Palestine Journal of Mathematics

Vol. 12(Special Issue III) (2023) , 13–18

Characterizations of *FI*-semi injective modules in terms of their endomorphism rings

S. Chase and M. K. Patel

MSC 2010 Classifications: 16D50, 16D70, 16D80.

Keywords and phrases: $FI - M-$p-injective modules; semi-injective modules; Endomorphism rings.

M. K. Patel thank the National Board for Higher Mathematics, for financial assistantship vide file No.: 02211/3/2019 NBHM (R.P.) RD-II/1439.

Abstract In this work, we explore some beautiful properties of $FI-$semi injective modules and $FI-$self-p-injective ring as a generalization of semi injective module and self-principally injective ring respectively. We also discuss various characterizations of $FI-$semi injective modules in terms of their endomorphism rings. Finally, we prove that the property of being $FI-$semi injective of a module is Morita invariant.

In this paper, R is considered to be an associative ring with unity and the modules considered are unital right modules over R. For a module M, if a submodule K has a non-trivial intersection with each non-trivial submodule of M, K is said to be a large submodule or an essential submodule. In this case we shall denote it by $K \subseteq_e M$. If we can get an onto homomorphism $M^{(I)} \to K$ for some index set I, we say the module K is $M-$generated. K is termed as finitely $M-$generated if this happens for a finite index set I. More specifically, if there is an isomorphism between a submodule K of an $R-$module M and M/N, for some submodule N of M, or equivalently, if $K = f(M)$ for some endomorphism f of M_R, then K is known as an $M-$cyclic submodule of M. If $f(U)$ is included in U for all $f \in End(M_R)$ for a submodule U of M, then U is fully invariant. It is easy to establish that 0 and M are trivial examples of submodules of M that are fully invariant, for a $R-$module M. It is worth noting that the submodules of R_R that are fully invariant, are the two-sided ideals of R. If all of the submodules of a module M are fully invariant, the module is called a duo. In the context of rings, a ring R is known to be a duo if it is a duo module when viewed as a module over itself. Some clear examples of duo rings are commutative rings and division rings. On the other hand, any matrix ring of order 2, over commutative rings or division rings, is not a duo ring. A module that generate all its submodules is called a self generator module.The Jacobson radical of a ring R is denoted by the symbol $J(R)$. $l(K)$ and $r(K)$ denote the left and right annihilators of any subset K of a module M, respectively. $Z(R_R)$ and $Z(_RR)$ will be used to denote the singular ideals, the kernel and image of an $R-$homomorphism f by $ker(f)$ and $im(f)$ respectively.

Lately, properties and characterizations related to principally injective rings and principally injective modules are being studied by several authors ([8], [11]). A principally injective ring R is one in which any homomorphism from a right ideal of R that is also principal to R can be expressed as a left-hand multiplication by an element of R. In the context of a module M, if each homomorphism $f : aR \to M$, $a \in R$ is extendable to R, then M is known as principally injective. This concept was extended to modules by Sanh et. al. in [13]. They generalized the idea to $M-$principal injective from that of principal injective, for some module M. If every homomorphism from $f(M)$ to N is extendable to some homomorphism from M to N, N is termed an $M-$principally injective module for modules M and N. If a module M is such that it is $M-$principally injective, then it is called semi-injective (refer [10], [13], [16]). In this paper, fully invariant $M-$cyclic submodules, $f(M)$ of M where f is an endomorphism of M_R, are considered instead of just M-cyclic submodules of M. This gives the idea of $FI-$semi-injective modules as an extension of semi-injective modules. By generalising the notions of $M-$principally injective and semi-injective modules, we introduce some concepts of $FI - M-$principally injective and $FI-$semi-injective modules. As a result, we can deduce the following conclusion:

Quasi-injective ⇒ Semi-injective ⇒ $FI-$semi-injective module .

