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On a solution of the Kolmogorov-Nikol'skii problem in class of $\overline{\psi}$- integrals

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Abstract In this paper, we investigate the problem of approximation of classes $C_{\infty}^{\overline{\psi}}$ introduced by A. I. Stepanets by the generalized Zygmund sums. Especially, we obtain asymptotic equalities that give a solution of the Kolmogorov-Nikol'skii problem for the generalized Zygmund sums on the classes $C_{\infty}^{\overline{\psi}}$ in several important cases.

1 Introduction

Assume that L denote the space of integrable 2π-periodic functions, and let

$$S[f] = \frac{a_o}{2} + \sum_{k=1}^{\infty}(a_k \cos kx + b_k \sin kx) \equiv \sum_{k=0}^{\infty} A_k(f;x) \tag{1.1}$$

be the Fourier series of a function $f \in L$ where

$$a_k = a_k(f) = \frac{1}{\pi}\int_{-\pi}^{\pi} f(t)\cos ktdt \quad , \quad fork = 0,1,2,\cdots$$

$$b_k = b_k(f) = \frac{1}{\pi}\int_{-\pi}^{\pi} f(t)\sin ktdt \quad , \quad fork = 0,1,2,\cdots .$$

It is known that $C_{\infty}^{\overline{\psi}}$ is class of 2π - periodic continuous functions expressed by

$$f(x) = \frac{a_0}{2} + \frac{1}{\pi}\int_{-\pi}^{\pi} \theta(x-t)\Psi(t)dt = \frac{a_0}{2} + (f^{\overline{\psi}} * \Psi)(x) \ ,$$

where $\Psi(x)$ is a a function that has the Fourier series

$$\sum_{k=1}^{\infty}(\psi_1(k)\cos kx + \psi_2(k)\sin kx),$$

$\overline{\psi} = (\psi_1, \psi_2)$ is a pair of arbitrary fixed systems of numbers $\psi_1(k)$ and $\psi_2(k)$, $k = 1,2,\cdots$ [10]. Here, the function θ is called $\overline{\psi}$ - derivative of function f, and is denoted by $f^{\overline{\psi}}(\cdot)$, $\operatorname{ess\,sup}_t |\theta(t)| \leq 1$, $\int_{-\pi}^{\pi} \theta(t)dt = 0$.

Let $\mathfrak{M}$ shows the set of continuous positive functions $\psi(t)$ convex downward for $t \geq 1$ and satisfying the condition $\lim\limits_{t\to\infty} \psi(t) = 0$, i.e., for $\Delta(\psi, t_1, t_2) = \psi(t_1) - 2\psi(\frac{t_1+t_2}{2}) + \psi(t_2)$,

$$\mathfrak{M} = \left\{\psi(t), t \geq 1 : \psi(t) > 0, \Delta(\psi, t_1, t_2) \geq 0, \forall t_1, t_2 \in [1, \infty) , \lim_{t\to\infty} \psi(t) = 0\right\},$$

$$\mathfrak{M}' = \left\{\psi(\cdot) \in \mathfrak{M} : \int_1^\infty \frac{\psi(t)}{t} dt < \infty\right\}.$$

We also set

$$\mathfrak{M}_o = \{\psi \in \mathfrak{M} : 0 < \zeta(\psi, t) \leq K < \infty, \forall t \geq 1\},$$

where

$$\zeta(\psi, t) = \frac{t}{\xi(\psi, t) - t},$$

$$\xi(\psi, t) = \psi^{-1}\left(\frac{\psi(t)}{2}\right),$$

$\psi^{-1}(\cdot)$ is the function inverse to $\psi(\cdot)$, and the constant K may depend on the function ψ.

In [10], if $\psi_1(v) = \psi(v)\cos\dfrac{\beta\pi}{2}$ and $\psi_2(v) = \psi(v)\sin\dfrac{\beta\pi}{2}$, then the classes $C^{\overline{\psi}}_\infty$ coincide with the classes $C^\psi_{\beta,\infty}$. Furthermore, if $\psi(v) = v^{-r}$, then the classes $C^{\overline{\psi}}_\infty$ coincide with the classes $W^r_{\beta,\infty}$-Weil-Nagy.

Let $f(x)$ be summable 2π-periodic function and let series (1.1) be its Fourier series. Consider polynomials of the form

$$Z^\varphi_n(f;x) = \frac{a_0}{2} + \sum_{k=1}^{n-1}\left(1 - \frac{\varphi(k)}{\varphi(n)}\right)(a_k coskx + b_k sinkx), \quad n \in \mathbb{N},$$

where $\varphi(k)$ are the values of a certain function $\varphi \in F$ at integer points, and F is the set of all continuous functions $\varphi(u)$ monotonically increasing to infinity on $[1, \infty)$. On the other hand, let F^+ denotes the class of functions which belong to F and satisfy the conditions $\varphi(u) \geq 0,\ u \geq 0$, such that $\varphi(0) = 0$ and $\varphi(u)$ is convex upwards or convex downwards on $[0, n]$ for any $n = 2, 3, \ldots$. The polynomials $Z^\varphi_n(f;x)$ were introduced in [6],[7] and are called the generalized Zygmund sums. Clearly, if $\varphi(t) = t^s,\ s > 0$, then $\varphi \in F^+$ and $Z^\varphi_n(f;x)$ coincide with the classical Zygmund sums $Z^s_n(f;x)$, i.e., with polynomials of the form

$$Z^s_n(f;x) = \frac{a_o}{2} + \sum_{k=1}^{n-1}(1 - (\frac{k}{n})^s)(a_k coskx + b_k sinkx), \quad n \in \mathbb{N}.$$

For $s = 1$, the Zygmund sums $Z^s_n(f;x)$ turn into the known Fejer sums $\sigma_n(f;x)$ of order $n-1$ for the function $f(x)$.

In [9], we know that the necessary and sufficient condition for the uniform convergence of the polynomials $Z^\varphi_n(f;x)$ to the function $f(x)$ in the entire space C is given in the following result:

Proposition 1.1. *Let $\varphi \in F^+$. Then the condition*

$$\frac{1}{\varphi(n)}\sum_{k=1}^{n-1}\frac{\varphi(n) - \varphi(k)}{n-k} \leq K$$

is necessary and sufficient for the uniform convergence of the polynomials $Z^\varphi_n(f;x)$ to the function $f(x)$ in the entire space C.

We are mainly interested in asymptotic equalities for the quantities

$$\mathcal{E}_n(\mathfrak{N}, U_n(f;x)) = \sup_{f\in\mathfrak{N}} \|f - U_n(f;x)\|_X$$

that realize solutions to the corresponding Kolmogorov-Nikol'skii problems. Recall that we say that, for a given method $U_n(f;\lambda)$ on the class $\mathfrak{N}$ in the space X, the Kolmogorov-Nikol'skii problem is solved if the function $\Omega(n) = \Omega(n,\lambda;\mathfrak{N})$ is determined in explicit form and is such that

$$\mathcal{E}_n(\mathfrak{N}, U_n(f;\lambda)) = \sup_{f\in\mathfrak{N}} \|f(x) - U_n(f;x;\lambda)\|_X = \Omega(n) + O(\Omega(n))$$

as $n \to \infty$, where $\lambda = ||\lambda_k^{(n)}||$ is a triangular matrices.

