σ is of the second kind and if $\mathfrak{R}$ is normal, then $\mathfrak{R}$ is commutative.*

Proof. Given that $\mathfrak{R}$ is normal, it follows that $hk = kh$ where $h \in \mathfrak{J}_H$ and $k \in \mathfrak{J}_S$. For any $x_1 \in \mathfrak{R}$, we have $x_1 - \sigma(x_1) \in \mathfrak{J}_S$

$$h(x_1 - \sigma(x_1)) = (x_1 - \sigma(x_1))h. \tag{2.1}$$

Since σ is of the second kind then we have $0 \neq s \in \mathfrak{J}_S \cap \mathfrak{Z}(\mathfrak{R})$ and $s(x_1 + \sigma(x_1)) \in \mathfrak{J}_S$ for all $x_1 \in \mathfrak{R}$, using normality of rings $\mathfrak{R}$ we have, $hs(x_1 + \sigma(x_1)) = s(x_1 + \sigma(x_1))h$ for all $x_1 \in \mathfrak{R}$, where $h \in \mathfrak{J}_H$, therefore last relation further implies

$$s\{h(x_1 + \sigma(x_1)) - (x_1 + \sigma(x_1))h\} = 0 \tag{2.2}$$

for all $x_1 \in \mathfrak{R}$. Applying Lemma 2.7, we find that either $s = 0$ or $h(x_1+\sigma(x_1)) = (x_1+\sigma(x_1))h$. First case is not possible by our supposition and later case together with (2.1), gives $hx_1 = x_1h$ for all $x_1 \in \mathfrak{R}$. On changing x_1 by x_2 for any $x_2 \in \mathfrak{R}$, we obtain

$$hx_2 = x_2h. \tag{2.3}$$

In view of the fact that $x_1 + \sigma(x_1) \in \mathfrak{J}_H$, replacing h by $x_1 + \sigma(x_1)$ in (2.3), we get

$$\{x_1 + \sigma(x_1)\}x_2 = x_2\{x_1 + \sigma(x_1)\} \tag{2.4}$$

for all $x_1, x_2 \in \mathfrak{R}$. Now, we take $0 \neq s \in \mathfrak{J}_S \cap \mathfrak{Z}(\mathfrak{R})$, then $s(x_1 - \sigma(x_1)) \in \mathfrak{J}_H$ and equation (2.3) implies that, $s\{(x_1 - \sigma(x_1))x_2 - x_2(x_1 - \sigma(x_1))\} = 0$ for all $x_1, x_2 \in \mathfrak{R}$. Using Lemma 2.7, we obtain

$$(x_1 - \sigma(x_1))x_2 = x_2(x_1 - \sigma(x_1)). \tag{2.5}$$

Last equation together with (2.4), gives us $x_1x_2 = x_2x_1$ for all $x_1, x_2 \in \mathfrak{R}$. Accordingly, we reach the prescribed result. □

Lemma 2.10. *Let $\mathfrak{R}$ be a σ-prime ring with involution σ, which is of the second kind, if σ is centralizing then $\mathfrak{R}$ is commutative.*

Proof. Based on the given criterion. For all $x_1 \in \mathfrak{R}$, we have

$$[x_1, \sigma(x_1)] \in \mathfrak{Z}(\mathfrak{R}). \tag{2.6}$$

Linearizing (2.6), we get

$$[x_1, \sigma(x_2)] + [x_2, \sigma(x_1)] \in \mathfrak{Z}(\mathfrak{R}). \tag{2.7}$$

For all $x_1, x_2 \in \mathfrak{R}$. Last relation further implies that

$$[[x_1, x_2], x_1] + [[\sigma(x_2), \sigma(x_1)], x_1] = 0. \tag{2.8}$$

Replacing x_2 by x_2x_1 in (2.8), we get

$$[[x_1, x_2], x_1]x_1 + \sigma(x_1)[[\sigma(x_2), \sigma(x_1)], x_1] + [\sigma(x_1), x_1][\sigma(x_2), \sigma(x_1)] = 0 \tag{2.9}$$

for all $x_1, x_2 \in \mathfrak{R}$. Combining (2.8) and (2.9), we obtain

$$[[x_1, x_2], x_1]x_1 - \sigma(x_1)[[x_2, x_1], x_1] + [\sigma(x_1), x_1][\sigma(x_2), \sigma(x_1)] = 0. \tag{2.10}$$