For usual definitions and standard notations, we refer to [1], [7] and [15]

1 $FI-M-$ Principal Injectivity

If there is an extended homomorphism from M to K for every homomorphism from $s(M)$ to K, $s \in End(M_R)$ and $s(M)$ is an $M-$cyclic submodule of M that is fully invariant, then K is termed fully invariant$-M-$principally injective (briefly, $FI-M-$principally injective or $FI-M-$p-injective).

$$\begin{array}{ccccc} O & \longrightarrow & s(M) & \xrightarrow{\;i\;} & M \\ & & \big\downarrow f & \swarrow g & \\ & & K & & \end{array}$$

In other words, K is $FI-M-$p-injective, if $s(M)$ is an $M-$cyclic submodule of M which is also fully invariant, every homomorphism $f : s(M) \to K$, can be splitted as $f = g \circ i$, where the homomorphism from M to K is denoted by g and the inclusion map to M from $s(M)$ is denoted by i. If K is $FI-R-$principally injective then K is called $FI-$principally injective. For examples, we may take Z_4 and Z_6 as modules over Z. It can be verified that Z_4 is $FI-Z_6-$p-injective and Z_6 is $FI-Z_4-$p-injective. If M is $FI-M-$p-injective, then M is called $FI-$semi-injective. For a ring R, if R_R is $FI-R-$p-injective then R is called right $FI-$self-p-injective. As examples, we can see that Z_4 and Z_6, considered as modules over Z, are $FI-$semi-injective modules, whereas Z is not $FI-$semi-injective over itself. In fact, examples of $FI-$ semi-injective modules consists of modules that are simple, semisimple, semi-injective, quasi-injective, $FI-$self-p-injective rings and direct summands of these.

We discuss some properties and characterizations in the following propositions.

Proposition 1.1. *The following statements are identical for two modules A and B:*

(i) B is $FI-A-$principally injective.

(ii) For any $A-$cyclic submodule M of A, B is $FI-M-$principally injective .

(iii) For any $A-$cyclic submodule M of A and N a direct summand of B, N is $FI-M-$principally injective.

Proof. $(i) \Rightarrow (ii)$ If an $M-$cyclic submodule K of M is fully invariant, then K is also a fully invariant $A-$cyclic submodule of A, according to Lemma 1.1 (iii) of [9]. Since, B is $FI-A-$principally injective, by restricting the existing homomorphism on M, we get the proof.
$(ii) \Rightarrow (iii)$ Suppose $f : K \to N$ is any homomorphism. Consider the embedding $j_1 : N \to B$, the projection $\pi_1 : B \to N$ and the inclusion map $i : K \to M$. The existence of an $R-$homomorphism $g : M \to B$ is guaranteed by B being $FI-M-$principally injective, such that $j_1 f = gi \Rightarrow \pi_1 j_1 f = \pi_1 gi \Rightarrow If = hi$ where $I = \pi_1 j_1$ and $h = \pi_1 g$, which is an $R-$homomorphism to N from M. Hence, we get, $f = hi$. Thus, N is $FI-M-$p-injective.
$(iii) \Rightarrow (i)$ This follows by exchanging the roles of B with N and A with M. □

Corollary 1.2. *If there are two modules A and B, with B being a $FI-A-$principally injective module. Consider direct summands M of A and N of B with M being fully invariant, then we have:*

(i) B is $FI-M-$p-injective module.

(ii) N is $FI-A-$p-injective module.

(iii) N is $FI-M-$p-injective module.

A homomorphism f from M to N, such that $f(M)$ is a fully invariant submodule of N, will be termed as a fully invariant homomorphism. We shall discuss some properties of $FI-$semi injective modules and $FI-$self-p-injective rings below.

Proposition 1.3. *The following assertions are equivalent if $S = End(M_R)$ and M is duo $R-$module:*

(i) M *is* $FI-$*semi injective.*

(ii) $l_S(ker(f)) = S_f, \forall f \in S$.

(iii) If $ker(g) \subseteq ker(f)$*, then* $S_f \subseteq S_g, \forall f, g \in S$.

(iv) $l_S[im(g) \cap ker(f)] = l_S(im(g)) + S_f, \forall f, g \in S$.