There are many studies focusing on the value $\mathcal{E}_n(\mathfrak{N}, Z_n^s)_C$. Some of these were investigated by A. Zygmund [12] in the event of $\mathfrak{N} = W_\infty^r$, $r > 0$; B. Nagy, S. A. Teljakovskií [[8], [11]] in the event of $\mathfrak{N} = W_{\beta,\infty}^r$ under various conditions on $\beta,\ s,\ r$; D. N. Bushev, A. I. Stepanets [[1], [10]] in the event of $\mathfrak{N} = C_{\beta,\infty}^{\psi}$ under the condition on function $\psi(\cdot)$; A. S. Fedorenko [[4], [5]] and Değer[[2], [3]] in the event of $\mathfrak{N} = C_\infty^{\overline{\psi}}$ under the various conditions on functions $\psi_1(\cdot)$ and $\psi_2(\cdot)$.

We will give some asymptotic equalities related to the estimation of the value

$$\mathcal{E}_n(C_\infty^{\overline{\psi}}, Z_n^{\varphi})_C = \sup_{f\in C_\infty^{\overline{\psi}}} \|f(.) - Z_n^{\varphi}(f;.)\|_C \tag{1.2}$$

under various conditions on functions $\varphi(\cdot),\ \psi_1(\cdot)$ and $\psi_2(\cdot)$, where $\|\varrho\|_C = \max_x |\varrho(x)|$.

The value of (1.2) depends on the functions $g_i(v) = \varphi(v)\psi_i(v)$, $i = 1, 2$, which are convex or concave downwards. There are five possible cases for functions $g_i(v)$, $i = 1, 2$:

a) $g_i(v)$ are convex functions with $\lim_{v\to\infty} g_i(v) = \infty$,
b) $g_i(v)$ are convex functions with $\lim_{v\to\infty} g_i(v) = C > 0$,
c) $g_i(v)$ are convex functions with $\lim_{v\to\infty} g_i(v) = 0$,
d) $g_i(v)$ are concave functions with $\lim_{v\to\infty} g_i(v) = c > 0$,
e) $g_i(v)$ are concave functions with $\lim_{v\to\infty} g_i(v) = \infty$.

In this study we have some asymptotic equalities in case of a), b), and c) for $\varphi \in F^+$, $\psi_1 \in \mathfrak{M}_{\circ}$ (or $-\psi_1 \in \mathfrak{M}_{\circ}$), and $\psi_2 \in \mathfrak{M}'$ (or $-\psi_2 \in \mathfrak{M}'$) about value (1.2).

2 Main Results

In this section, some main results will be given concerning the generalized Zygmund sums for the states a), b), and c). Throughout this paper, $O(1)$ denotes a properly bounded identity with respect to n and $\overline{\psi}(n) = (\psi_1^2(n) + \psi_2^2(n))^{1/2}$.

Theorem 2.1. *Assume that $\varphi \in F^+$, $\psi_1 \in \mathfrak{M}_{\circ}$, $\psi_2 \in \mathfrak{M}'$ and $g_i(v) = \varphi(v)\psi_i(v),\ \ i = 1, 2$, be convex functions on $v \geq b \geq 1$ with $\lim_{v\to\infty} g_i(v) = 0$ or $\lim_{v\to\infty} g_i(v) = c > 0$. Then as $n \to \infty$, we obtain*

$$\mathcal{E}_n(C_\infty^{\overline{\psi}}, Z_n^{\varphi})_C = \frac{2}{\pi\varphi(n)} \int_1^n \frac{\varphi(v)\psi_2(v)}{v} dv + \frac{2}{\pi} \int_n^\infty \frac{\psi_2(v)}{v} dv + O(\frac{1}{\varphi(n)}). \tag{2.1}$$

Proposition 2.2. *Let $\varphi \in F^+$, $\psi_1 \in \mathfrak{M}_{\circ}$ and $g_1(v) = \varphi(v)\psi_1(v)$ be convex functions on $v \geq b \geq 1$ with $\lim_{v\to\infty} g_1(v) = 0$ or $\lim_{v\to\infty} g_1(v) = c > 0$. Then as $n \to \infty$, we have*

$$\int_{-\infty}^{\infty} |\frac{1}{\pi} \int_0^\infty \tau_1(v) cosvt dv| dt = O(\frac{1}{\varphi(n)}) \tag{2.2}$$

where

$$\tau_1(v)=\begin{cases}\dfrac{\varphi(v)\psi_1(1)}{\varphi(n)} & ,0\le v\le 1\\ \dfrac{\varphi(v)\psi_1(v)}{\varphi(n)} & ,1\le v\le n\\ \psi_1(v) & ,v\ge n\end{cases}.$$

Proposition 2.3. *Let $\varphi\in F^+$, $\psi_2\in\mathfrak{M}'$ and $g_2(v)=\varphi(v)\psi_2(v)$ be convex functions on $v\ge b\ge 1$ with $\lim_{v\to\infty} g_2(v)=0$ or $\lim_{v\to\infty} g_2(v)=c>0$. Then as $n\to\infty$, we have*

$$\int_{-\infty}^{\infty}|\frac{1}{\pi}\int_0^{\infty}\tau_2(v)sinvtdv|dt=\frac{2}{\pi\varphi(n)}\int_1^n\frac{\varphi(v)\psi_2(v)}{v}dv+\frac{2}{\pi}\int_n^{\infty}\frac{\psi_2(v)}{v}dv+O(\frac{1}{\varphi(n)}) \tag{2.3}$$

where

$$\tau_2(v)=\begin{cases}\dfrac{\varphi(v)\psi_2(1)}{\varphi(n)} & ,0\le v\le 1\\ \dfrac{\varphi(v)\psi_2(v)}{\varphi(n)} & ,1\le v\le n\\ \psi_2(v) & ,v\ge n\end{cases}.$$

Proof of Proposition 2.2. By partial integration, we have

$$\int_0^{\infty}\tau_1(v)cosvtdv=\frac{1}{t}\int_0^{\infty}(-\tau_1^{'}(v))sinvtdv.$$

Hence, we can write

$$\int_{-\infty}^{\infty}|\frac{1}{\pi}\int_0^{\infty}\tau_1(v)cosvtdv|dt=2\int_0^{\infty}|\frac{1}{\pi}\int_0^{\infty}\tau_1(v)cosvtdv|dt\le$$

$$\le 2\int_0^{\infty}|\frac{1}{\pi t}\int_0^{n}(-\tau_1^{'}(v))sinvtdv|dt+2\int_0^{\infty}|\frac{1}{\pi t}\int_n^{\infty}(-\tau_1^{'}(v))sinvtdv|dt. \tag{2.4}$$