Taking x_2x_1 in place of x_2 in the above equation, we obtain

$$[[x_1, x_2], x_1]x_1^2 - \sigma(x_1)[[x_2, x_1], x_1]x_1 + [\sigma(x_1), x_1]\sigma(x_1)[\sigma(x_2), \sigma(x_1)] = 0 \tag{2.11}$$

for all $x_1, x_2 \in \mathfrak{R}$. Using equations (2.10) and (2.11) and substituting $\sigma(x_1)$ for x_1 and $\sigma(x_2)$ for x_2, we obtain

$$[x_1, \sigma(x_1)]\{x_1[x_2, x_1] - [x_2, x_1]\sigma(x_1)\} = 0. \tag{2.12}$$

For all $x_1, x_2 \in \mathfrak{R}$. Replacing x_2 by x_2x_1 in (2.12), we attain

$$[x_1, \sigma(x_1)]\{x_1[x_2, x_1]x_1 - [x_2, x_1]x_1\sigma(x_1)\} = 0 \tag{2.13}$$

for all $x_1, x_2 \in \mathfrak{R}$. When (2.12) is utilized in (2.13), it yields

$$[x_1, \sigma(x_1)][x_2, x_1]\{-x_1\sigma(x_1) + \sigma(x_1)x_1\} = 0. \tag{2.14}$$

For all $x_1, x_2 \in \mathfrak{R}$. In such a way that we obtain

$$[x_1, \sigma(x_1)]^2\mathfrak{R}[x_2, x_1] = 0. \tag{2.15}$$

For all $x_1, x_2 \in \mathfrak{R}$. Replacing x_1 by $\sigma(x_1)$ and x_2 by $\sigma(x_2)$ in the above equation, we obtain

$$(0) = [x_1, \sigma(x_1)]^2\mathfrak{R}[x_2, x_1] = [x_1, \sigma(x_1)]^2\mathfrak{R}\,\sigma\{[x_2, x_1]\}. \tag{2.16}$$

For all $x_1, x_2 \in \mathfrak{R}$. Accordingly, with the definition of a σ-prime ring, we get

$$\text{either} \quad [x_1, \sigma(x_1)]^2 = 0, \quad \text{or} \quad [x_1, x_2] = 0. \tag{2.17}$$

For all $x_1, x_2 \in \mathfrak{R}$. Later case implies that $\mathfrak{R}$ is commutative, first case implies that

$$[x_1, \sigma(x_1)]^2 = 0. \tag{2.18}$$

With the help of Lemma 2.7, we obtain

$$[x_1, \sigma(x_1)] = 0. \tag{2.19}$$

For all $x_1 \in \mathfrak{R}$. Hence we obtain $\mathfrak{R}$ is normal. Using Lemma 2.9, the desired conclusion is achieved. □

Theorem 2.11. *Let $\mathfrak{R}$ be a 2-torsion free σ-prime ring, and $\psi \neq 0$ be a derivation on $\mathfrak{R}$ commutes with σ. If for any $a \in \mathfrak{R}$ satisfying $a\psi(x_1) = \psi(x_1)a$ for all $x_1 \in \mathfrak{R}$, then $a \in \mathfrak{Z}(\mathfrak{R})$.*