Proof. $(i) \Rightarrow (ii)$ Suppose M is an $FI-$semi-injective module. We shall show that $l_S(ker(f)) \subseteq S_f$ and $S_f \subseteq l_S(ker(f))$. Let $g \in l_S(ker(f))$, we have $g(ker(f)) = 0 \Rightarrow ker(f) \subseteq ker(g)$. Let $f' : M \to f(M)$ and $g' : M \to g(M)$ be induced $R-$homomorphisms from f and $g : M \to M$ respectively and consider the embeddings $i_1 : f(M) \to M$ and $i_2 : g(M) \to M$. As f' is an epimorphism, we obtain an $R-$homomorphism $\alpha : f(M) \to g(M)$ satisfying $\alpha f' = g'$. Hence, there is an $R-$ homomorphism $\beta \in End(M_R)$ satisfying $\beta i_1 = i_2\alpha$, because M is $FI-$semi injective. Therefore, $g = \beta f \Rightarrow g \in S_f$ i.e., $l_S(ker(f)) \subseteq S_f$. For the other part, since $f \in l_S(ker(f))$, we get $S_f \subseteq l_S(ker(f))$. Thus combining these two parts, we have, $l_S(ker(f)) = S_f$.
$(ii) \Rightarrow (iii)$ Assume that $l_S(ker(f)) = S_f$. If $ker(g)$ is contained in $ker(f)$, then $l_S(ker(f)) \subseteq l_S(ker(g)) \Rightarrow S_f \subseteq S_g$ by our assumption.
$(iii) \Rightarrow (iv)$ Assume that, $ker(g) \subseteq ker(f) \Rightarrow S_f \subseteq S_g$. We shall show that $l_S[im(g) \cap ker(f)]$ is in $l_S(im(g)) + S_f$ and also $l_S(im(g)) + S_f$ is in $l_S[im(g) \cap ker(f)]$. If $\alpha \in l_S[im(g) \cap ker(f)]$, then $\alpha(im(g) \cap ker(f)) = 0$. Let $x \in ker(fg) \Rightarrow (fg)(x) = 0 \Rightarrow im(g) \subseteq ker(f) \Rightarrow (im(g)) = 0 \Rightarrow x \in ker(\alpha g)$, because $\alpha(im(g) \cap ker(f)) = 0$. Thus showing that $ker(fg)$ is a subset of $ker(\alpha g)$. Hence, by our assumption $S_{\alpha g} \subseteq S_{fg} \Rightarrow \alpha g = \beta fg$ for some $\beta \in S$. This implies that $g(\alpha - \beta f) = 0 \Rightarrow (\alpha - \beta f) \in l_S(im(g))$, i.e.,$\alpha \in l_S(im(g)) + S_f$. Thus $l_S[im(g) \cap ker(f)] \subseteq l_S(im(g)) + S_f$.
Conversely, let $y \in l_S(im(g)) + S_f \Rightarrow y = p + q$, where $p[im(g)] = 0$ and $q[ker(f)] = 0$. This asserts that $y \in l_S[im(g) \cap ker(f)]$. Thus $l_S(im(g)) + S_f \subseteq l_S[im(g) \cap ker(f)]$.
Combining these two parts we get the required result.
$(iv) \Rightarrow (ii)$ This part follows by taking $g = I_M$, we get $l_S(ker(f)) = S_f$.
$(iii) \Rightarrow (i)$ Assume that $ker(g) \subseteq ker(f)$, then $S_f \subseteq S_g$, let $f \in S$ induce a map $f' : M \to f(M)$ and consider the embedding $i_1 : f(M) \to M$. Let $\phi : f(M) \to M$ where $f(M)$ is an $M-$cyclic submodule of M that is fully invariant. Then $\phi f'$ is an $R-$endomorphism of M with $ker(f) \subseteq ker(\phi f') \Rightarrow S_{\phi f'} \subseteq S_f \Rightarrow \phi f' = \theta f$ for some $\theta \in S$. Thus showing M to be an $FI-$semi-injective module. □

Corollary 1.4. *The following assertions are equivalent for a commutative ring* R*:*

(i) R *is right* $FI-$*self-p-injective.*

(ii) $lr(u) = Ru$ *for all* u *in* R.

(iii) $r(v) \subseteq r(u)$ *for* $u, v \in R \Rightarrow Ru \subseteq Rv$.

(iv) $l(vR \cap r(u)) = l(vR) + Ru$ *for all* $u, v \in R$.