Now let us estimate the first integral on the right side of inequality (2.4):

$$2\int_0^{\infty}|\frac{1}{\pi t}\int_0^{n}(-\tau_1^{'}(v))sinvtdv|dt\le$$

$$\le 2\int_0^{\infty}|\frac{1}{\pi t}\int_0^{1}\tau_1^{'}(v)sinvtdv|dt+2\int_0^{\infty}|\frac{1}{\pi t}\int_1^{n}(-\tau_1^{'}(v))sinvtdv|dt \tag{2.5}$$

Since the function $\tau_1^{'}(v)$ is a continuous function that is nonnegative and nonincreasing on interval $[0,1]$ for all $t\ge 0$, the following inequality is true:

$$\frac{1}{t}\int_0^1\tau_1^{'}(v)sinvtdv>0 \tag{2.6}$$

For the first integral on the right side of inequality (2.5), if we consider the statement of (2.6) and change the order of integration, we obtain

$$\frac{2}{\pi}\int_0^{\infty}|\frac{1}{t}\int_0^1\tau_1^{'}(v)sinvtdv|dt=\frac{2}{\pi}\int_0^1\tau_1^{'}(v)\int_0^{\infty}\frac{sinvt}{t}dtdv=O(\frac{1}{\varphi(n)}). \tag{2.7}$$

Let us estimate the second integral on the right side of (2.5):

$$\frac{2}{\pi}\int_0^\infty |\frac{1}{t}\int_1^n (-\tau_1^{'}(v))sinvtdv|dt \le$$

$$\le \frac{2}{\pi}\int_0^\pi |\frac{1}{t}\int_1^n (-\tau_1^{'}(v))sinvtdv|dt + \frac{2}{\pi}\int_\pi^\infty |\frac{1}{t}\int_1^n (-\tau_1^{'}(v))sinvtdv|dt =$$

$$= \frac{2}{\pi}\int_0^\pi |J_1|dt + \frac{2}{\pi}\int_\pi^\infty |J_1|dt$$

$v_k = \frac{k\pi}{t}$, $k \in Z$, are the zeros of $\sin vt$. Then we can write the following equality for J_1:

$$J_1 = \frac{1}{t}\int_1^n (-\tau_1^{'}(v))sinvtdv = \frac{1}{t}\int_1^{\pi/t} (-\tau_1^{'}(v))sinvtdv + \frac{1}{t}\int_{\pi/t}^n (-\tau_1^{'}(v))sinvtdv =$$

$$= J_{11} + J_{12}$$

Hence, for $0 \le t \le \pi$ and $1 \le v \le \frac{\pi}{t}$, $J_{11} \ge 0$, and for $0 \le t \le \pi$ and $\frac{\pi}{t} \le v \le n$, $J_{12} \le 0$, since $(-\tau_1^{'}(v))$ is nonnegative and nonincreasing on $[1, n]$. If we consider $J_1 = J_{11} + J_{12}$, we can write

$$\frac{2}{\pi}\int_0^\pi |J_1|dt \le \frac{2}{\pi}\int_0^\pi |J_{11}|dt + \frac{2}{\pi}\int_0^\pi |J_{12}|dt. \tag{2.8}$$

Firstly, we will estimate the first integral on the right side of (2.8):

$$\frac{2}{\pi}\int_0^\pi |J_{11}|dt = \frac{2}{\pi}\int_0^\pi \frac{1}{t}\int_1^{\pi/t} (-\tau_1^{'}(v))sinvtdvdt =$$

$$\frac{2}{\pi}\int_1^{\pi/t} (-\tau_1^{'}(v))\int_0^\pi \frac{sinvt}{t}dtdv = \frac{2}{\pi}\int_1^{\pi/t} (-\tau_1^{'}(v))\int_0^{v\pi} \frac{sinu}{u}dudv = O(\frac{1}{\varphi(n)})$$

Therefore, we get

$$\frac{2}{\pi}\int_0^\pi |J_{11}|dt = O(\frac{1}{\varphi(n)}). \tag{2.9}$$

Now let us estimate the second integral on the right side of (2.8):

$$\frac{2}{\pi}\int_0^\pi |J_{12}|dt = -\frac{2}{\pi}\int_0^\pi \frac{1}{t}\int_{\pi/t}^n (-\tau_1^{'}(v))sinvtdvdt = -\frac{2}{\pi}\int_{\pi/t}^n (-\tau_1^{'}(v))\int_0^\pi \frac{sinvt}{t}dtdv =$$

$$= -\frac{2}{\pi}\int_{\pi/t}^n (-\tau_1^{'}(v))\int_0^{v\pi} \frac{sinu}{u}dudv = O(\frac{1}{\varphi(n)})$$

Hence, we have

$$\frac{2}{\pi}\int_0^\pi |J_{12}|dt = O(\frac{1}{\varphi(n)}). \tag{2.10}$$

Owing to (2.9) and (2.10), we obtain

$$\frac{2}{\pi}\int_0^\pi |J_1|dt = O(\frac{1}{\varphi(n)}). \tag{2.11}$$

Now we will estimate that

$$\frac{2}{\pi}\int_{\pi}^{\infty}|J_1|dt = O(\frac{1}{\varphi(n)}).$$

Thus, we consider the function

$$\eta_x(t) = \int_{x}^{n}\mu(v)sinvtdv, x > 0, t > 0, \tag{2.12}$$

where $\mu(v)$ is nonnegative and nonincreasing function for all $v \geq 1$. The function $\eta_x(t)$ is a continuous function for every fixed t. Further, on each interval between the successive zeros v_k and v_{k+1} of the function $sinvt$, the function $\eta_x(t)$ has one simple zero x_k[10]. Therefore let's suppose that $x_k^{'}$ is zero the nearest from the right of point 1. In view of this, if we set $\mu(v) = -\tau_1^{'}(v)$ on interval $[1, n]$ in (2.12), we have

$$J_1 = \frac{1}{t}\int_{1}^{x_k^{'}}(-\tau_1^{'}(v))sinvtdv.$$

Hence the following result is obtained:

$$\frac{2}{\pi}\int_{\pi}^{\infty}|J_1|dt \leq \frac{2}{\pi}\int_{\pi}^{\infty}\frac{1}{t}\int_{1}^{1+\frac{2\pi}{t}}|\tau_1^{'}(v)|dvdt \leq$$

$$\leq \frac{2}{\pi}\int_{\pi}^{\infty}\frac{2\pi}{t^2}|\tau_1^{'}(1)|dt = O(\frac{1}{\varphi(n)})$$

$$\frac{2}{\pi}\int_{\pi}^{\infty}|J_1|dt = O(\frac{1}{\varphi(n)}) \tag{2.13}$$

Therefore from (2.11) and (2.13), we get

$$\frac{2}{\pi}\int_{0}^{\infty}|\frac{1}{t}\int_{1}^{n}(-\tau_1^{'}(v))sinvtdv|dt = O(\frac{1}{\varphi(n)}).$$