Proof. Let on contrary $a \notin \mathfrak{Z}(\mathfrak{R})$ and we have $[a, \psi(x_1)] = 0$ for all $x_1 \in \mathfrak{R}$. Putting x_1x_2 in place of x_1, we obtain

$$[a, x_1]\psi(x_2) + \psi(x_1)[a, x_2]. \tag{2.20}$$

For all $x_1, x_2 \in \mathfrak{R}$ and assume that $x_2 \in \mathfrak{R}$ commutes with a, we have

$$C_{\mathfrak{R}}(a) = \{x_2 \in \mathfrak{R} \mid ax_2 = x_2a\}. \tag{2.21}$$

Invoking last relation in (2.20), we get

$$[a, x_1]\psi(x_2) = 0. \tag{2.22}$$

For all $x_1, x_2 \in \mathfrak{R}$, when we substitute rx_1 for x_1 in (2.22), we acquire

$$[a, r]\mathfrak{R}\psi(x_2) = 0. \tag{2.23}$$

For all $r, x_2 \in \mathfrak{R}$. Since, by assumption x_2 commutes with a i.e., $[a, x_2] = 0$, for any $a \in \mathfrak{R}$, we can write $2a = h + k$ where $h \in \mathfrak{J}_H$ and $k \in \mathfrak{J}_S$. Therefore we obtain

$$[h + k, x_2] = 0 \ \text{ where, } h \in \mathfrak{J}_H, \ \ k \in \mathfrak{J}_S \text{ and } \ \ x_2 \in C_{\mathfrak{R}}(a). \tag{2.24}$$

Taking involution σ both side, we get

$$[h - k, \sigma(x_2)] = 0 \ \text{ where, } \ k \in \mathfrak{J}_S, \ \ h \in \mathfrak{J}_H \text{ and } x_2 \in C_{\mathfrak{R}}(a). \tag{2.25}$$

Using (2.24) and (2.25), we get

$$[h, x_2 + \sigma(x_2)] = 0 \text{ where, } k \in \mathfrak{J}_S, \quad h \in \mathfrak{J}_H \text{ and } x_2 \in C_{\mathfrak{R}}(a). \tag{2.26}$$

Again using (2.24) and (2.25), we get

$$[k, x_2 - \sigma(x_2)] = 0 \text{ where, } k \in \mathfrak{S}(\mathfrak{R}), \quad h \in \mathfrak{J}_H \text{ and } x_2 \in C_{\mathfrak{R}}(a). \tag{2.27}$$

Taking involution σ both side, yields

$$[k, \sigma(x_2) - x_2] = 0 \text{ where } k \in \mathfrak{J}_S \text{ and } x_2 \in C_{\mathfrak{R}}(a). \tag{2.28}$$

Using (2.26) and (2.28) and $char(\mathfrak{R}) \neq 2$, we obtain

$$[h + k, \sigma(x_2)] = 0 \text{ where } k \in \mathfrak{J}_S, \quad h \in \mathfrak{J}_H \text{ and } x_2 \in C_{\mathfrak{R}}(a). \tag{2.29}$$

Finally, we have

$$[a, \sigma(x_2)] = 0 \text{ where, } x_2 \in C_{\mathfrak{R}}(a) \text{ and } a \in \mathfrak{R}. \tag{2.30}$$

Implies that

$$\sigma(x_2) \in C_{\mathfrak{R}}(a), \quad \text{for } x_2 \in C_{\mathfrak{R}}(a). \tag{2.31}$$

Replacing x_2 by $\sigma(x_2)$ in (2.23), we obtain

$$[a, r]\mathfrak{R}\psi(\sigma(x_2)) = (0) \quad \text{for } x_2 \in C_{\mathfrak{R}}(a) \text{ and } a \in \mathfrak{R}. \tag{2.32}$$

By utilizing the condition that σ commutes with ψ, we deduce

$$[a, r]\mathfrak{R}\psi(x_2) = [a, r]\mathfrak{R}\sigma(\psi(x_2)) = (0) \quad \text{for } x_2 \in C_{\mathfrak{R}}(a) \text{ and } a \in \mathfrak{R}. \tag{2.33}$$