Proposition 1.5. *For a duo* $FI-$*semi-injective module* M *and* $f, g \in S = End(M_R)$*, we have:*

(i) Let $\alpha : f(M) \to g(M)$ *be monomorphism, then there exist an epimorphism* $\beta : S_g \to S_f$.

(ii) Let $\alpha : f(M) \to g(M)$ *be epimorphism, then there exist monomorphism* $\beta : S_g \to S_f$.

(iii) Let $f(M)$ *be isomorphic to* $g(M)$*, then* S_f *is isomorphic to* S_g.

Proof. (i) If $\alpha : f(M) \to g(M)$ is an injective homomorphism and $i_1 : f(M) \to M$ and $i_2 : g(M) \to M$ are embeddings, Then $f \in S$ induces a map $f' : M \to f(M)$ (i.e., $i_1 f' = f$). Because M is $FI-$semi injective, an endomorphism $\alpha' : M \to M$ exists which extends the homomorphism $i_2\alpha : f(M) \to M$ such that $\alpha' i_1 = i_2\alpha$. Now define $S-$homomorphism $\beta : S_g \to S_f$ as $\beta(sg) = s\alpha' f$, for all $s \in S$. Since $im(\alpha' f) \subseteq f(M) \subseteq g(M) = im(g)$ implying that β is well defined, also in other way if $s_1 g = s_2 g \Rightarrow s_1\alpha' f = s_2\alpha' f \Rightarrow \beta(s_1 g) = \beta(s_2 g)$. For any $t \in S$, homomorphism $ti_1 : f(M) \to M$ is extendable to an $R-$homomorphism $\theta \in S$, such

that $\theta i_2\alpha = ti_1$. Consequently, we have $\beta(\theta g) = \theta\alpha' f = \theta\alpha' i_1 f' = \theta i_2 \alpha f' = ti_1 f' = tf$. Thus showing that $\beta : S_g \to S_f$ is an onto homomorphism.
(ii) Adopting the same notations used in proving (i) except that we let $\alpha : f(M) \to g(M)$ to be an onto homomorphism instead of an injective homomorphism. Since, M is $FI-$semi injective, $i_2\alpha : f(M) \to M$ is extendable to $\alpha' \in S$ such that $\alpha' i_1 = i_2\alpha$. Now define $S-$homomorphism $\beta : S_g \to Sf$ as $\beta(sg) = s\alpha' f$, for all $s \in S$. Since $im(\alpha' f) = im(g)$, β is properly defined and it is clearly seen that β is an $S-$homomorphism which is injective. Thus, S_g can be embedded into S_f.
(iii) This immediately follows from the above two statements. □

Corollary 1.6. *For any $u, v \in R$, where R is a commutative $FI-$self-p-injective ring, the following statements holds:*

(i) Ru can be embedded into Rv if uR is an image of vR.

(ii) Rv is an image of Ru if vR is embedded in uR.

(iii) $Ru \cong Rv$ if $vR \cong uR$.

If a module M generates all its submodules, then it is called a self-generator module [15]. Let M be self-generator, then for every $m \in M$, we have $mR = \Sigma_{s\in I} s(M)$ for some $I \subseteq S$. Let $\Delta = \{s \in S | ker(s) \subseteq_e M\}$, a generalization of Theorem 2.13 [13] is provided in the theorem below.

Theorem 1.7. *For a self-generator $FI-$semi injective module $\Delta = J(S)$.*

Proof. We shall establish this result by proving that Δ and $J(S)$ are contained in each other. So, for any $f \in \Delta$, $ker(f) \cap ker(1 - f) = 0$ implies that $ker(1 - f) = 0$. Thus by Proposition 1.3 (ii), $S = l_S(ker(1 - f)) = S(1 - f)$. Thus $f \in J(S)$. this proves that $\Delta \subseteq J(S)$. For other direction, let $f \in J(S)$, if $g(M) \cap ker(f) = 0$ for some $g \in S$, then $g = 0$. We have $S = l_S[im(g) \cap ker(f)] = l_S(im(g)) + S_f$ by Proposition 1.3(iv). Hence, $l_S(im(g)) = S$, i.e., $g = 0$. Since by hypothesis, M is self generator, we have $mR = \Sigma_{g\in I} g(M)$ for some $I \subseteq S$ and for any $m \in M$. If $ker(f) \cap mR = 0$, then $ker(f) \cap g(M) = 0$ for all $g \in I$ and hence, $mR = 0$. Thus showing that $ker(f) \subseteq_e M$, i.e., $f \in \Delta$ implying $J(S)$ is contained in Δ. Thus proving that $J(S) = \Delta$. □

