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Generalized Derivations on State Residuated Lattices

Mbarek Zaoui and Driss Gretete

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Abstract In this paper, we introduce the notion of a generalized state derivation D on a state residuated lattices, determined by a derivation d and a state operator τ. Also, we discuss some related properties of isotone (resp. strong) generalized derivations and give some characterization of (good) ideal generalized derivations. Moreover, we obtain that the fixed point set of ideal generalized derivations in a Heyting algebra is a residuated lattice.

1 Introduction

States on MV-algebras were introduced by Mundici [17] with the intent of measuring the average truth-value of propositions in Łukasiewicz logic, which is a generalization of probability measures on Boolean algebras. From then on, the notion of states has been deeply investigated in other logical algebras and many profound results have been achieved, Such as BL-algebras, MTL-algebras, and residuated lattices [12].

The notion of derivation is a very interesting and important area of research, also helpful for studying structures and properties in algebraic systems. In 1957, Posner [15] introduced the notion of derivation in a prim ring $(R, +, .)$. In 2004, Jun and Xin [10] applied the notion of derivations to BCI-algebras. In 2005, Zhan and Liu [20] introduced the notion of f-derivation of BCI-algebras. In 2008, Xin et al. [23] proposed the concept of a derivation on a lattice $(L, \wedge, \vee)$. In the same year, Çeven and Özturk [24] introduced the notion of an f-derivation on a lattice. In 2016, He et al. [7] introduced the concept of derivation in a residuated lattice, and they characterized some special types of residuated lattices in terms of derivations. In 2018, Rachunek and Salunova [19] introduced the concept of derivations and a complete description of all derivations on a non-commutative generalization of MV-algebras. In the same year, Liang et al. [13] presented the notions of derivations on EQ-algebras and obtained many special types of them. In addition, Wang et al. [26] introduced the notion of derivations of commutative multiplicative semilattices, they investigated the related properties of some special derivations and gave some characterizations. In 2019, Wang et al. [27] gave some representations of MV-algebras in terms of derivations. Rasheed and Majeed [18] studied some results of (α, β)-derivations on prime seeding. Dey et al. [5] considered generalized orthogonal derivations of semiprimary rings. Ciungu [4] studied the properties of implicit derivations in pseudo-BCI-algebras. Chaudhuri [3] discussed (σ, τ)-derivations of group rings. In 2020, Guven [6] proposed the notion of (σ, τ)-derivations generalized on rings and discussed some related aspects. Hosseini and Fosner [9] studied the image of left Jordan derivations on algebras. Ali and Rahaman [1] studied a pair of generalized derivations in rings. Zhu et al. [21] introduced the notion of a generalized derivation and investigated some related properties of them. In 2021, Ling and Zhu [14] proposed a generalization of a derivation in a residuated lattice and some related properties were investigated.

This paper aims to combine derivations, state operators, and residuated lattices. The notion of generalized derivation on a state residuated lattices D from L to L is introduced, determined by a derivation d and state operator τ. More precisely, for any $x, y \in L$, we propose the following formula: $D(x \otimes y) = (D(x) \otimes \tau(y)) \vee (\tau(x) \otimes d(y))$. Meanwhile, we discuss and investigate some related properties.

This paper is organized as follows. In section 2, we recall some concepts and results on residuated lattices, state operators, and derivations. In section 3, we propose the notion of generalized derivation on a state residuated lattices (L, τ) and investigate some related properties of isotone, strong, ideal, and good ideal generalized derivations. Also, we define the notion of fixed point. In particular, we obtain that the fixed point set in a Heyting algebra is still a residuated lattice.

2 Preliminaries

We assume that the reader is familiar with the classical results concerning residuated lattices, but to make this work more self-contained, we briefly introduce some basic notions used in the rest of the work.

Definition 2.1. [25] An algebraic structure $(L, \wedge, \vee, \otimes, \rightarrow, 0, 1)$ of type $(2, 2, 2, 2, 0, 0)$ is called a bounded commutative residuated lattice (simply called a residuated lattice) if:

(i) $(L, \wedge, \vee, 0, 1)$ is a bounded lattice;

(ii) $(L, \otimes, 1)$ is a monoid with unit element 1;

(iii) For all $x, y, z \in L$, $x \otimes y \leq z$ if and only if $x \leq y \to z$.

We denote by L a residuated lattice $(L, \wedge, \vee, \otimes, \to, 0, 1)$.