Thus for the first integral on the right side of (2.4), we get that

$$2\int_{0}^{\infty}|\frac{1}{\pi t}\int_{0}^{n}(-\tau_1^{'}(v))sinvtdv|dt = O(\frac{1}{\varphi(n)}). \tag{2.14}$$

Now we will show that

$$2\int_{0}^{\infty}|\frac{1}{\pi t}\int_{n}^{\infty}(-\tau_1^{'}(v))sinvtdv|dt = O(\frac{1}{\varphi(n)}). \tag{2.15}$$

Firstly by partial integration, we have

$$\frac{1}{t}\int_{n}^{\infty}\tau_1^{'}(v)sinvtdv = \frac{1}{t^2}[-\tau_1^{'}(n+0)cosnt - \int_{n}^{\infty}(-\tau_1^{''}(v))cosvtdv].$$

We know that $\tau_1^{''}(v) > 0$. Then we get

$$|\frac{1}{t}\int_{n}^{\infty}\tau_1^{'}(v)sinvtdv| \leq \frac{1}{t^2}[|\tau_1^{'}(n+0)cosnt| + |\int_{n}^{\infty}\tau_1^{''}(v)cosvtdv|] \leq$$

$$\leq \frac{1}{t^2}[|\psi_1^{'}(n)| + |\psi_1^{'}(n)|] = \frac{2}{t^2}|\psi_1^{'}(n)|.$$

Hence since $\psi_1 \in \mathfrak{M}_0$, we obtain

$$\frac{1}{\pi}\int\limits_{t\geq\frac{1}{n}}|\frac{1}{t}\int\limits_{n}^{\infty}\tau_1^{'}(v)sinvtdv|dt \leq \frac{1}{\pi}\int\limits_{t\geq\frac{1}{n}}\frac{2}{t^2}|\psi_1^{'}(n)|dt = O(\frac{1}{\varphi(n)}). \tag{2.16}$$

After this estimation, we will show that

$$\frac{1}{\pi}\int\limits_{t\leq\frac{1}{n}}|\frac{1}{t}\int\limits_{n}^{\infty}(-\tau_1^{'}(v))sinvtdv|dt = O(\frac{1}{\varphi(n)}).$$

By partial integration, we obtain

$$\int\limits_{n}^{\infty}\tau_1(v)\cos vtdv = -\psi_1(n)\frac{sinnt}{t} - \frac{1}{t}\int\limits_{n}^{\infty}\tau_1^{'}(v)sinvtdv.$$

$$|\frac{1}{t}\int\limits_{n}^{\infty}\tau_1^{'}(v)sinvtdv| \leq \psi_1(n)|\frac{sinnt}{t}| + |\int\limits_{n}^{\infty}\tau_1(v)\cos vtdv|.$$

From here, we have

$$\frac{1}{\pi}\int\limits_{t\leq\frac{1}{n}}|\frac{1}{t}\int\limits_{n}^{\infty}\tau_1^{'}(v)sinvtdv|dt \leq$$

$$\leq \frac{2}{\pi}\psi_1(n)\int\limits_{0}^{\frac{1}{n}}|\frac{sinnt}{t}|dt + \frac{2}{\pi}\int\limits_{0}^{\frac{1}{n}}|\int\limits_{n}^{\infty}\tau_1(v)\cos vtdv|dt.$$

$\int\limits_{0}^{\frac{1}{n}}|\frac{sinnt}{t}|dt \leq K_1$ and owing to similar estimation of the integral in [3] we know that

$$\frac{2}{\pi}\int\limits_{0}^{\frac{1}{n}}|\int\limits_{n}^{\infty}\tau_1(v)cosvtdv|dt = O(\frac{1}{\varphi(n)}).$$

Thus we find that

$$\int\limits_{t\leq\frac{1}{n}}|\frac{1}{\pi t}\int\limits_{n}^{\infty}\tau_1^{'}(v)sinvtdv|dt = O(\frac{1}{\varphi(n)}).$$

Therefore the proof of the proposition is completed.

□

Proof of Proposition 2.3. By applying two times partial integration, we have

$$\frac{1}{\pi}\int\limits_{0}^{\infty}\tau_2(v)sinvtdv = \frac{1}{\pi t^2}[(\tau_2^{'}(1-0) - \tau_2^{'}(1+0))sint + (\tau_2^{'}(n-0) - \tau_2^{'}(n+0))sinnt -$$

$$-(\int\limits_{0}^{1}\tau_2^{''}(v)sinvtdv + \int\limits_{1}^{n}\tau_2^{''}(v)sinvtdv + \int\limits_{n}^{\infty}\tau_2^{''}(v)sinvtdv)] \tag{2.17}$$

From (2.17), since the functions φ' and ψ_2 are nonincreasing, we get

$$|\frac{1}{\pi}\int_0^\infty \tau_2(v)sinvtdv| \leq \frac{-2\varphi(1)\psi_2^{'}(1)+2\varphi'(1)\psi_2(1)}{\pi t^2\varphi(n)} \tag{2.18}$$

Hence, accordingly (2.18) we obtain

$$\int_{|t|\geq\pi/2} |\frac{1}{\pi}\int_0^\infty \tau_2(v)sinvtdv|dt = 2\int_{\pi/2}^\infty |\frac{1}{\pi}\int_0^\infty \tau_2(v)sinvtdv|dt \leq$$

$$\leq \int_{\pi/2}^\infty \frac{-2\varphi(1)\psi_2^{'}(1)+2\varphi'(1)\psi_2(1)}{\pi t^2}dt = \frac{-8\varphi(1)\psi_2^{'}(1)+8\varphi'(1)\psi_2(1)}{\pi^2\varphi(n)} = O(\frac{1}{\varphi(n)}). \tag{2.19}$$

Let's estimate the following integral:

$$\int_{\pi/2n}^{\pi/2} |\frac{1}{\pi}\int_0^\infty \tau_2(v)sinvtdv|dt \leq$$

$$\leq \int_{\pi/2n}^{\pi/2} |\frac{1}{\pi}\int_0^1 \tau_2(v)sinvtdv|dt + \int_{\pi/2n}^{\pi/2} |\frac{1}{\pi}\int_1^n \tau_2(v)sinvtdv|dt +$$

$$+\int_{\pi/2n}^{\pi/2} |\frac{1}{\pi}\int_n^\infty \tau_2(v)sinvtdv|dt := I_1 + I_2 + I_3$$

. Firstly we consider the first integral on the right hand:

$$I_1 := \int_{\pi/2n}^{\pi/2} |\frac{1}{\pi}\int_0^1 \tau_2(v)sinvtdv|dt \leq \frac{1}{\pi}\int_{\pi/2n}^{\pi/2} \tau_2(1)dt = O(\frac{1}{\varphi(n)}).$$