Now, based on the definition of a σ-prime ring, we derive $\psi(x_2) = 0$ or $[a, r] = 0$ for all $r \in \mathfrak{R}$. The latter case is not valid according to our assumption, leaving us with the first case $\psi(x_2)$=0 for $x_2 \in C_{\mathfrak{R}}(a)$. That implies ψ vanishes on the element of $C_{\mathfrak{R}}(a)$ and $\psi(x_1) \in C_{\mathfrak{R}}(a)$ for all $x_1 \in \mathfrak{R}$, certainly, we can obtain $\psi^2(x_1) = 0$ for all $x_1 \in \mathfrak{R}$. According to Lemma 2.6, we deduce that $\psi = 0$. This contradicts our initial assumption. Thus $a \in \mathfrak{Z}(\mathfrak{R})$. □

Theorem 2.12. *Let $\mathfrak{R}$ be a σ-prime ring of characteristics different from* 2 *and $\psi \neq 0$ be a derivation on $\mathfrak{R}$ commutes with σ. If $[\psi(x_1), \sigma(x_1)] = 0$ for all $x_1 \in \mathfrak{R}$ and $\psi(\mathfrak{Z}(\mathfrak{R})) \neq \{0\}$, then $\mathfrak{R}$ is commutative.*

Proof. In accordance with our assumption.

$$[\psi(x_1), \sigma(x_1)] = 0. \tag{2.34}$$

Linearizing (2.34), results in the following.

$$[\psi(x_1), \sigma(x_2)] + [\psi(x_2), \sigma(x_1)] = 0. \tag{2.35}$$

For all $x_1, x_2 \in \mathfrak{R}$. In accordance with our assumption there exists $0 \neq z \in \mathfrak{Z}(\mathfrak{R})$ such that $\psi(z) \neq 0$. Replacing x_2 by $x_2 z$, we obtain

$$[\psi(x_1), \sigma(x_2)]\sigma(z) + [\psi(x_2), \sigma(x_1)]z + [x_2, \sigma(x_1)]\psi(z) = 0. \tag{2.36}$$

For all $x_1, x_2 \in \mathfrak{R}$ and $z \in \mathfrak{Z}(\mathfrak{R})$. Invoking (2.35) in (2.36), yields

$$[\psi(x_2), \sigma(x_1)](z - \sigma(z)) + [x_2, \sigma(x_1)]\psi(z) = 0. \tag{2.37}$$

Case(1): Let σ is of the first kind i.e., $\sigma(z) = z$ for all $z \in \mathfrak{Z}(\mathfrak{R})$, we get

$$[x_2, \sigma(x_1)]\psi(z) = 0 \text{ for all } x_1, x_2 \in \mathfrak{R} \text{ and } z \in \mathfrak{Z}(\mathfrak{R}). \tag{2.38}$$

Given that σ and ψ commutes, thus $\psi(z) \in \mathfrak{J}_H \cap \mathfrak{Z}(\mathfrak{R})$ and using Lemma 2.7, we obtain either $\psi(z) = 0$ or $[x_2, \sigma(x_1)] = 0$. By given condition $\psi(z) = 0$ is not possible, therefore latter case implies

$$[x_2, \sigma(x_1)] = 0. \tag{2.39}$$

For all $x_1, x_2 \in \mathfrak{R}$ replacing x_2 by $\sigma(x_2)$ in the above equation, we obtain

$$[x_2, x_1] = 0 \quad \text{for all} \quad x_1, x_2 \in \mathfrak{R}. \tag{2.40}$$

Thus $\mathfrak{R}$ is commutative.

Case(2): When σ is of the second kind, replacing x_1 by $x_1 z$ in (2.34), where $0 \neq z \in \mathfrak{Z}(\mathfrak{R})$.

$$[\psi(x_1), \sigma(x_1)]z\sigma(z) + [x_1, \sigma(x_1)]\sigma(z)\psi(z) = 0. \tag{2.41}$$

For all $x_1 \in \mathfrak{R}$. Using (2.34) in the above equation, we have

$$[x_1, \sigma(x_1)]\sigma(z)\psi(z) = 0 \quad \text{for all} \ \ x_1 \in \mathfrak{R} \quad \text{and } z \in \mathfrak{Z}(\mathfrak{R}). \tag{2.42}$$