For any $x \in L$ and a natural number n, we define $x^{'} = x \to 0$, which is a negation in a sense. $x^{''} = (x^{'})^{'}$, $x^0 = 1$, $x^n = x^{n-1} \otimes x$ for all $n \geq 1$.

Proposition 2.2. *[25] For all $x, y, z, w \in L$, we have:*

(i) $1 \to x = x$, $x \to 1 = 1$*;*

(ii) $x \leq y$ *if and only if* $x \to y = 1$*;*

(iii) If $x \leq y$*, then* $z \to x \leq z \to y$ *and* $y \to z \leq x \to z$*;*

(iv) If $x \leq y$ *and* $z \leq w$ *then* $x \otimes z \leq y \otimes w$*;*

(v) $x \otimes y \leq x \wedge y$, $x \otimes x^{'} = 0$*;*

(vi) $0^{'} = 1$, $1^{'} = 0$, $x \leq x^{''}$*;*

(vii) $x \otimes y = 0$ *if and only if* $x \leq y^{'}$*;*

(viii) $x \otimes (y \vee z) = (x \otimes y) \vee (x \otimes z)$*;*

(ix) $x \to (y \to z) = (x \otimes y) \to z = y \to (x \to z)$*.*

The notion of a state BL-algebra was introduced by Ciungu. In 2015, as a generalization of the notion of a state BL-algebra, He et al. [8] introduced the notion of a state residuated lattice as follows.

Definition 2.3. [8] A mapping: $\tau : L \longrightarrow L$ is called a state operator on L if it satisfies the following conditions, for all $x, y, z \in L$,

(i) $\tau(0) = 0$;

(ii) $x \to y = 1$ implies $\tau(x) \to \tau(y) = 1$;

(iii) $\tau(x \to y) = \tau(x) \to \tau(x \wedge y)$;

(iv) $\tau(x \otimes y) = \tau(x) \otimes \tau(x \to (x \otimes y))$;

(v) $\tau(\tau(x) \otimes \tau(y)) = \tau(x) \otimes \tau(y)$;

(vi) $\tau(\tau(x) \to \tau(y)) = \tau(x) \to \tau(y)$;

(vii) $\tau(\tau(x) \vee \tau(y)) = \tau(x) \vee \tau(y)$;

(viii) $\tau(\tau(x) \wedge \tau(y)) = \tau(x) \wedge \tau(y)$.

In what follows, we denote by (L, τ) a state residuated lattice.

Example 2.4. [8] Let $L = [0, 1]$ be a reel unit interval. For all $x, y \in L$, we define $x \otimes y = min\{x, y\}$ and

$$x \longrightarrow y = \begin{cases} 1, & x \leq y, \\ y, & \text{otherwise}. \end{cases}$$

Then $(L, min, max, \otimes, \longrightarrow, 0, 1)$ becomes a residuated lattice, which is called Gödel structure. Now, for any $a \in L$, we define a map τ_a on L as follows:

$$\tau_a(x) = \begin{cases} x, & x \leq a, \\ 1, & \text{otherwise}. \end{cases}$$

One can check that τ_a is a state operator on L. Therefore, (L, τ_a) is a state residuated lattice.

Next, we present some properties of state operators on residuated lattices.

Proposition 2.5. *[8] Let (L, τ) be a state residuated lattice. Then, for all $x, y, z \in L$, the following properties hold.*

(i) $\tau(1) = 1$, $\tau(x^{'}) = \tau(x)^{'}$, $\tau(\tau(x)) = \tau(x)$*;*

(ii) If $x \leq y$*, then* $\tau(x) \leq \tau(y)$*;*

Definition 2.6. A mapping $f : L \longrightarrow L$ is called a homomorphism if it satisfies the following conditions:

(i) $f(1) = 1$, $f(0) = 0$;

(ii) $f(x * y) = f(x) * f(y)$, for all $x, y \in L$ and $* \in \{\wedge, \vee, \otimes, \longrightarrow\}$.

Next, we recall a class of residuated lattices, which are Heyting algebras.

Definition 2.7. [2] A lattice $(L, \vee, \wedge)$ is called to be a Heyting algebra if for any $x, y \in L$, there exists $x \to y \in L$ such that $z \leq x \to y$ if and only if $z \wedge x \leq y$ for all $z \in L$.

We have the following characterization for Heyting algebras.