Secondly, we estimate the following integral:

$$I_2 := \int_{\pi/2n}^{\pi/2} |\frac{1}{\pi}\int_1^n \tau_2(v)sinvtdv|dt \leq$$

$$\leq \int_{\pi/2n}^{\pi/2} |\frac{1}{\pi}\int_1^{\pi/2t} \tau_2(v)sinvtdv|dt + \int_{\pi/2n}^{\pi/2} |\frac{1}{\pi}\int_{\pi/2t}^{n} \tau_2(v)sinvtdv|dt := I_{21} + I_{22}$$

$$I_{21} := \int_{\pi/2n}^{\pi/2} |\frac{1}{\pi}\int_1^{\pi/2t} \tau_2(v)sinvtdv|dt = \int_{\pi/2n}^{\pi/2} \frac{1}{\pi}\int_1^{\pi/2t} \tau_2(v)sinvtdvdt =$$

$$= \frac{1}{\pi}\int_1^n\int_{\pi/2n}^{\pi/2v} \tau_2(v)sinvtdtdv = \frac{1}{\pi}\int_1^n \frac{\tau_2(v)}{v}cos\frac{\pi v}{2n}dv =$$

$$= \frac{1}{\pi\varphi(n)}\int_1^n \frac{\varphi(v)\psi_2(v)}{v}cos\frac{\pi v}{2n}dv$$

Now let's show that

$$\frac{1}{\varphi(n)}\int\limits_1^n \frac{\varphi(v)\psi_2(v)}{v}cos\frac{\pi v}{2n}dv = \frac{1}{\varphi(n)}\int\limits_1^n \frac{\varphi(v)\psi_2(v)}{v}dv + O(\frac{1}{\varphi(n)}). \tag{2.20}$$

For proof of (2.20), we will obtain the necessary estimation of the following difference

$$\frac{1}{\varphi(n)}\int\limits_1^n \frac{\varphi(v)\psi_2(v)}{v}(1-cos\frac{\pi v}{2n})dv =$$

$$= \frac{2}{\varphi(n)}\int\limits_1^n \frac{\varphi(v)\psi_2(v)}{v}v\frac{sin\pi v/4n}{\pi v/4n}\frac{\pi}{4n}sin\frac{\pi v}{4n}dv \le$$

$$\le \frac{2\varphi(1)\psi_2(1)}{\varphi(n)}\frac{\pi}{4n}\int\limits_1^n sin\frac{\pi v}{4n}dv = O(\frac{1}{\varphi(n)})$$

Therefore we have

$$I_{21} := \frac{1}{\pi\varphi(n)}\int\limits_1^n \frac{\varphi(v)\psi_2(v)}{v}dv + O(\frac{1}{\varphi(n)}).$$

Now we will estimate I_{22}:

$$I_{22} := \int\limits_{\pi/2n}^{\pi/2} |\frac{1}{\pi}\int\limits_{\pi/2t}^{n} \tau_2(v)sinvtdv|dt = \int\limits_{\pi/2n}^{\pi/2} \frac{1}{\pi}\int\limits_{\pi/2t}^{n} \tau_2(v)sinvtdvdt =$$

$$= \frac{1}{\pi}\int\limits_1^n\int\limits_{\pi/2v}^{\pi/2} \tau_2(v)sinvtdtdv = -\frac{1}{\pi}\int\limits_1^n \frac{\tau_2(v)}{v}cos\frac{\pi v}{2}dv =$$

$$= -\frac{1}{\pi\varphi(n)}\int\limits_1^n \frac{\varphi(v)\psi_2(v)}{v}cos\frac{\pi v}{2}dv$$

Let us show that

$$-\frac{1}{\pi\varphi(n)}\int\limits_1^n \frac{\varphi(v)\psi_2(v)}{v}cos\frac{\pi v}{2}dv = \frac{1}{\pi\varphi(n)}\int\limits_1^n \frac{\varphi(v)\psi_2(v)}{v}dv + O(\frac{1}{\varphi(n)}) \tag{2.21}$$

For proof of (2.21) we will estimate the following difference:

$$\frac{1}{\pi\varphi(n)}\int\limits_1^n \frac{\varphi(v)\psi_2(v)}{v}(-cos\frac{\pi v}{2}-1)dv \le$$

$$\frac{1}{\pi}\frac{\varphi(1)\psi_2(1)}{\varphi(n)}\int\limits_1^n(-cos\frac{\pi v}{2}-1)dv = O(\frac{1}{\varphi(n)})$$

Hence we obtain that

$$I_{22} := \frac{1}{\pi\varphi(n)}\int\limits_1^n \frac{\varphi(v)\psi_2(v)}{v}dv + O(\frac{1}{\varphi(n)}).$$

Let us estimate the following integral:

$$I_3 := \int_{\pi/2n}^{\pi/2} |\frac{1}{\pi}\int_n^{\infty} \tau_2(v) sinvtdv| dt = O(\frac{1}{\varphi(n)}).$$

By partial integral, we have

$$\frac{1}{\pi}\int_n^{\infty} \tau_2(v) sinvtdv = \frac{1}{\pi}(-\tau_2(v)\frac{cosvt}{t}|_n^{\infty} + \frac{1}{t}\int_n^{\infty} \tau_2'(v) cosvtdv) =$$

$$= \frac{1}{\pi}\psi_2(n)\frac{cosnt}{t} + \frac{1}{\pi t}\int_n^{\infty} \tau_2'(v) cosvtdv.$$

$$I_3 \leq \int_{\pi/2n}^{\pi/2} |\frac{1}{\pi}\psi_2(n)\frac{cosnt}{t}| dt + \int_{\pi/2n}^{\pi/2} |\frac{1}{\pi t}\int_n^{\infty} \tau_2'(v) cosvtdv| dt$$

As similar estimate in [2],

$$\int_{\pi/2n}^{\pi/2} |\frac{1}{\pi t}\int_n^{\infty} \tau_2'(v) cosvtdv| dt = O(\frac{1}{\varphi(n)}).$$

Therefore, we get

$$I_3 \leq \frac{1}{\pi}\psi_2(n)\int_{\pi/2n}^{\pi/2} |\frac{cosnt}{t}| dt + O(\frac{1}{\varphi(n)}) = -\frac{1}{\pi}\psi_2(n)\int_{\pi/2n}^{\pi/2} \frac{cosnt}{t} dt + O(\frac{1}{\varphi(n)}) =$$

$$= -\frac{1}{\pi}\psi_2(n)\int_{\pi/2}^{\pi n/2} \frac{cosz}{z} dz + O(\frac{1}{\varphi(n)}) \leq \frac{\psi_2(n)}{\pi} ci(\frac{\pi}{2}) + O(\frac{1}{\varphi(n)}) \leq$$

$$\leq \frac{\varphi(1)\psi_2(1)}{\pi\varphi(n)} ci(\frac{\pi}{2}) + O(\frac{1}{\varphi(n)}) = O(\frac{1}{\varphi(n)}).$$

And then, we obtain

$$\int_{\pi/2n}^{\pi/2} |\frac{1}{\pi}\int_0^{\infty} \tau_2(v) sinvtdv| dt = \frac{2}{\pi\varphi(n)}\int_1^{n} \frac{\varphi(v)\psi_2(v)}{v} dv + O(\frac{1}{\varphi(n)}) \tag{2.22}$$