In particular, taking $z \in \mathfrak{Z}(\mathfrak{R}) \cap \mathfrak{J}_S$, above equation reduces to

$$[x_1, \sigma(x_1)]\sigma(z)\psi(z) = 0 \quad \text{for all} \ \ x_1 \in \mathfrak{R}, \quad \text{for all} \ \ z \in \mathfrak{Z}(\mathfrak{R}) \cap \mathfrak{J}_S. \tag{2.43}$$

Since σ commutes with ψ, therefore $\psi(z)\sigma(z) \in \mathfrak{J}_H \cap \mathfrak{Z}(\mathfrak{R})$ and using Lemma 2.7, we obtain either, $\psi(z)\sigma(z) = 0$ or $[x_1, \sigma(x_1)] = 0$. By employing the given condition we can say that $\psi(z)\sigma(z) = 0$ is not possible and later case implies

$$[x_1, \sigma(x_1)] = 0 \tag{2.44}$$

for all $x_1 \in \mathfrak{R}$. Hence $\mathfrak{R}$ is normal, on using the Lemma 2.9, commutativity of $\mathfrak{R}$ holds. □

Theorem 2.13. *Let $\mathfrak{R}$ be a σ-prime ring of characteristics different from 2 and $\psi \neq 0$ be a derivation on $\mathfrak{R}$ commutes with σ. If $[\psi(x_1), \sigma(x_1)] \in \mathfrak{Z}(\mathfrak{R})$ for all $x_1 \in \mathfrak{R}$ and $\psi(\mathfrak{Z}(\mathfrak{R})) \neq \{0\}$, then $\mathfrak{R}$ is commutative.*

Proof. By the given condition, we have $[\psi(x_1), \sigma(x_1)] \in \mathfrak{Z}(\mathfrak{R})$ for all $x_1 \in \mathfrak{R}$. Using Proposition 2.8, gives that $[\psi(x_1), \sigma(x_1)] = 0$ for all $x_1 \in \mathfrak{R}$. Now applying Theorem 2.12, we infer that, $\mathfrak{R}$ is commutative. □

Theorem 2.14. *Let $\mathfrak{R}$ be a 2-torsion free σ-prime ring with involution σ of the second kind and $\psi \neq 0$ be a derivation on $\mathfrak{R}$ which commute with σ. If $\psi([x_1, \sigma(x_1)]) = 0$ for all $x_1 \in \mathfrak{R}$, then $\mathfrak{R}$ is commutative.*

Proof. By the given condition, we have

$$\psi([x_1, \sigma(x_1)]) = 0 \quad \text{forall} \quad x_1 \in \mathfrak{R}. \tag{2.45}$$

Linearization of (2.45), gives us

$$\psi([x_1, \sigma(x_2)]) + \psi([x_2, \sigma(x_1)]) = 0. \tag{2.46}$$

For all $x_1, x_2 \in \mathfrak{R}$. Replacing x_2 by $x_2 h$ where $h \in \mathfrak{J}_H \cap \mathfrak{Z}(\mathfrak{R})$, we obtain

$$\{\psi([x_1, \sigma(x_2)]) + \psi([x_2, \sigma(x_1)])\}h + \{[x_1, \sigma(x_2)] + [x_2, \sigma(x_1)]\}\psi(h) = 0 \tag{2.47}$$

for all $x_1, x_2 \in \mathfrak{R}$. Invoking (2.46) in (2.47), yields

$$\{[x_1, \sigma(x_2)] + [x_2, \sigma(x_1)]\}\psi(h) = 0. \tag{2.48}$$

Since σ and ψ commute with each other so $\psi(h) \in \mathfrak{J}_H \cap \mathfrak{Z}(\mathfrak{R})$, using Lemma 2.7, we have either $\psi(h) = 0$ or $[x_1, \sigma(x_2)] + [x_2, \sigma(x_1)] = 0$. But $\psi(h) = 0$ is not possible because σ is of the second kind. Thus we have

$$[x_1, \sigma(x_2)] + [x_2, \sigma(x_1)] = 0. \tag{2.49}$$

For all $x_1, x_2 \in \mathfrak{R}$. Replacing x_2 by x_1 and using $char(\mathfrak{R}) \neq 2$, we get

$$[x_1, \sigma(x_1)] = 0 \quad \text{for all} \ \ x_1 \in \mathfrak{R}. \tag{2.50}$$