Theorem 2.8. *[16] Let $(L, \vee, \wedge, \otimes, 0, 1)$ be a residuated lattice. Then, the following statements are equivalent:*

(i) L is a Heyting algebra;

(ii) $x \otimes y = x \wedge y = x \otimes (x \to y)$ for all $x, y \in L$.

An element $x \in L$ is called complemented if there exists an element $y \in L$ such that $x \wedge y = 0$ and $x \vee y = 1$. By $B(L)$, we mean the set of all complemented elements of L, i.e.,

$$B(L) = \{x \in L : \exists y \in L, \ x \wedge y = 0, \ x \vee y = 1\}.$$

Proposition 2.9. *[11] For a residuated lattice L we have*

(i) $x \in B(L)$ if and only if $x \vee x' = 1$;

(ii) If $x \in B(L)$, then $x \wedge y = x \otimes y$ for all $y \in L$;

(iii) If $x \in B(L)$, then $x \otimes x = x$.

At the end of this section, we give the notion of derivation in a residuated lattice L as follows.

Definition 2.10. [7] A mapping d: $L \longrightarrow L$ is called a multiplicative derivation on L if it satisfies the following conditions: for any $x, y \in L$,

$$d(x \otimes y) = (d(x) \otimes y) \vee (x \otimes d(y)).$$

3 Generalized derivations of a state Residuated Lattices

In this section, we give the notion of generalized derivations on state Residuated Lattices (L, τ). then, we study some properties of those generalized derivations.

Definition 3.1. Let (L, τ) be a state residuated lattice and $d : L \longrightarrow L$ a derivations on L. A mapping D: $L \longrightarrow L$ is called a generalized derivation determined by d if

$$D(x \otimes y) = (D(x) \otimes \tau(y)) \vee (\tau(x) \otimes d(y))$$

for any $x, y \in L$.

Now, we present some examples for generalized derivations on state residuated lattice (L, τ).

Example 3.2. Let (L, τ) be a state residuated lattice and d a zero derivation. Define a map $D : L \longrightarrow L$ by $D(x) = 0$ for all $x \in L$, then D is a generalized derivation on (L, τ), which is called a zero generalized derivation.

Moreover, we define a map $D : L \longrightarrow L$ by $D(x) = x$ for all $x \in L$, $\tau : L \longrightarrow L$ by $\tau(x) = x$ for all $x \in L$ and d an identity derivation on L. Then D is a generalized derivation on (L, τ), which is called an identity generalized derivation on (L, τ).

If $D(x) = d(x)$ for all $x \in L$, and τ a state operator on L. Then, D is a generalized derivation on (L, τ).

Example 3.3. Let $L = \{0, a, b, 1\}$ be a chain and the operations $\otimes$, $\to$ be defined as follows:

$\otimes$	0	a	b	1
0	0	0	0	0
a	0	0	a	a
b	0	a	b	b
1	0	a	b	1

$\to$	0	a	b	1
0	1	1	1	1
a	a	1	1	1
b	0	a	1	1
1	0	a	b	1

Then it is easy to verify that $L = \{0, a, b, 1\}$ is a residuated lattice, where $x \wedge y = min\{x, y\}$ and $x \vee y = max\{x, y\}$.

We define a mapping $\tau : L \longrightarrow L$ as follows: for all $x \in L$,

$$\tau(x) = \begin{cases} 0, & x = 0, \\ a, & x = a, \\ 1, & x = b, 1. \end{cases}$$

It is easy to see that (L, τ) is a state residuated lattice. Moreover, we define a mapping $d : L \longrightarrow L$ as follows: for all $x \in L$,

$$d(x) = \begin{cases} 0, & x = 0, a, \\ a, & x = b, 1. \end{cases}$$

One can check that d is a derivation of L. Based on d, we define a mapping $D : L \longrightarrow L$ as follows: for all $x \in L$,

$$D(x) = \begin{cases} 0, & x = 0, \\ a, & x = a, \\ 1, & x = b, 1. \end{cases}$$

We can see that D is a generalized state derivation on (L, τ).

Now, we show some properties of generalized derivation on state residuated lattices (L, τ).