Now let's investigate in neighborhood of origin: Since $\tau_2(v) = \psi_2(v)$ on $[n,\infty)$, in [10], there exist $a > 0$ for all $n \geq 1$, such that we have

$$\int_{|t|\leq a/n} |\frac{1}{\pi}\int_n^{\infty} \tau_2(v) sinvtdv| dt = \frac{2}{\pi}\int_n^{\infty} \frac{\psi_2(v)}{v} dv + O(1)\overline{\psi}(n). \tag{2.23}$$

After that, we estimate the following integral:

$$\int_{|t|\leq a/n} |\frac{1}{\pi}\int_0^{n} \tau_2(v) sinvtdv| dt \leq \tag{2.24}$$

$$\leq \int_{|t|\leq a/n} |\frac{1}{\pi}\int_0^{1} \tau_2(v) sinvtdv| dt + \int_{|t|\leq a/n} |\frac{1}{\pi}\int_1^{n} \tau_2(v) sinvtdv| dt =$$

$$= \frac{2}{\pi}\int_0^{a/n} |\int_0^1 \tau_2(v) sinvtdv|dt + \frac{2}{\pi}\int_0^{a/n} |\int_1^n \tau_2(v) sinvtdv|dt \le$$

$$\le \frac{2}{\pi}\int_0^{a/n} \tau_2(1)dt + \frac{2}{\pi}\int_0^{a/n} \tau_2(1)(n-1)dt =$$

$$= \frac{2\tau_2(1)}{\pi}\frac{a}{n} + \frac{2\tau_2(1)}{\pi}\frac{a(n-1)}{n} = O(\frac{1}{\varphi(n)})$$

Now let's estimate the following integral:

$$2\frac{1}{\pi}|\int_{a/n}^{\pi/2n}\int_0^n \tau_2(v) sinvtdv|dt \le \tag{2.25}$$

$$2\frac{1}{\pi}|\int_{a/n}^{\pi/2n}\int_0^1 \tau_2(v) sinvtdv|dt + 2\frac{1}{\pi}|\int_{a/n}^{\pi/2n}\int_1^n \tau_2(v) sinvtdv|dt \le$$

$$\le 2\frac{1}{\pi}\int_{a/n}^{\pi/2n} \tau_2(1)\int_0^1 dvdt + 2\frac{1}{\pi}\int_{a/n}^{\pi/2n} \tau_2(1)\int_1^n dvdt =$$

$$= \frac{2\tau_2(1)}{\pi}(\frac{\pi}{2n} - \frac{a}{n}) + \frac{2\tau_2(1)}{\pi}(n-1)(\frac{\pi}{2n} - \frac{a}{n}) = O(\frac{1}{\varphi(n)})$$

After that, we obtain that

$$2\frac{1}{\pi}|\int_{a/n}^{\pi/2n}\int_n^\infty \tau_2(v) sinvtdv|dt \le \frac{2}{\pi}\int_{a/n}^{\pi/2n} |\int_n^\infty \psi_2(v) sinvtdv|dt \le \tag{2.26}$$

$$\le \frac{2}{\pi}\int_{a/n}^{\pi/2n}\int_n^{n+\frac{2\pi}{t}} \psi_2(v)dvdt = O(\frac{1}{\varphi(n)}).$$

Therefore, by using (2.19), (2.22), (2.23), (2.24), (2.25) and (2.26) for $n \ge 1$, we get (2.3). The proof of the proposition is completed. □

Proof of Theorem 2.1. From [2], it is known that

$$\mathcal{E}_n(C_\infty^{\overline{\psi}}, Z_n^\varphi)_C = \int_{-\infty}^{\infty} |\hat{\tau}_n(t)|dt + \gamma(n), \tag{2.27}$$

where $\gamma(n) \le 0$,

$$|\gamma(n)| = O(\int_{|t|\ge\frac{\pi}{2}} |\hat{\tau}_n(t)|dt)$$

and

$$\hat{\tau}_n(t) = \frac{1}{\pi}\int_0^\infty \tau_1(v) cosvtdv + \frac{1}{\pi}\int_0^\infty \tau_2(v) sinvtdv.$$

By using (2.27) and Proposition 2.2-2.3, we will obtain (2.1). At first, we estimate $\gamma(n)$:

$$|\gamma(n)| \le O(1)\int_{|t|\ge\frac{\pi}{2}} |\hat{\tau}_n(t)|dt \le O(1)\int_{|t|\ge\frac{\pi}{2}} |\frac{1}{\pi}\int_0^\infty \tau_1(v) cosvtdv|dt +$$

$$O(1)\int\limits_{|t|\geq\frac{\pi}{2}} |\frac{1}{\pi}\int\limits_{0}^{\infty}\tau_2(v)sinvtdv|dt := \gamma_1+\gamma_2.$$

We know that $\gamma_1 = O(\frac{1}{\varphi(n)})$ and $\gamma_2 = O(\frac{1}{\varphi(n)})$ from similar estimation in [2]. Therefore, we obtain $|\gamma(n)| = O(\frac{1}{\varphi(n)})$. Consequently, according to Proposition 2.2-2.3, we have (2.1). Hence, the proof is completed. □

Theorem 2.4. *Assume that* $\varphi\in F^+$, $\psi_1\in\mathfrak{M}_o$, $\psi_2\in\mathfrak{M}'$ *and* $g_i(v)=\varphi(v)\psi_i(v)$, $i=1,2$, *be convex functions on* $v\geq b\geq 1$ *with* $\lim\limits_{v\to\infty} g_i(v)=\infty$. *Then as* $n\to\infty$, *we have*

$$\mathcal{E}_n(C^{\overline{\psi}}_{\infty}, Z^{\varphi}_n)_C = \frac{2}{\pi\varphi(n)}\int\limits_{1}^{n}\frac{\varphi(v)\psi_2(v)}{v}dv+\frac{2}{\pi}\int\limits_{n}^{\infty}\frac{\psi_2(v)}{v}dv+O(1)\overline{\psi}(n). \quad (2.28)$$

Proposition 2.5. *Let* $\varphi\in F^+$, $\psi_1\in\mathfrak{M}_o$ *and* $g_1(v)=\varphi(v)\psi_1(v)$ *be convex functions on* $v\geq b\geq 1$ *with* $\lim\limits_{v\to\infty} g_1(v)=\infty$. *Then as* $n\to\infty$, *we get*

$$\int\limits_{-\infty}^{\infty}|\frac{1}{\pi}\int\limits_{0}^{\infty}\tau_3(v)cosvtdv|dt = O(1)\psi_1(n) \quad (2.29)$$

where

$$\tau_3(v)=\begin{cases}\dfrac{v\varphi(1)\psi_1(1)}{\varphi(n)} & ,0\leq v\leq 1\\ \dfrac{\varphi(v)\psi_1(v)}{\varphi(n)} & ,1\leq v\leq n\\ \psi_1(v) & ,v\geq n\end{cases}.$$