Corollary 1.8. *If R is an $FI-$self-p-injective commutative ring, then $Z(R_R) = J(R)$. Consequently, if R is both sided $FI-$self-p-injective ring, then $Z(_RR) = Z(R_R)$.*

Theorem 1.9. *If M is $FI-$semi-injective module and $s_i \in S = End(M_R), (1 \leq i \leq n)$, where s_i's are fully invariant endomorphisms, such that $\Sigma_{i=1}^n S_{s_i}$ is direct, then any homomorphism $\phi : \Sigma_{i=1}^n S_i(M) \to M$ is extendable to a homomorphism $\theta \in S$.*

Proof. Since each s_i's are fully invariant endomorphisms, for each $i = 1, 2, ..., n$, $s_i(M)$ are $M-$cyclic and fully invariant submodules of M. Since, M is $FI-$semi injective, homomorphism $\theta_i : M \to M$ exists such that $\theta_i s_i = \phi s_i$. It follows that $\Sigma_{i=1}^n \theta_i s_i = \Sigma_{i=1}^n \phi s_i$. Since $(\Sigma_{i=1}^n s_i)(M) \subseteq \Sigma_{i=1}^n s_i(M)$, ϕ is extendable to $\theta \in SM$ such that, for any $m \in M, \theta(\Sigma_{i=1}^n s_i)(m) = \phi(\Sigma_{i=1}^n s_i)(m)$, i.e.,$\Sigma_{i=1}^n \theta s_i = \Sigma_{i=1}^n \phi s_i$. Thus, $\Sigma_{i=1}^n \theta s_i = \Sigma_{i=1}^n \theta_i s_i$. From ΣS_{s_i} being direct, $\theta s_i = \theta_i s_i$ for all $i = 1, 2, ..., n$. Hence, for any $x \in \Sigma_{i=1}^n S_i(M)$, we get $\phi(x) = \theta(x)$, completing the proof. □

Corollary 1.10. *If R is a right $FI-$self-p-injective ring, the sum $\Sigma_{i=1}^n Rx_i$ is direct and x_iR is fully invariant ideal for all $x_i \in R$, then any linear map $\phi : \Sigma_{i=1}^n x_iR \to R$ is extendable to $\theta : R \to R$.*

If there is an isomorphism from a module M to every submodule of M, then it is called a self-similar module [12].

Proposition 1.11. *The following assertions are equivalent for a duo self-similar module M:*

(i) M *is* $FI-$*semi-injective module.*

(ii) M *is semisimple.*

Proof. Proof is straightforward. □

Corollary 1.12. *R_R being semisimple and R_R being $FI-$semi-injective coincide whenever R_R is duo self-similar.*

Thus by the above corollary, it can be concluded that the R is commutative $FI-$self-p-injective ring if R_R is duo self-similar and semisimple.

For Morita equivalent rings R and S, the property P of a module M in $modR$ is called Morita invariant if, whenever M has P, $\alpha(M)$ has P, for every additive equivalence $\alpha : modR \to modS$.

Proposition 1.13. *The property of being $FI - N-$principally injective of a module is Morita invariant.*