Proposition 3.4. *Let D be a generalized derivation on (L, τ) determined by a derivation d, then the following statements hold:*

(i) $D(0) = 0$;

(ii) $D(1) \otimes \tau(x) \leq D(x)$ *and* $d(x) \leq D(x)$ *for all* $x \in L$;

(iii) $d(1) \otimes \tau(x) \leq D(x)$ *for all* $x \in L$;

(iv) $D(x) \leq \tau(x)^{''}$ *and* $d(x^{'}) \leq \tau(x^{'}) \leq (D(x))^{'}$ *for all* $x \in L$;

(v) $D(x) \otimes \tau(x^{n-1}) \leq D(x^n)$ *and* $\tau(x)^{n-1} \otimes d(x) \leq D(x^n)$ *for all* $x \in L$, $n \geq 1$;

(vi) *if* $x \leq y^{'}$, *then* $D(x) \leq \tau(y)^{'}$ *and* $d(y) \leq \tau(x)^{'}$ *for all* $x, y \in L$.

Proof. (1) It follows from definition that $D(0) = d(0 \otimes 0) = (D(0) \otimes \tau(0)) \vee (\tau(0) \otimes d(0)) = (0 \otimes 0) \vee (0 \otimes 0) = (0 \otimes 0) = 0$, i.e. $D(0) = 0$.

(2) Let $x \in L$. Then $D(x) = D(1 \otimes x) = (D(1) \otimes \tau(x)) \vee (\tau(1) \otimes d(x)) = (D(1) \otimes \tau(x)) \vee d(x)$. Hence, $D(1) \otimes \tau(x) \leq D(x)$ and $d(x) \leq D(x)$.

(3) Let $x \in L$. Then $D(x) = D(x \otimes 1) = (D(x) \otimes \tau(1)) \vee (\tau(x) \otimes d(1)) = D(x) \vee (\tau(x) \otimes d(1))$. Hence, $d(1) \otimes \tau(x) \leq D(x)$.

(4) Let $x \in L$. Since $x \otimes x^{'} = 0$. Then,

$$\begin{aligned} D(0) &= D(x \otimes x^{'}) \\ &= (D(x) \otimes \tau(x^{'})) \vee (\tau(x) \otimes d(x^{'})) \\ &= 0, \end{aligned}$$

which implies $D(x) \otimes \tau(x^{'}) = D(x) \otimes \tau(x)^{'} = 0$ and $\tau(x) \otimes d(x^{'}) = 0$, then $D(x) \leq \tau(x)^{''}$ and $d(x^{'}) \leq \tau(x^{'}) \leq (D(x))^{'}$.

(5) It follows from Definition 3.1 that $D(x^2) = D(x \otimes x) = (D(x) \otimes \tau(x)) \vee (\tau(x) \otimes d(x))$ for all $x \in L$, which implies $D(x) \otimes \tau(x) \leq D(x^2)$ and $\tau(x) \otimes d(x) \leq D(x^2)$. By induction, we can obtain $D(x) \otimes \tau(x^{n-1}) \leq D(x^n)$ and $\tau(x) \otimes d(x^{n-1}) = \tau(x) \otimes \tau(x^{n-2}) \otimes d(x) \leq D(x^n)$. Since $\tau(x)^{n-2} \leq \tau(x^{n-2})$ we have $\tau(x)^{n-1} \otimes d(x) \leq D(x^n)$ for all $n \geq 1$.

(6) Let $x, y \in L$ and $x \leq y^{'}$. Then, $x \otimes y = 0$. Thus, $D(x \otimes y) = (D(x) \otimes \tau(y)) \vee (\tau(x) \otimes d(y)) = 0$, which implies $D(x) \otimes \tau(y) = 0$ and $\tau(x) \otimes d(y) = 0$. Therefore, $D(x) \leq \tau(y)^{'}$ and $d(y) \leq \tau(x)^{'}$. □

In what follows, we introduce ideal generalized derivations in a state residuated lattice and investigate some related properties of them.

Definition 3.5. Let (L, τ) be a state residuated lattice and D be a generalized derivation on (L, τ).

(i) D is called an isotone generalized derivation provided that $x \leq y$ implies $D(x) \leq D(y)$ for all $x, y \in L$;

(ii) D is called a strong generalized derivation provided that $D(x) \leq \tau(x)$ for all $x \in L$.

In particular, if D is both isotone and strong, we call D an ideal generalized derivation on (L, τ).

Example 3.6. Let L and τ in the example Defined in example 3.3, and a derivation $d : L \longrightarrow L$ by $d(x) = o$ for all $x \in L$. Let $k \in L$. We define a mapping $D : L \longrightarrow L$ by $d(x) = k \otimes \tau(x)$ for all $x \in L$. We can check that D is an ideal generalized derivation on (L, τ).