Hence $\mathfrak{R}$ is normal. Invoking Lemma 2.9, we get our required result. □

Theorem 2.15. *Let $\mathfrak{R}$ be a 2-torsion σ-prime rings with involution σ which is of the second kind and ψ be a nonzero derivation on $\mathfrak{R}$, which commutes with σ. If $\psi([x_1, \sigma(x_1)]) \in \mathfrak{Z}(\mathfrak{R})$ for all $x_1 \in \mathfrak{R}$. Then $\mathfrak{R}$ is commutative.*

Proof. By the assumption, we have

$$\psi([x_1, \sigma(x_1)]) \in \mathfrak{Z}(\mathfrak{R}) \text{ for all } x_1 \in \mathfrak{R}. \tag{2.51}$$

Linearizing above equation, we find that

$$\psi([x_1, \sigma(x_2)]) + \psi([x_2, \sigma(x_1)]) \in \mathfrak{Z}(\mathfrak{R}) \text{ for all } x_1, x_2 \in \mathfrak{R}. \tag{2.52}$$

Replacing x_2 by $x_2 h$ where $h \in \mathfrak{J}_H \cap \mathfrak{Z}(\mathfrak{R})$, we obtain

$$\{\psi([x_1, \sigma(x_2)]) + \psi([x_2, \sigma(x_1)])\}h + \{[x_1, \sigma(x_2)] + [x_2, \sigma(x_1)]\}\psi(h) \in \mathfrak{Z}(\mathfrak{R}). \tag{2.53}$$

For all $x_1, x_2 \in \mathfrak{R}$. Invoking (2.52) in (2.53), yields

$$\{[x_1, \sigma(x_2)] + [x_2, \sigma(x_1)]\}\psi(h) \in \mathfrak{Z}(\mathfrak{R}) \quad \text{for all } x_1, x_2 \in \mathfrak{R} \text{ and } h \in \mathfrak{Z}(\mathfrak{R}) \cap \mathfrak{J}_H. \tag{2.54}$$

Implies that

$$\{[x_1, \sigma(x_2)] + [x_2, \sigma(x_1), r]\}\psi(h) = 0 \quad \text{for all } r, x_1, x_2 \in \mathfrak{R}. \tag{2.55}$$

As σ and ψ commutes. Hence $\psi(h) \in \mathfrak{J}_H \cap \mathfrak{Z}(\mathfrak{R})$ and in σ-prime ring $\mathfrak{J}_H \cap \mathfrak{Z}(\mathfrak{R})$ is free from zero divisor, we have either $\psi(h) = 0$ or $[x_1, \sigma(x_2)] + [x_2, \sigma(x_1)] \in \mathfrak{Z}(\mathfrak{R})$. But $\psi(h) = 0$ is not possible because σ is of the second kind. Thus, we have

$$[x_1, \sigma(x_2)] + [x_2, \sigma(x_1)] \in \mathfrak{Z}(\mathfrak{R}). \tag{2.56}$$

For all $x_1, x_2 \in \mathfrak{R}$. Replacing x_2 by x_1 and using $char(\mathfrak{R}) \neq 2$, we obtain

$$[x_1, \sigma(x_1)] \in \mathfrak{Z}(\mathfrak{R}). \tag{2.57}$$

Applying Lemma 2.10, we can establish that $\mathfrak{R}$ is a commutative ring. □

The example below illustrates that the condition requiring σ to be of the second kind is essential in Theorems 2.14 and 2.15.