Proposition 2.6. *Let* $\varphi\in F^+$, $\psi_2\in\mathfrak{M}'$ *and* $g_2(v)=\varphi(v)\psi_2(v)$ *be convex functions on* $v\geq b\geq 1$ *with* $\lim\limits_{v\to\infty} g_2(v)=\infty$. *Then as* $n\to\infty$, *we get*

$$\int\limits_{-\infty}^{\infty}|\frac{1}{\pi}\int\limits_{0}^{\infty}\tau_4(v)sinvtdv|dt = \frac{2}{\pi\varphi(n)}\int\limits_{1}^{n}\frac{\varphi(v)\psi_2(v)}{v}dv+\frac{2}{\pi}\int\limits_{n}^{\infty}\frac{\psi_2(v)}{v}dv+O(1)\psi_2(n) \quad (2.30)$$

where

$$\tau_4(v)=\begin{cases}\dfrac{\varphi(v)\psi_2(1)}{\varphi(n)} & ,0\leq v\leq 1\\ \dfrac{\varphi(v)\psi_2(v)}{\varphi(n)} & ,1\leq v\leq n\\ \psi_2(v) & ,v\geq n\end{cases}.$$

Proof of Proposition 2.5. Let's consider the following function:

$$H_n(v)=\begin{cases}\varphi(v)\psi_1'(n)+\psi_1(n)-\varphi(n)\psi_1'(n) & ,0\leq v\leq n\\ \psi_1(v) & ,v\geq n\end{cases}$$

$H_n(v)$ is a continuous function that defined on the interval $[0,\infty)$. This function is convex downwards and monotony decreasing on $[0,\infty)$. In addition, it coincides with function $\tau_3(v)$ on interval $[n,\infty)$. $\tau_3(v)$ is increasing on interval $[0,n]$ and decreasing on interval $[n,\infty)$. Also $\tau_3'(v)$ is continuous on interval $[0,1]$ and $[1,n]$, and let $\lim\limits_{v\to\infty}\tau_3(v)=\lim\limits_{v\to\infty}\tau_3'(v)=0$ on interval $[n,\infty)$.If we apply two times partial integration on the integral $\int\limits_{0}^{\infty}\tau_3(v)cosvtdv$, then we get

$$\int\limits_{0}^{\infty}\tau_3(v)cosvtdv = \frac{1}{t^2}[(\frac{\varphi(1)\psi_1(1)}{\varphi(n)}-\frac{\varphi'(1)\psi_1(1)+\varphi(1)\psi_1'(1)}{\varphi(n)})cost+$$

$$+\frac{\varphi'(n)\psi_1(n)}{\varphi(n)}cosnt - \frac{\varphi(1)\psi_1(1)}{\varphi(n)}] - \frac{1}{t^2}\int_1^n \tau_3''(v)cosvtdv - -\frac{1}{t^2}\int_n^\infty \tau_3''(v)cosvtdv. \tag{2.31}$$

From (2.31) and the definition $g_1(v)$,

$$|\frac{1}{\pi}\int_0^\infty \tau_3(v)cosvtdv| \leq \frac{2}{\pi t^2\varphi(n)}[\varphi(1)\psi_1(1) - \varphi(1)\psi_1'(1) + \varphi'(n)\psi_1(n)]. \tag{2.32}$$

Hence, according to (2.32), we obtain

$$\int_{|t|\geq\frac{n}{2n-1}} |\frac{1}{\pi}\int_0^\infty \tau_3(v)cosvtdv|dt = 2\int_{\frac{n}{2n-1}}^\infty |\frac{1}{\pi}\int_0^\infty \tau_3(v)cosvtdv|dt = O(1)\psi_1(n). \tag{2.33}$$

Let's estimate the following asymptotic statement:

$$2\int_0^{1/n} |\frac{1}{\pi}\int_0^\infty \tau_3(v)cosvtdv|dt = O(1)\psi_1(n) \tag{2.34}$$

$$2\int_{1/n}^{n/2n-1} |\frac{1}{\pi}\int_0^\infty \tau_3(v)cosvtdv|dt = O(1)\psi_1(n). \tag{2.35}$$

Now we will estimate the statement (2.34) :

$$2\int_0^{1/n} |\frac{1}{\pi}\int_0^\infty \tau_3(v)cosvtdv|dt \leq 2\int_0^{1/n} |\frac{1}{\pi}\int_0^n \tau_3(v)cosvtdv|dt+$$

$$+2\int_0^{1/n} |\frac{1}{\pi}\int_n^\infty \tau_3(v)cosvtdv|dt := L_1 + L_2$$

$g_1(v)$ is convex function on $[0, n]$. Therefore we can write $|\tau_3(v)| \leq \psi_1(n)$ on interval $[0, n]$. Thus for integral L_1, we have

$$L_1 \leq 2\int_0^{1/n} \frac{1}{\pi}n\psi_1(n)dt = O(1)\psi_1(n) \tag{2.36}$$

For integral L_2, according to definition of the function $H_n(v)$;

$$L_2 = 2\int_0^{1/n} |\frac{1}{\pi}\int_n^\infty H_n(v)cosvtdv|dt \leq$$

$$\leq 2\int_0^{1/n} |\frac{1}{\pi}\int_0^\infty H_n(v)cosvtdv|dt + 2\int_0^{1/n} |\frac{1}{\pi}\int_0^n H_n(v)cosvtdv|dt := L_{21} + L_{22}$$

Firstly we will estimate that $L_{21} = O(1)\psi_1(n)$. For simplicity we will show $L_{21} := 2\int_0^{1/n} |L_{211}|dt$ where

$$L_{211} = \frac{1}{\pi}\int_0^\infty H_n(v)cosvtdv.$$

By applying partial integration for L_{211}, we get

$$L_{211} = \frac{1}{\pi t}\int_0^\infty (-H_n'(v)) sinvtdv.$$

Since $H_n(v)$ is a nonincreasing and convex, $(-H_n'(v))$ is a nonnegative and nonincreasing. Therefore for any $t > 0$,

$$L_{211} = \frac{1}{t}\int_0^\infty (-H_n'(v)) sinvtdv > 0. \tag{2.37}$$

Hence, owing to (2.37), we get

$$L_{21} = 2\int_0^{1/n} |L_{211}|dt = 2\int_0^{1/n} \frac{1}{\pi t}\int_0^\infty (-H_n'(v)) sinvtdvdt.$$

By using Fubini's theorem, we have

$$L_{21} = \frac{2}{\pi}\int_0^\infty (-H_n'(v)) \int_0^{1/n} \frac{sinvt}{t} dtdv \leq H_n(0) = \psi_1(n) - \varphi(n)\psi_1'(n) \leq$$

$$\leq \psi_1(n) + \varphi'(n)\psi_1(n) = O(1)\psi_1(n) \tag{2.38}$$

Secondly, we will show that $L_{22} = O(1)\psi_1(n)$. Since $H_n(v)$ is monotony decreasing, then we get

$$L_{22} = 2\int_0^{1/n} |\frac{1}{\pi}\int_0^n H_n(v) cosvtdv|dt \leq \frac{2}{\pi}\int_0^{1/n}\int_0^n |H_n(v)|dvdt \leq$$

$$\leq \frac{2n}{\pi} H_n(0)\int_0^{1/n} dt = \frac{2}{\pi}(\psi_1(n) - \varphi(n)\psi_1'(n)) = O(1)\psi_1(n) \tag{2.39}$$