Proposition 3.7. *Let D be a generalized derivation on (L, τ). If D is strong and $\tau(x) \leq x$ for all $x \in L$, then d is contractive (i.e, $d(x) \leq x$ for all $x \in L$).*

Proof. We know that $d(x) \leq D(x)$ for all $x \in L$. Since D is strong (i.e., $D(x) \leq \tau(x)$ for all $x \in L$), it holds that $d(x) \leq \tau(x) \leq x$ for all $x \in L$. Thus, d is contractive. □

Proposition 3.8. *Let D be a generalized derivation on (L, τ). If d is isotone, then D is isotone.*

Proof. Assume that d is isotone and let $x, y \in L$ such that $x \leq y$, then $d(x) \leq d(y)$ and $\tau(x) \leq \tau(y)$. Thus, $D(x) = (D(1) \otimes \tau(x)) \vee d(x) \leq (D(1) \otimes \tau(y)) \vee d(y) = d(y)$. Consequently, D is isotone. □

Proposition 3.9. *Let D be an isotone generalized derivation on (L, τ). Then the following statements hold:*

(i) *If* $z \leq x \to y$, *then* $\tau(z) \leq D(x) \to D(y)$ *and* $\tau(x) \leq d(z) \to D(y)$ *for all* $x, y, z \in L$;

(ii) $\tau(x \to y) \leq D(x) \to D(y)$ *and* $d(x \to y) \leq \tau(x) \to D(y)$ *for all* $x, y \in L$;

(iii) $\tau(x) \leq d(y) \to D(x)$ *for all* $x, y \in L$.

Proof. (1) Let $x, y, z \in L$ and $z \leq x \to y$. Then $x \otimes z \leq y$. Since D is an isotone generalized derivation on (L, τ), we have $(D(x) \otimes \tau(z)) \vee (\tau(x) \otimes d(z)) \leq D(y)$, then $D(x) \otimes \tau(z) \leq D(y)$ and $\tau(x) \otimes d(z) \leq D(y)$. Therefore $\tau(z) \leq D(x) \to D(y)$ and $\tau(x) \leq d(z) \to D(y)$.

(2) Since $x \otimes (x \to y) \leq y$, we have $D(x \otimes (x \to y)) \leq D(y)$. It follows that $(D(x) \otimes \tau(x \to y)) \vee (\tau(x) \otimes d(x \to y)) \leq D(y)$, which implies $D(x) \otimes \tau(x \to y) \leq D(y)$ and $\tau(x) \otimes d(x \to y) \leq D(y)$, Therefore $\tau(x \to y) \leq D(x) \to D(y)$ and $d(x \to y) \leq \tau(x) \to D(y)$ for all $x, y \in L$.

(3) Since $x \otimes y \leq x$ for all $x, y \in L$, we have $D(x \otimes y) \leq D(x)$. It follows that $(D(x) \otimes \tau(y)) \vee (\tau(x) \otimes d(y)) \leq D(x)$. Thus, $\tau(x) \otimes d(y) \leq D(x)$. Therefore, $\tau(x) \leq d(y) \to D(x)$. □

Proposition 3.10. *Let D be a strong generalized derivation on (L, τ). Then the following statements hold:*

(i) $D(x) \otimes d(y) \leq D(x \otimes y) \leq D(x) \vee D(y)$ *for all* $x, y \in L$*;*

(ii) $D(1) = 1$ *if and only if* $D(x) = \tau(x)$ *for all* $x \in L$.

Proof. (1) Let $x, y \in L$. Since D is a strong generalized derivation on (L, τ), we have $D(x) \otimes d(y) \leq \tau(x) \otimes d(y)$ and $D(x) \otimes d(y) \leq D(x) \otimes D(y) \leq D(x) \otimes \tau(y)$. Then, $D(x) \otimes d(y) \leq (D(x) \otimes \tau(y)) \vee (\tau(x) \otimes d(y)) = D(x \otimes y)$. On the other hand, since $D(x) \otimes \tau(y) \leq D(x)$ and $\tau(x) \otimes d(y) \leq d(y) \leq D(y)$, we have $D(x \otimes y) \leq D(x) \vee D(y)$. Therefore, $D(x) \otimes d(y) \leq D(x \otimes y) \leq D(x) \vee D(y)$.