Example 2.16. Let us take $\mathfrak{R} = \left\{ \begin{bmatrix} a & b \\ c & d \end{bmatrix} \middle| a, b, c, d \in \mathbb{Z} \right\}$, define σ on $\mathfrak{R}$ in such away, $\sigma\left(\begin{bmatrix} a & b \\ c & d \end{bmatrix}\right) = \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$ and set $\psi \neq 0$ as follows $\psi\left(\begin{bmatrix} a & b \\ c & d \end{bmatrix}\right) = \begin{bmatrix} 0 & b \\ 0 & 0 \end{bmatrix}$. It is straightforward to verify that $\mathfrak{R}$ is a σ-prime ring with the first kind of involution, ψ is non zero derivation fulfilling the condition of Theorems 2.14 and 2.15, however $\mathfrak{R}$ is not commutative.

It is a widely recognized that zero-divisors cannot exist in the centre of a prime ring. However, in σ-prime rings, it's important to note that the centre may not be devoid of zero-divisors. The follwing example explain the above Lemma.

Example 2.17. Let us consider $\mathfrak{R} = \left\{ \begin{bmatrix} a & 0 \\ 0 & b \end{bmatrix} \middle| a, b, \in \mathbb{Z} \right\}$, define σ on $\mathfrak{R}$ in such away, $\sigma\left(\begin{bmatrix} a & 0 \\ 0 & b \end{bmatrix}\right) = \begin{bmatrix} b & 0 \\ 0 & a \end{bmatrix}$. It is easy to verify that $\mathfrak{R}$ is σ-prime ring. For any non-zero a, $\begin{bmatrix} a & 0 \\ 0 & 0 \end{bmatrix} \in \mathfrak{Z}(\mathfrak{R})$ and for any non-zero b, $\begin{bmatrix} 0 & 0 \\ 0 & b \end{bmatrix} \in \mathfrak{R}$ and $\begin{bmatrix} a & 0 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} 0 & 0 \\ 0 & b \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$. This shows the Lemma.

References

[1] S. Ali and A. Alhazmi: *Some commutativity theorems in prime rings with involution and derivation*, Journal of Advances in Mathematics and Computer Science, **24(5)**,1-6, (2017).

[2] S. Ali and N. A. Dar: *On $*$-centralizing mappings in rings with involution*, Georgian Math. J. **1**, 25-28, (2014).

[3] S. Ali and N. A. Dar: *On left centralizers of prime rings with involution*, Palestine Journal of Mathematics, **3(1)**, 505-511, (2014).

[4] S. Ali, N. A. Dar and M. Asci: *On derivation and commutativity of prime rings with involution*, Georgian Math. J. **23 (1)**, 9-14, (2016).

[5] I. N. Herstein: *Rings with Involution*, University of Chicago Press, Chicago, 1976.

[6] I. N. Herstein: *A note on derivation*, Canad. Math. Bull. **22(4)**, 509-511, (1979).

[7] M. Hongan and N. U. Rehman : *On generalized Jordan $*$-derivations in semiprime rings with involution*, Palestine Journal of Mathematics, **1(2)**, 74-75, (2012).

[8] A. Maha, A. Alaa and M. Najat: *Some commutativity theorems for rings with involution involving generalized derivations*, Palestine Journal of Mathematics, **8(2)**, 169-176, (2019).

[9] J. H. Mayne: *Centralizing mappings of prime rings*, Canad. Math. Bull. **27**, 122-126, (1984).

[10] B. Nejjar, A. Kacha, A.Mamouni and L.Oukhtite: *Commutativity theorems in rings with involution*, Comm. Algebra, **45(2)**, 698-708, (2017).

[11] E. C. Posner: *Derivations in prime rings*, Proc. Amer. Math. Soc, **8**, 1093-1100, (1957).

Author information

Muzibur Rahman Mozumder, Department of Mathematics, Aligarh Muslim University, India.
E-mail: muzibamu81@gmail.com

Md Arshad Madni, Department of Mathematics, Aligarh Muslim University, India.
E-mail: arshadmadni7613@gmail.com

Wasim Ahmed, Department of Mathematics, Aligarh Muslim University, India.
E-mail: wasim10041994@gmail.com

Received: August 22, 2022.
Accepted: July 27, 2023.