According to (2.36), (2.38) and (2.39), we obtain (2.34). Let's estimate the asymptotic statement (2.35). By applying partial integration, we have

$$\frac{1}{\pi}\int_0^\infty \tau_3(v) cosvtdv = -\frac{1}{\pi t}\int_0^n \tau_3'(v) sinvtdv + \frac{1}{\pi t}\int_n^\infty (-\tau_3'(v)) sinvtdv := J_1' + J_2'.$$

Let's estimate that

$$2\int_{1/n}^{n/2n-1} |J_1'|dt = O(1)\psi_1(n). \tag{2.40}$$

Thus, we take into account the function

$$f_t(x) = \int_0^x \kappa(v) sinvtdv \ , \ x > 0, \ t > 0 \tag{2.41}$$

where $\kappa(v)$ is nonnegative and nondecreasing function for all $v \geq 1$. The function $f_t(x)$ is a continuous function for every fixed t. Moreover, on each interval between the consecutive zeros v_k and v_{k+1} of the function $sinvt$ the function $f_t(x)$ has one simple zero x_k, [7, cht. IV]. Therefore let's suppose that x_k' is zero nearest from the left of the point n, we have $n - v_k \leq \frac{\pi}{t}$. Therefore, by setting $\kappa(v) = \tau_3'(v)$ on interval $[0, n]$ in (2.41) , we find

$$|J_1'| = |\frac{1}{\pi t}\int_{x_k'}^n \tau_3'(v) sinvtdv|.$$

Therefore since $\tau_3'(v)$ is nondecreasing on [0,n], we have

$$2\int_{1/n}^{n/2n-1}|J_1'|dt \leq \frac{2}{\pi}\int_{1/n}^{n/2n-1}\frac{\tau_3'(n)(n-x_k')}{t}dt \leq 2\tau_3'(n)\int_{1/n}^{n/2n-1}\frac{dt}{t^2} =$$

$$= 2\tau_3'(n)\frac{(n-1)^2}{n} = O(1)\psi_1(n). \tag{2.42}$$

Hence we have (2.40). Now we will show that

$$2\int_{1/n}^{n/2n-1}|J_2'|dt = O(1)\psi_1(n). \tag{2.43}$$

Similarly to (2.40), we take into account the function,

$$h_t(y) = \int_y^\infty \mu(v) sinvtdv \;\;,\; x>0,\; t>0 \tag{2.44}$$

where $\mu(v)$ is nonnegative and nonincreasing function for all $v \geq 1$. The function $h_t(y)$ is continuous for every fixed t. Moreover, on each interval between the consecutive zeros v_k and v_{k+1} of the function $sinvt$ the function $h_t(y)$ has one simple zero y_k. Therefore, let's suppose that y_k' is zero nearest from the right of the point n, we get $n \leq y_k' \leq n + \frac{2\pi}{t}$. Since the function $(-\tau_3'(v))$ is nonnegative and nonincreasing, by taking $\mu(v) = -\tau_3'(v)$ in (2.44) , we get

$$|J_2'| = |\frac{1}{\pi t}\int_n^\infty(-\tau_3'(v))sinvtdv| \leq |\frac{1}{\pi t}\int_n^{y_k'}(-\tau_3'(v))sinvtdv| + |\frac{1}{\pi t}\int_{y_k'}^\infty(-\tau_3'(v))sinvtdv| =$$

$$= |\frac{1}{\pi t}\int_n^{y_k'}(-\tau_3'(v))sinvtdv| \leq \frac{1}{\pi t}\int_n^{y_k'}|\tau_3'(v)|dv = \frac{1}{\pi t}(-\psi_1'(n))(y_k'-n) \leq \frac{|\psi_1'(n)|}{t^2}$$

Thus

$$2\int_{1/n}^{n/2n-1}|J_2'|dt \leq 2|\psi_1'(n)|\int_{1/n}^{n/2n-1}\frac{dt}{t^2} \leq 2n|\psi_1'(n)| = O(1)\psi_1(n). \tag{2.45}$$

Hence we obtain (2.43). By combining (2.42) and (2.45), we have (2.35). According to (2.34) and (2.35), (2.29) is proved.

□

Proof of Proposition 2.6. Here we will estimate only the following integral. Because the rest of the proof of this proposition is getting similar to proof of Proposition 2 in [2].

$$\frac{1}{t}\int_0^n \tau_4'(v)cosvtdv = \frac{1}{t}\int_0^{\pi/2t}\tau_4'(v)cosvtdv + \frac{1}{t}\int_{\pi/2t}^n \tau_4'(v)cosvtdv. \tag{2.46}$$

From [2], we know that

$$\int_{\pi/2n}^{\pi/2}|\frac{1}{t}\int_0^{\pi/2t}\tau_4'(v)cosvtdv|dt = \frac{1}{\varphi(n)}\int_1^n\psi_2(v)dv + O(1)\psi_2(n).$$

Now by considering the second integral on the right side of equality (2.46), we will estimate the following asymptotic statement:

$$\int_{\pi/2n}^{\pi/2} |\frac{1}{t} \int_{\pi/2t}^{n} \tau_4'(v) cosvtdv| dt = O(1)\psi_2(n). \tag{2.47}$$

Thus, we take into account the function

$$\rho_t(x) = \int_{\pi/2t}^{x} \kappa(v) cosvtdv \ \ , \ x > 0, \ t > 0 \tag{2.48}$$

where $\kappa(v)$ is nonnegative and nondecreasing function for all $v \geq 1$. The function $\rho_t(x)$ is a continuous function for every fixed t. Further, on each interval between the successive zeros v_k and v_{k+1} of the function $cosvt$ the function $\rho_t(x)$ has one simple zero x_k. Thus by assuming that x_k' is zero the nearest from the left of the point n. Therefore, by setting $\kappa(v) = \tau_4'(v)$ on interval $[1, n]$ in (2.48) , we find

$$\frac{1}{t} \int_{\pi/2t}^{n} \tau_4'(v) cosvtdv = \frac{1}{t} \int_{x_k'}^{n} \tau_4'(v) cosvtdv$$

$$\int_{\pi/2n}^{\pi/2} |\frac{1}{t} \int_{\pi/2t}^{n} \tau_4'(v) cosvtdv| dt \leq \int_{\pi/2n}^{\pi/2} \frac{1}{t} \tau_4'(n)(n - x_k') dt = O(1)\psi_2(n)$$

Hence we have (2.47). Therefore we get (2.30).

□

Proof of Theorem 2.4. The proof of Theorem 2.4 is get similar to the proof of Theorem 2.1. □

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