(2) Let $x \in L$. From proposition 3.4, we have $D(1) \otimes \tau(x) \leq D(x)$ for all $x \in L$, then $\tau(x) = D(1) \otimes \tau(x) \leq D(x) \leq \tau(x)$. Therefore, $D(x) = \tau(x)$ for all $x \in L$. On the other hand, if $D(x) = \tau(x)$ for all $x \in L$, then $D(1) = \tau(1) = 1$. □

Proposition 3.11. *Let D be an ideal generalized derivation on (L, τ), then: $d(x \to y) \leq D(x) \to D(y)) \leq D(x) \to \tau(y)$ for all $x, y \in L$.*

Proof. Let $x, y \in L$. Since $x \otimes (x \to y) \leq y$ and D is isotone, we have $D(x \otimes (x \to y)) \leq D(y)$, and from statement (1) of proposition 3.10, we have $D(x) \otimes d(x \to y) \leq D(x \otimes (x \to y))$, which implies $D(x) \otimes d(x \to y) \leq D(y)$, then $d(x \to y) \leq D(x) \to D(y)$. On the other hand, since D is strong, we have $D(y) \leq \tau(y)$, then $D(x) \to D(y) \leq D(x) \to \tau(y)$. Therefore, $d(x \to y) \leq D(x) \to D(y)) \leq D(x) \to \tau(y)$. □

Theorem 3.12. *Let D be a generalized derivation on (L, τ). Then the following statements hold:*

(i) If $D(x) \to D(y) = D(x) \to \tau(y)$, for all $x, y \in L$, then D is an ideal generalized derivation on (L, τ);

(ii) The converse holds if we assume that $D(1) = 1$.

Proof. (1) Assume that $D(x) \to D(y) = D(x) \to \tau(y)$ for all $x, y \in L$. Since $D(x) \otimes 1 \leq D(x)$, we have $1 \leq D(x) \to D(x) = D(x) \to \tau(x)$, then $D(x) = D(x) \otimes 1 \leq \tau(x)$ for all $x \in L$, which implies D is strong. Moreover, for all $x, y \in L$, let $x \leq y$, we have $\tau(x) \leq \tau(y)$. Thus, $D(x) \otimes 1 \leq D(x) \leq \tau(x) \leq \tau(y)$, i.e., $D(x) \otimes 1 \leq \tau(y)$, which implies $1 \leq D(x) \to \tau(y) = D(x) \to D(y)$, that is, $D(x) \leq D(y)$. Hence, D is isotone. Therefore, D is an ideal generalized derivation on (L, τ).

(2) Let $x, y \in L$ and D be an ideal generalized derivation on (L, τ). Since $D(y) \leq \tau(y)$, it holds that $D(x) \to D(y) \leq D(x) \to \tau(y)$. On the other hand, if $D(1) = 1$ then, $\tau(y) = D(1) \otimes \tau(y) \leq D(y)$, which implies that $D(x) \to \tau(y) \leq D(x) \to D(y)$. Therefore, we obtain $D(x) \to D(y) = D(x) \to \tau(y)$, for all $x, y \in L$. □

Theorem 3.13. *Let (L, τ) be a state-morphism residuated lattice and D be a strong generalized derivation on (L, τ). If $D(1) \in B(L)$, then the following statements are equivalent: for all $x, y \in L$,*

(i) D *is an ideal generalized derivation on* (L, τ)*;*

(ii) $D(x) \leq D(1)$*;*

(iii) $D(x) = D(1) \otimes \tau(x)$*;*

(iv) $D(x \wedge y) = D(x) \wedge D(y)$*;*

(v) $D(x \vee y) = D(x) \vee D(y)$*;*

(vi) $D(x \otimes y) = D(x) \otimes D(y)$.

Proof. (1) $\Rightarrow$ (2) Let $x \in L$. Since $x \leq 1$, and D is isotone, then $D(x) \leq D(1)$.
(2) $\Rightarrow$ (3) Suppose that $D(x) \leq D(1)$ for all $x \in L$. Notice that $D(1) \in B(L)$, we obtain $D(x) = D(1) \wedge D(x) = D(1) \otimes D(x) \leq D(1) \otimes \tau(x)$. On the other hand, since $D(1) \otimes \tau(x) \leq D(x)$. We obtain, $D(x) = D(1) \otimes \tau(x)$ for all $x \in L$.
(3) $\Rightarrow$ (4) Let $D(x) = D(1) \otimes \tau(x)$ for all $x \in L$. Then, for all $x, y \in L$, it holds that

$$\begin{aligned} D(x \wedge y) &= D(1) \otimes \tau(x \wedge y) \\ &= D(1) \wedge \tau(x \wedge y) \\ &= D(1) \wedge (\tau(x) \wedge \tau(y)) \\ &= (D(1) \wedge \tau(x)) \wedge (D(1) \wedge \tau(y)) \\ &= (D(1) \otimes \tau(x)) \wedge (D(1) \otimes \tau(y)) \\ &= D(x) \wedge D(y). \end{aligned}$$