Proof. For Morita equivalent rings R and S, with equivalence $\alpha : modR \to modS$ and $\beta : modS \to modR$, with natural isomorphism $\sigma : \beta\circ\alpha \to 1_{modR}$ and $\sigma^{-1} : \alpha\circ\beta \to 1_{modS}$, let $M \in modR$ be $FI-N-$principally injective. We claim that $\alpha(M)$ is $FI-\alpha(N)-$principally injective object in $modS$. Taking N'_S to be $\alpha(N)-$ cyclic submodule of $\alpha(N)$ in $modS$, that is fully invariant, and $i : N'_S \to \alpha(N)$ be the injective homomorphism. For $\alpha(M)$ is $FI - \alpha(N)-$principally injective, any $S-$morphism $f : N'_S \to \alpha(M)$ is extendable to an $S-$morphism $g : \alpha(N) \to \alpha(M)$ such that $g \circ i = f$. Since any category equivalence preserve injective homomorphism and fully invariant properties, applying β implies $\beta(i)$ is injective homomorphism. So, there exist $h : N \to M$ such that $h \circ \sigma_N \circ \beta(i) = \sigma_M \circ \beta(f)$. We claim that $g = \alpha(h)$ satisfies $f = \alpha(h) \circ i$. Since β is faithful, it suffices to show that $\beta(f) = \beta \circ \alpha(h) \circ \beta(i)$. However, $\beta \circ \alpha(h) = \sigma_M^{-1} \circ h \circ \sigma_N$ because σ is natural, which gives $\beta(f) = \sigma_M^{-1} \circ h \circ \sigma_N \circ \beta(i)$, which shows that M is $FI - N-$principally injective. It follows since for some $R-$module M_R, every $S-$submodule is isomorphic to $\alpha(M)$.

□

Corollary 1.14. *The property of being $FI-$semi injective of a module is Morita invariant.*

References

[1] F. W. Anderson, K. R. Fuller, *Rings and Categories of Modules*, Springer-Verlag, New York, (1974).

[2] G. F. Birkenmeier, B. J. Muller and S. T. Rizvi, Modules in which every fully invariant submodule is essential in a direct summand, *Communication in Algebra*, Volume 30, 1395-1415, (2002).

[3] H. Cartan, S. Eilenberg, *Homological Algebra*, Princeton University Press, (1956).

[4] V. Kumar, A. J. Gupta, B. M. Pandeya, M. K. Patel, C-pseudo injective modules, *East-West J. of Mathematics*, Vol. 14, No 1, 68-76,(2012).

[5] V. Kumar, A. J. Gupta, B. M. Pandeya, M. K. Patel, M-SP-Injective module, *Asian European Joural of Mathematics*, Vol 5, 1-11, (2011).

[6] R. P. Kurshan, Rings whose cyclic module has finitely generated socle, *Journal of Algebra*, Vol. 15, 376-386, (1970).

[7] S. H. Mohamed, B. J. Muller, *Continuous and Discrete module*, Cambridge University Press, (1990).

[8] W. K. Nicholson, M. F. Yousif, Principally Injective Rings, *Journal of Algebra*, 174, 77-93, (1995).

[9] M. K. Patel, S. Chase, $FI-$ Semi-Injective Modules, *Palestine Journal of Mathematics*, Vol. 11(1), 182-190, (2022).

[10] M. K. Patel, B. M. Pandeya, A. J. Gupta, V. Kumar, Quasi-P-Injective modules, *International Journal of Algebra*, Vol. 4, 1255-1259, (2010).

[11] G. Puninski, R. Wisbauer, M. F. Yousif, On P-injective ring, *Glasgow Math. Journal*, Vol. 37, 373-378, (1995).

[12] V. S. Rodrigues, A. A. Santana, A note on a problem due to Zelmanowitz, *Algebra and Discrete Mathematics*, 3, 85-93, (2009).

[13] N. V. Sanh, K. P. Shum, S. Dhompongsa, S. Wongwai, On Quasi Principally Injective Modules, *Algebra Colloquium*, 6(3), 269-276, (1999).

[14] K. Varadarajan, Properties of endomorphism rings, *Acta Math. Hunger.*, 74(1-2), 83-92, (1997).

[15] R. Wisbauer, *Foundation of Module and Ring Theory*, Gordon and Breach, London-tokyo, (1991).

[16] S. Wongwai, On the endomorphism ring of a semi-injective module, *Acta. Math. Univ. Comenianae*, Vol. 71, 27-33, (2002).

Author information

S. Chase and M. K. Patel, Department of Mathematics, National Institute of Technology Nagaland, Dimapur-797103, Nagaland, India.
E-mail: sedevikho.chase@gmail.com mkpitb@gmail.com