(4) $\Rightarrow$ (1) Assume that $x \leq y$, then, $x \wedge y = x$. An consequence of (4) is that $D(x) = D(x \wedge y) = D(x) \wedge D(y)$, which implies $D(x) \leq D(y)$ for all $x, y \in L$. Thus, D is an ideal generalized derivation on (L, τ).
(3) $\Rightarrow$ (5) For all $x, y \in L$, it follows from (3) that

$$\begin{aligned} D(x \vee y) &= D(1) \otimes \tau(x \vee y) \\ &= D(1) \otimes (\tau(x) \vee \tau(y)) \\ &= (D(1) \otimes \tau(x)) \vee (D(1) \otimes \tau(y)) \\ &= D(x) \vee D(y). \end{aligned}$$

(5) $\Rightarrow$ (1) Let $x, y \in L$ such that $x \leq y$. It follows from (5) that $D(y) = D(x \vee y) = D(x) \vee D(y)$. Then we conclude that $D(x) \leq D(y)$ for all $x, y \in L$. Therefore, D is an ideal generalized derivation on (L, τ).
(3) $\Rightarrow$ (6) For all $x, y \in L$, it follows from (3) that

$$\begin{aligned} D(x \otimes y) &= D(1) \otimes \tau(x \otimes y) \\ &= D(1) \otimes (\tau(x) \otimes \tau(y)) \\ &= (D(1) \otimes \tau(x)) \otimes (D(1) \otimes \tau(y)) \\ &= D(x) \otimes D(y). \end{aligned}$$

(6) $\Rightarrow$ (2) Let $x \in L$. Then it holds that

$$\begin{aligned} D(x) &= D(x \otimes 1) \\ &= D(x) \otimes D(1) \\ &= D(x) \wedge D(1). \end{aligned}$$

Thus, $D(x) \leq D(1)$ for all $x \in L$. □

An ideal generalized derivation is said to be good if $D(1) \in B(L)$.

Proposition 3.14. *Let D be a good ideal generalized derivation on (L, τ), then:*
$D(x) = \tau(x) \otimes D(1)$ for all $x \in L$.

Proof. Let $x \in L$. We have $\tau(x) \otimes D(1) \leq D(x)$. On the other hand, since $D(x) \leq D(1)$ and $D(x) \leq \tau(x)$, we have $D(x) \leq D(1) \wedge \tau(x) = D(1) \otimes \tau(x)$, which implies $D(x) = \tau(x) \otimes D(1)$. □

Let (L, τ) be a state-morphism residuated lattice and $a \in L$. We define a map $D_a : L \longrightarrow L$ as follows: $D_a(x) = a \otimes \tau(x)$ for all $x \in L$, determined by a state derivation $d_a(x) = a \otimes \tau(x)$ for all $x \in L$ [22].

Proposition 3.15. *Let (L, τ) be a state-morphism residuated lattice and $a \in L$. Then the map D_a is an ideal generalized derivation on (L, τ), which is called a principal ideal generalized derivation.*

Proof. Let $x, y \in L$, then

$$\begin{aligned} D_a(x \otimes y) &= a \otimes \tau(x \otimes y) \\ &= (a \otimes \tau(x \otimes y)) \vee (a \otimes \tau(x \otimes y)) \\ &= (a \otimes \tau(x) \otimes \tau(y)) \vee (a \otimes \tau(x) \otimes \tau(y)) \\ &= (D_a(x) \otimes \tau(y)) \vee (\tau(x) \otimes d_a(y)). \end{aligned}$$

Then, D_a is a generalized derivation.
For all $x, y \in L$, let $x \leq y$, we have $\tau(x) \leq \tau(y)$. Thus, $D_a(x) = a \otimes \tau(x) \leq a \otimes \tau(y) = D_a(y)$, that is D_a is isotone. Moreover, since $a \leq 1$, we have $D_a(x) = a \otimes \tau(x) \leq \tau(x)$ for all $x \in L$, which implies that D_a is strong. Therefore, D_a is an ideal generalized derivation on (L, τ). □

Next, we discuss the structures and properties of the fixed point set of ideal generalized derivation. Firstly, we give the concept of the fixed point set of a generalized derivation in a state residuated lattice (L, τ) as follows.

Definition 3.16. Let D be an ideal generalized derivation on (L, τ). Define a set $Fix_D(L) = \{x \in L : D(x) = x\}$. $Fix_D(L)$ is called the set of fixed elements of L for D.

Now, we investigate some operations of $Fix_D(L)$.

Proposition 3.17. *Let D be an ideal generalized derivation on (L, τ) and $\tau(x) \leq x$ for all $x \in L$. Then we have:*
for all $x, y \in Fix_D(L)$: $x \otimes y,\ x \vee y\ \in Fix_D(L)$.

Proof. Let $x, y \in Fix_D(L)$, we have $D(x) = x$ and $D(y) = y$. Then, $x \otimes y = D(x) \otimes D(y) \leq D(x) \otimes \tau(y) \leq D(x \otimes y)$. On the other hand, since D is an ideal generalized derivation on (L, τ), we have $D(x \otimes y) \leq \tau(x \otimes y) \leq x \otimes y$, which implies $D(x \otimes y) = x \otimes y$. Therefore, $x \otimes y \in Fix_D(L)$. Moreover, since D is an ideal generalized derivation on (L, τ), we have $x \vee y = D(x) \vee D(y) \leq D(x \vee y) \leq \tau(x \vee y) \leq x \vee y$, then we have $D(x \vee y) = x \vee y$, which implies that $x \vee y \in Fix_D(L)$. □

Theorem 3.18. *Let L be a Heyting algebra, D an ideal generalized derivation on (L, τ) and $\tau(x) \leq x$ for all $x \in L$. Then $(Fix_D(L), \wedge, \vee, \otimes, \longmapsto, 0, \overline{1})$ is a residuated lattice, where $x \longmapsto y = D(x \longrightarrow y)$ and $\overline{1} = D(1)$ for all $x, y \in L$.*

Proof. We complete the proof by three steps.

1. First, we show that $(Fix_D(L), \wedge, \vee, \otimes, \longmapsto, 0, \overline{1})$ is a bounded lattice with 0 as the smallest element and $\overline{1}$ as the greatest element. From Proposition 3.17 and Theorem 2.8, we have $Fix_D(L)$ is closed under $\vee$ and $\wedge$. Therefore, $(Fix_D(L), \wedge, \vee)$ is a lattice. Let $x \in Fix_D(L)$, we have, $x \wedge 0 = 0$ and

$$\begin{aligned} x \vee D(1) &= D(x) \vee D(1) \\ &= D(x \vee 1) \\ &= D(1). \end{aligned}$$

Therefore, 0 the smallest element and $\overline{1} = D(1)$ is the greatest element in $Fix_D(L)$.

2. Next,we prove that $(Fix_D(L), \otimes, \overline{1})$ is a commutative monoid with $\overline{1} = D(1)$ as neutral element. It follows from Proposition 3.17 that $(Fix_D(L), \otimes)$ is closed under $\otimes$, and easy to show that it satisfies associative laws. Thus, $(Fix_D(L), \otimes)$ is a commutative semigroup. Let $x \in Fix_D(L)$,

$$\begin{aligned} x \otimes \overline{1} &= D(x) \otimes D(1) \\ &= D(x \otimes 1) \\ &= D(x) \\ &= x, \end{aligned}$$

which implies $\overline{1} = D(1)$ is a neutral element.

3. Finally, we show that $x \otimes y \leq z$ if and only if $y \leq x \mapsto z$ for all $x, y \in Fix_D(L)$. We have for all $x, y, z \in Fix_D(L)$

$$\begin{aligned} x \otimes y \leq z &\Leftrightarrow y \leq x \rightarrow z \\ &\Leftrightarrow D(y) \leq D(x \rightarrow z) \\ &\Leftrightarrow D(y) \leq x \mapsto z \\ &\Leftrightarrow y \leq x \mapsto z. \end{aligned}$$

Therefore, $(Fix_D(L), \wedge, \vee, \otimes, \longmapsto, 0, \overline{1})$ is a residuated lattice. □

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Author information

Mbarek Zaoui, University of Ibn Tofail, National school of Applied Sciences ENSAK, Morocco.
E-mail: zaouimbarek@yahoo.fr, mbarek.zaoui@uit.ac.ma

Driss Gretete, University of Ibn Tofail, National school of Applied Sciences ENSAK, Morocco.
E-mail: driss.gretete@uit.ac.ma

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